Relaxation approximations to second-order traffic flow models by high-resolution schemes

I.K. Nikolos, A.I. Delis and M.Papageorgiou

School of Production Engineering & Management, Technical University of Crete, University Campus, Chania 73100, Crete, Greece.

Abstract. A relaxation-type approximation of second-order non-equilibrium traffic models, written in conservation or balance law form, is considered. Using the relaxation approximation, the nonlinear equations are transformed to a semi-linear diagonilizable problem with linear characteristic variables and stiff source terms with the attractive feature that neither Riemann solvers nor characteristic decompositions are in need. In particular, it is only necessary to provide the flux and source term functions and an estimate of the characteristic speeds. To discretize the resulting relaxation system, high-resolution reconstructions in space are considered. Emphasis is given on a fifth-order WENO scheme and its performance. The computations reported demonstrate the simplicity and versatility of relaxation schemes as numerical solvers.

Keywords: Second-order traffic flow models; Numerical modeling; Relaxation schemes; High-resolution
PACS: 47.11.Df; 47.35.-i; 47.10.ab

INTRODUCTION

Over the past years several approaches have been followed to model the dynamics of traffic flow, the most successful being microscopic car-following models, gas-kinetic models, and macroscopic traffic models. Comprehensive descriptions of such models can be found, for example, in [1, 2, 3]. Macroscopic models consider the traffic as an anisotropic continuum, where vehicle dynamics are described in terms of spatial vehicle density and average velocity, as functions of location and time.

The main motivation behind the present work is the development of a common numerical framework for the macroscopic simulation of traffic flow, transparent to the traffic flow model used by considering second-order non-equilibrium traffic flow models. The requirement for the use of different macroscopic models (existing or new developments) under the same computational environment imposes the avoidance of developing new complex numerical solvers. The approach followed here includes the adoption of the relaxation approach proposed in [4] along with finite volume discretizations of a linearized system of differential equations, which renders the methodology independent of the use of Riemann solvers. Within this approach, the differences between the various traffic flow models are taken into account through the corresponding flux and source term functions, plus their different parameters and upper bounds of the corresponding eigenvalues of the flux's Jacobian. Spatial discretizations of different order of accuracy are incorporated; a first-order upwind scheme, a second-order MUSCL and a fifth-order weighted essentially non-oscillatory (WENO) scheme are introduced. Implicit-explicit (IMEX) Runge-Kutta (RK) schemes are also introduced for time discretization. The relaxation approach of [4], has found wide application in fluid dynamics problems, we refer, for example, to [5, 6, 7], among others. The main advantage of such schemes is that neither Riemann solvers, nor the explicit computation of eigenvalues are needed, which renders this methodology ideal for problems where an analytic expression for the eigenvalues of the systems' Jacobian matrix may not be possible or is computationally tedious to obtain, or the Riemann problems are difficult to approximate [7]. However, the work on relaxation schemes for traffic flow problems includes, thus far, few works on second- or higher-order schemes for the Lighthill-Whitham-Richards (LWR) model and its multi-lane variants, and low-order schemes on standard second-order traffic flow models such as the Aw and Rascle [8] see, for example, [7, 9, 10, 11, 12].

The proposed computational framework is formulated as to have a generalized description of variables and fluxes and to deal with different traffic flow models and future modifications. Thus, we depart by considering the general formulation of two-equation traffic flow models, with a possible source term present, all written as nonlinear systems of conservation laws with initial data, as

$$\begin{aligned} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) &= \mathbf{s}(\mathbf{u}), \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \end{aligned} \tag{1}$$

where the functions $\mathbf{u}$, $\mathbf{f}(\mathbf{u})$ and $\mathbf{s}(\mathbf{u}) \in \mathbb{R}^2$. When the source term $\mathbf{s}(\mathbf{u})$ is present system (1) is called a balance law. In what follows, we will denote, as functions in space x and time t, $\rho(x,t)$ the car or traffic density, $u(x,t)$ the average speed and $q = \rho u$ the traffic flow rate. Systems in the form of (1) can be rewritten in quasi-linear form $\partial_t \mathbf{u} + \mathbf{J}(\mathbf{u})\partial_x \mathbf{u} = \mathbf{s}(\mathbf{u})$, where $\mathbf{J}(\mathbf{u}) = \partial \mathbf{f}/\partial \mathbf{u}$ is the Jacobian matrix of the system. Solving for the eigenvalues, from $\det[\mathbf{J}(\mathbf{u}) - \lambda \mathbf{I}] = 0$, certain realistic features of the traffic models can be verified or not, e.g. the anisotropic nature of traffic flow. Real and distinct eigenvalues, $\lambda_{1,2}$, guarantee hyperbolicity. The solutions related to the $k-$th eigenvalue are called $k-$family wave solutions. These elementary wave solutions are constituted by shock waves, rarefaction waves and contact discontinuities. The correct numerical representation of these elementary solutions is a requirement by any numerical scheme approximating (1).

TRAFFIC FLOW MODELS AND THE RELAXATION MODEL

The Aw-Rascle (AR) model. Aw and Rascle [8] proposed a two-equation traffic flow model which in conservation law form for $\mathbf{u} = [\rho,\ \rho(u + P(\rho))]^{\mathrm{T}}$ reads as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0; \\ \partial_t\left(\rho(u + P(\rho))\right) + \partial_x\left(\rho u(u + P(\rho)\right) = 0, \end{cases} \tag{2}$$

where $P(\rho)$ is an increasing function of density, so as to insure that the model caries the anisotropic property, given as $P(\rho) = C_0^2 \rho^\gamma,\ \ \gamma > 0$. Solving for the eigenvalues two distinct, except for $\rho = 0$, i.e. at vacuum, and real ones can be found namely, $\lambda_1 = u - \rho P'(\rho)$ and $\lambda_2 = u$. In this work, we use the values $C_0 = 1$ and $\gamma = 2$.

The non-local Gas-Kinetic based traffic model. In [13] (see also [3]) a macroscopic gas-kinetic-based traffic flow model (GKT model) was proposed which, in balance law form for $\mathbf{u} = [\rho,\ \rho u]^{\mathrm{T}}$, reads as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0; \\ \partial_t\left(\rho u\right) + \partial_x\left(\rho u^2 + \Theta\rho\right) = \rho\left(\dfrac{V^e(\rho) - u}{\tau}\right), \end{cases} \tag{3}$$

where $\Theta = A(\rho)u^2$, with $A(\rho)$ being a density dependent variance factor with $A(\rho) = A_0 + \delta A\left[1 + \tanh\left(\frac{\rho - \rho_{cr}}{\delta\rho}\right)\right]$ in which ρ_{cr} is the critical density and $A_0, \delta A$ and $\delta\rho$ are constant values given in [13, 3, 14] along with other typical values for this model. $V^e(\rho, u, \rho_a, u_a)$ denotes the (non-local and dynamic) equilibrium speed, which depends not only on the local density ρ and mean speed u but also on the non-local density ρ_a and mean speed u_a. Model (3) has two distinct eigenvalues $\lambda_{1,2} = u \pm C$, with $C = \sqrt{\partial\Theta/\partial\rho}$, which indicate that the model is also strictly hyperbolic.

The relaxation model. The class of relaxation models introduced in [4] is obtained by introducing the artificial variable $\mathbf{v}$ (relaxation variable) and the corresponding to (1) (now linear) relaxation system is given as

$$\begin{aligned} \partial_t \mathbf{u} + \partial_x \mathbf{v} &= \mathbf{s}(\mathbf{u}), \\ \partial_t \mathbf{v} + \mathbf{C}^2 \partial_x \mathbf{u} &= -\frac{1}{\varepsilon}\left(\mathbf{v} - \mathbf{f}(\mathbf{u})\right), \end{aligned} \tag{4}$$

with initial data $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ and $\mathbf{v}(x,0) = \mathbf{v}_0(x) = \mathbf{f}(\mathbf{u}_0(x))$, where the parameter ε is the *relaxation rate* ($0 < \varepsilon \ll 1$) and $\mathbf{C}^2 = \mathrm{diag}\{c_1^2, c_2^2\}$ matrix to be chosen. For small ε, applying the Chapman-Enskog expansion in system (4), see for example [4], we can derive the following approximation for $\mathbf{u}$

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u}) + \varepsilon\partial_x\left[\mathbf{J}(\mathbf{u})\mathbf{s}(\mathbf{u})\right] + \varepsilon\partial_x\left[\left(\mathbf{C}^2 - \mathbf{J}^2\right)\partial_x \mathbf{u}\right] + O(\varepsilon^2). \tag{5}$$

Equation (5) governs the first-order behavior of (4), with the third term on the right-hand side being an $O(\varepsilon)$ dominant dissipation term in the model and $\left(\mathbf{C}^2 - \mathbf{J}^2\right)$ being the diffusion coefficient matrix. Model (4) is well-posed only if $\left(\mathbf{C}^2 - \mathbf{J}^2\right)$ is positive semi-definite for all $\mathbf{u}$. This requirement on the diffusion matrix is the well-known *sub-characteristic condition*, [4], which ensures the dissipative nature of (5) and in 1D it is equivalent to

$$\lambda^2 \le c^2, \quad \text{where} \quad \lambda = \max_{1\le i\le 2}|\lambda_i| \quad \text{and} \quad c = \min_{1\le i\le 2}|c_i|. \tag{6}$$

It is important that this condition is satisfied also at the discrete level presented next.

NUMERICAL SCHEMES AND TEST RESULTS

To discretize system (4), let $x_i = i\Delta x$, $x_{i\pm\frac{1}{2}} = (i\pm\frac{1}{2})\Delta x$, where, for simplicity, we assume that Δx is a uniform spatial step. The approximate cell average of the variable $\mathbf{u}$ in the cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time t is denoted as $\mathbf{u}_i(t)$, i.e., $\mathbf{u}_i(t) = \frac{1}{\Delta x}\int_{I_i}\mathbf{u}(x,t)dx$ and the approximate value of $\mathbf{u}$ at $(x_{i+\frac{1}{2}},t)$ by $\mathbf{u}_{i+\frac{1}{2}}(t)$. Integrating (4) the semi-discrete relaxation system can be reformulated as a system of autonomous ordinary differential equations as

$$\frac{d\mathscr{Y}}{dt} = \mathscr{F}(\mathscr{Y}) - \frac{1}{\varepsilon}\mathscr{G}(\mathscr{Y}), \tag{7}$$

with the time-dependent vector functions given as

$$\mathscr{Y} = \begin{pmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{pmatrix}, \quad \mathscr{F}(\mathscr{Y}) = \begin{pmatrix} \mathbf{s}(\mathbf{u})_i - \mathfrak{D}_x\mathbf{v}_i \\ -\mathbf{C}^2\mathfrak{D}_x\mathbf{u}_i \end{pmatrix} \text{ and } \mathscr{G}(\mathscr{Y}) = \begin{pmatrix} 0 \\ \mathbf{v}_i - \mathbf{f}(\mathbf{u})_i \end{pmatrix},$$

with

$$\mathfrak{D}_x\mathbf{v}_i = \frac{\mathbf{v}_{i+\frac{1}{2}} - \mathbf{v}_{i-\frac{1}{2}}}{\Delta x} \quad \text{and} \quad \mathfrak{D}_x\mathbf{u}_i = \frac{\mathbf{u}_{i+\frac{1}{2}} - \mathbf{u}_{i-\frac{1}{2}}}{\Delta x}$$

and $\mathbf{s}(\mathbf{u})_i$ and $\mathbf{f}(\mathbf{u})_i$ are space averages of the source term and flux function, respectively. To complete the spatial discretization, it is necessary to evaluate the flux values $\mathbf{u}_{i\pm\frac{1}{2}}$ and $\mathbf{v}_{i\pm\frac{1}{2}}$. To this end three spatial discretizations of different order have been applied, a first-order upwind one, a second order MUSCL [15] and a fifth-order WENO [16].

The time marching scheme is based on implicit-explicit (IMEX) Runge-Kutta (RK) splitting. The non-stiff stage of the splitting in $\mathscr{F}(\mathscr{Y})$ in treated by an explicit RK scheme, while the stiff stage for $\mathscr{G}(\mathscr{Y})$ by diagonally implicit RK (DIRK) schemes, we refer to [17, 10, 14] for more details. By denoting with Δt^n the current time step and $\mathscr{Y}^n$ the approximate solution at time $t = t_n$, high-order relaxation schemes are under the usual CFL condition,

$$CFL = \max\left((\max_{i,k} c_k^n)\frac{\Delta t^n}{\Delta x}, \frac{\Delta t^n}{\Delta x}\right) \leq 1/2,$$

where the values of $c_k, k = 1,2$ are adjusted according to the solutions behavior by considering a global selection on every time step Δt^n. Based on the global maximum of each eigenvalue of the system's Jacobian matrices over the grid cells I_i, their choice is made as $c_k^n = \max_i|\lambda_k^n| + e$, $k = 1,2$ where e is a small correction parameter of $O(10^{-2})$, added to avoid the characteristic speeds from vanishing. In the numerical results presented next, the relaxation rate ε was set to 10^{-8} and the CFL value used was set to 0.4.

For the AR model we consider the situation of faster vehicles downstream followed by slower ones from behind. The initial condition for this problem is $[\rho(x,0),\ u(x,0)] = [0.4,\ 0.1]$ for $x \leq 8$ and $[\rho(x,0),\ u(x,0)] = [0.1,\ 0.9]$ for $x > 8$. The exact solution of this problem is given by a rarefaction wave connected to a vacuum state, which is followed by a contact discontinuity going downstream. This is a demanding test case due to the appearance of the vacuum and it is a challenge for any numerical scheme to correctly resolve the transition wave that connects the rarefaction wave to the vacuum state. To this end a relatively fine grid has to be used with $\Delta x = 0.02$. Results are presented in Fig. 1 for the MUSCL and WENO schemes. The WENO scheme almost perfectly predicts the appearance of the vacuum and the transition wave that connects the rarefaction with the vacuum state.

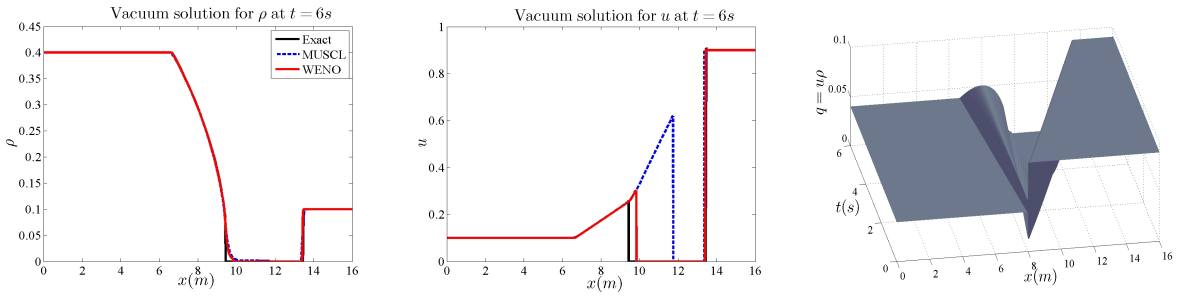

FIGURE 1. AR model: Vacuum solution for ρ (left), u (middle) and spatio-temporal evolution of the flow rate q (right) computed with the WENO scheme

Next, and for the GKT model, the development of traffic instabilities, starting with almost homogeneous initial traffic, is considered. Following [13, 14], we consider a dipole-like initial variation of the average density $\bar{\rho}$. For

$\bar{\rho} = 37$ a cascade of traffic jams emerges, i.e. stop-and-go traffic. This is a case which can be difficult to resolve with a low-order scheme. Referring to Fig. 2, the first-order relaxation scheme fails to reproduce all the stop-and-go waves when 400 grid points were used. When massively increasing the $npts$ to 4000 all emerging waves can be captured by this scheme but, obviously, with a high computational cost. Applying the WENO scheme, the correct wave structured is computed only with 400 grid points. This shows the significant advantage that can be gained when one implements a high-resolution scheme in the simulation. Finally, in Fig. 3, numerical results obtained with the WENO scheme for

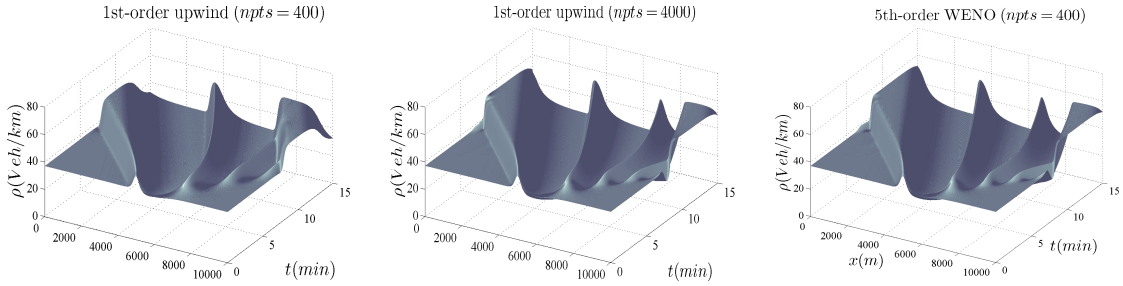

FIGURE 2. GKT model: Localized Perturbation and stop and go waves for $\bar{\rho} = 37$s

$\bar{\rho} = 26$ are shown. In this scenario a single density cluster is formed.

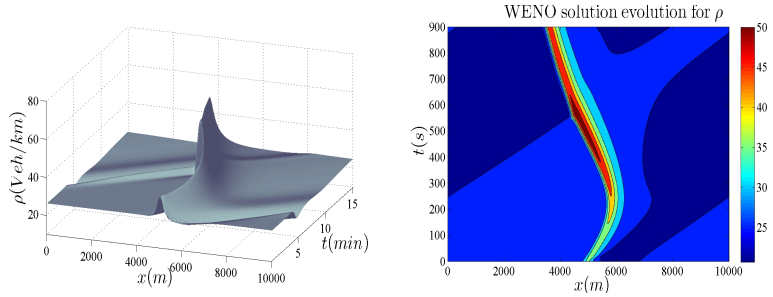

FIGURE 3. GKT model: Localized Perturbations $\bar{\rho} = 25$, periodic boundary conditions

Acknowledgments

This work was supported by TRAffic MANagement for the 21st century (TRAMAN21) ERC Advanced Investigator Grand under the European Union's Seventh Framework Programme (FP/2007-20013)

REFERENCES

1. S. Hoogendoorn, and P. Bovy, *Proceedings of the Institution of Mechanical Engineering. Part I: Journal of Systems and Control Engineering* **215**, 283 (2001).
2. N. Bellomo, and C. Dogbe, *SIAM Rev* **53**, 409 (2011).
3. M. Treiber, and A. Kesting, *Traffic flow dunamics: data, models and simulation*, Springer, 2013.
4. S. Jin, and Z. Xin, *Comm. Pure Appl. Math.* **48**, 235 (1995).
5. A. I. Delis, and T. Katsaounis, *Int. J. Numer. Meth. Fluids* **41**, 695 (2003).
6. M. Banda, and M. Seaid, *J. Numer. Math.* **13**, 171 (2005).
7. M. Seaid, *Applied Mathematics and Computation* **175**, 383 (2006).
8. A. Aw, and M. Rascle, *SIAM J. App. Math.* **60**, 916 (2000).
9. M. Herty, L. Pareschi, and M. Seaid, *SIAM J. Sci. Comput.* **28**, 1582 (2006).
10. J.-Z. Chen, Z.-K. Shi, and Y.-M. Hu, *J. Zejiang Univ Sci A* **10**, 1835 (2009).
11. J.-Z. Chen, Z.-K. Shi, and Y.-M. Hu, *J. Zejiang Univ Sci A* **13**, 29 (2012).
12. J. Ankamah, *Relaxation scheme for macroscopic traffic flow models*, Department of Mathematics TU-Kaiserlautern and TU-Eindhoven, M.Sc. thesis, 2012.
13. M. Treiber, A. Hennecke, and D. Helbing, *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics* **59**, 239 (1999).
14. A. I. Delis, I. K. Nikolos, and M. Papageorgiou, *Trans Res Part C* **to appear** (2014).
15. B. van Leer, *J. of Comp. Phys* **32**, 101 (1979).
16. R. Borges, M. Carmona, B. Costa, and W.-S. Don, *J. of Comp. Phys* **227**, 3191 (2008).
17. L. Pareschi, and G. Russo, *J. Sci. Comput* **25**, 129 (2005).