Nonlinear Adaptive Control Scheme for Discrete-Time Systems with Application to Freeway Traffic Flow Networks

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***Abstract*— This paper is devoted to the development of adaptive control schemes for uncertain discrete-time systems, which guarantee robust, global, exponential convergence to the desired equilibrium point of the system. The proposed control scheme consists of a nominal feedback law, which achieves robust, global, exponential stability properties when the vector of the parameters is known, in conjunction with a nonlinear, dead-beat observer. The obtained results are applicable to highly nonlinear, uncertain, discrete-time systems with unknown constant parameters. The applicability of the obtained results to real control problems is demonstrated by the rigorous application of the proposed adaptive control scheme to uncertain freeway models. A provided example demonstrates some features of the approach.**

I. INTRODUCTION

Adaptive control for discrete-time systems has been studied in many works (see, for instance, [1], [2], [3], [4]) and in many cases it is a direct extension of adaptive control schemes for continuous-time systems (see, [5]). Although discrete-time systems allow a direct study of the limitations of adaptive control schemes (see, for example, [6]), the major shortcoming of many adaptive control methodologies is that the closed-loop system does not exhibit an exponential convergence rate to the desired equilibrium point of the system, even if the nominal feedback law achieves global exponential stability properties when the parameters are precisely known.

This work is devoted to the development of adaptive control schemes for uncertain, discrete-time systems, which guarantee robust, global, exponential convergence to the desired equilibrium point of the system. The idea is simple: use a nominal feedback law, which achieves robust, global, exponential stability properties when the vector of the parameters is known, in conjunction with a nonlinear, dead-beat observer. The dead-beat observer (designed using an extension of the methodology described in [7]) achieves the precise knowledge of the vector of unknown parameters after a transient period; then the states of the closed-loop system are robustly led to the desired equilibrium point with an exponential rate by the nominal feedback law. The proposed adaptive scheme does not require the knowledge of a Lyapunov function for the closed-loop system under the action of the nominal feedback stabilizer.

The obtained results are applicable to highly nonlinear, uncertain discrete-time systems with unknown constant parameters. The applicability of the obtained results to real control problems is demonstrated by the rigorous application of the proposed adaptive control scheme to uncertain freeway models. However, it can also be applied to uncertain discrete-time systems arising from other application fields, such as fluid flow networks and robotics.

Traffic congestion in freeways leads to serious degradation of the infrastructure causing excessive delays, impacting traffic safety and the environment. Extensive research has been conducted to investigate and develop traffic control measures which can tackle this phenomenon. It is well known, that the efficiency of traffic operations can be enhanced by explicit feedback control approaches applied via ramp-metering or other control measures. For instance, the pioneering I-type regulator ALINEA [8] and its extensions [9], [10], as well as other proposed feedback control algorithms (see, e.g., [11], [12]) are explicit feedback control strategies that should guarantee local stability properties for the desired Uncongested Equilibrium Point (UEP) of a freeway model.

A Lyapunov approach was adopted in [13], which led to the robust, global exponential stabilization of the UEP of a nonlinear freeway model via an explicit feedback law. However, the nonlinear feedback stabilizer demands the knowledge of several model parameters, which are usually unknown. The present work proposes an adaptive control scheme, which is based on a dead-beat nonlinear observer and guarantees the robust, global exponential convergence rate to the desired UEP of the freeway model. The nonlinear freeway model considered in this work is a generalization of various freeway models (see [14], [15]).

The structure of the paper is as follows. Firstly, in Section II the robust, global, exponential, adaptive control scheme for nonlinear, uncertain, discrete-time systems is described. Then, in Section III, the obtained results are applied to uncertain freeway models while in Section IV, an illustrating example of a freeway model is presented, where it is shown that the proposed adaptive control scheme is robust, even if the vector of the unknown parameters is not constant. The concluding remarks of the paper are given in Section V.

Due to space limitations, all proofs are omitted and can be found in [16].

<u>*Definitions and Notation:*</u> Throughout this manuscript, we adopt the following notation and terminology:

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- For every set S, $S^n = \overbrace{S \times ... \times S}^{\text{n times}}$ for every positive integer n. For certain sets $S_1, S_2, ..., S_n$, the set $S_1 \times S_2 \times ... \times S_n$ is denoted by $\prod_{i=1}^{n} S_i$. Moreover, if $\Re_+ := [0, +\infty)$, then, $\Re_+^n := (\Re_+)^n$.
- Let $x \in \Re^n$. By $|x|$ we denote the Euclidean norm of $x \in \Re^n$ and by x' we denote the transpose of $x \in \Re^n$.
- When R is an index set, then by $(x_i; i \in R)$ we denote a vector with components all $x_i \in \Re$ with $i \in R$, in increasing order. For example, if $R = \{2, 5, 10\}$, then $(x_i; i \in R) = (x_2, x_5, x_{10})'$.
- We denote by $y^{(p)}(t) = (y(t-1), y(t-2), ..., y(t-p))$ for certain positive integer $p > 0$ the "p-history" of the signal $y(t)$ (defined for all $t \geq p$).

Consider the discrete-time system:

$$z^+ = F(d, z), z \in X \subseteq \Re^n, d \in D, \tag{1}$$

where $X \subseteq \Re^n$ is a non-empty closed set with $z^* \in X$, $D \subseteq \Re^l$ is a non-empty set, $F : D \times X \to X$ is a locally bounded mapping with $F(d, z^*) = z^*$ for all $d \in D$. In this work we adopt the following robust, exponential stability notion (see similar notions in [17], [18], [19]).

Definition 1.1: *We say that $z^* \in X$ is Robustly Globally Exponentially Stable (RGES) for system (1) if there exist constants $M, \sigma > 0$ such that for every $z_0 \in X$, $\{d_i \in D\}_{i=0}^{\infty}$, the solution $z(t)$ of (1) with $z(0) = z_0$ corresponding to $\{d_i \in D\}_{i=0}^{\infty}$ satisfies $|z(t) - z^*| \leq M \exp(-\sigma t)|z_0 - z^*|$ for all $t \geq 0$.*

II. Exponential Stabilization Of Systems With Unknown Parameters

We consider discrete-time systems with uncertain constant parameters and outputs. Consider the discrete-time system:

$$x^+ = f(d, \theta, x, u), x \in S, d \in D, u \in U, \tag{2}$$

where $S \subseteq \Re^n$, $D \subseteq \Re^l$, $U \subseteq \Re^m$, $\Theta \subseteq \Re^q$ are non-empty sets and $f : D \times \Theta \times S \times U \to S$ is a locally bounded mapping. In this setting, $x \in S$ denotes the state of the system (2), $d \in D$ is an unknown, time-varying input, $u \in U$ is the control input and $\theta \in \Theta$ denotes the vector of unknown, constant parameters. The measured output of the system is given by

$$y(t) = h(d(t), \theta, x(t)), \tag{3}$$

where $h : D \times \Theta \times S \to \Re^k$ is a locally bounded mapping. We assume that $x^* \in S$ is an equilibrium point for system (2) and $d \in D$ is a vanishing perturbation, i.e., there exist vectors $y^* \in h(D \times \{\theta\} \times S)$ such that $f(d, \theta, x^*, u^*) = x^*$, $y^* = h(d, \theta, x^*)$ for all $d \in D$. Moreover, let $Y \subseteq \Re^k$ be a set with $h(D \times \Theta \times S) \subseteq Y$.

The main result of this section provides sufficient conditions for dynamic, robust, global, exponential stabilization of the equilibrium point $x^* \in S$. The stabilizer is constructed under the following assumptions.

(H1) *Suppose that there exists a mapping $k : \Theta \times Y \to U$ such that $x^* \in S$ is RGES for the closed-loop system (2), (3) with $u = k(\theta, y)$.*

(H2) *Suppose that there exist a positive integer $p > 0$, a mapping $\Psi : Y \times A \to \Theta$ and a set $A \subseteq Y^p$ which contains a neighborhood of $(y^*, ..., y^*)$, such that for every sequence $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^{\infty}$ and for every $x_0 \in S$, the solution $x(t)$ of (2), (3) with $u = k(\hat{\theta}, y)$, initial condition $x(0) = x_0$ corresponding to inputs $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^{\infty}$ satisfies $\theta = \Psi(y(t), y^{(p)}(t))$ for all $t \geq p$ with $y^{(p)}(t) \in A$.*

(H3) *There exists a positive integer $m > 0$, such that for every sequence $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^{\infty}$ and for every $x_0 \in S$, the solution $x(t)$ of (2), (3) with $u = k(\hat{\theta}, y)$, initial condition $x(0) = x_0$ corresponding to inputs $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^{\infty}$ satisfies $y^{(p)}(t - i(t)) \in A$ for some $i(t) \in \{0, 1, ..., m\}$ and for all $t \geq m + p$.*

Assumption (H1) is a standard assumption, which guarantees the existence of a robust, global, exponential stabilizer when the vector of the parameters $\theta \in \Theta$ is known. Assumptions (H2), (H3) are equivalent to complete, robust observability assumption of θ from the output given by (3) (see also [7]).

Now, we are ready to state the main result of this section.

Theorem 2.1: *Consider system (2) with output given by (3) under assumption (H1), (H2), (H3). Moreover, suppose that the sets $f(D \times \Theta \times S \times U)$, Y, Θ are bounded. Finally, assume that there exist a constant $L \geq 0$, neighborhoods $N_1 \subseteq \Re^n$ of x^*, $N_2 \subseteq \Re^k$ of y^*, $N_3 \subseteq \Re^q$ of θ, such that the inequalities $|f(d, \theta, x, k(\hat{\theta}, h(d, \theta, x))) - x^*| + |h(d, \theta, x) - y^*| \leq L|x - x^*| + L|\hat{\theta} - \theta|$ and $|\Psi(h(d, \theta, x), w) - \theta| \leq L|x - x^*| + L\sum_{i=1}^{p} |w_i - y^*|$ hold for all $x \in N_1 \cap S$, $d \in D$, $\hat{\theta} \in N_3 \cap \Theta$, $w_i \in N_2 \cap Y (i = 1, ..., p)$ with $w = (w_1, ..., w_p)$. Then, the dynamic feedback stabilizer*

$$\begin{aligned} w_1^+ &= y, \\ w_2^+ &= w_1, \\ &\vdots \\ w_p^+ &= w_{p-1}, \\ \hat{\theta}^+ &= \begin{cases} \hat{\theta} & \text{if } w \notin A \\ \Psi(y, w) & \text{if } w \in A \end{cases} \\ u &= k(\hat{\theta}, y), \end{aligned} \tag{4}$$

where $w = (w_1, ..., w_p) \in Y^p$ and $\hat{\theta} \in \Theta$, achieves the following:

1) There exist constants $M, \sigma > 0$ such that for every sequence $\{d(i) \in D\}_{i=0}^{\infty}$ and for every $(x_0, w_0, \hat{\theta}_0) \in S \times Y^p \times \Theta$, the solution $(x(t), w(t), \hat{\theta}(t))$ of the closed-loop system (2), (3) with (4), initial condition $(x(0), w(0), \hat{\theta}(0)) = (x_0, w_0, \hat{\theta}_0)$ corresponding to input $\{d(i) \in D\}_{i=0}^{\infty}$ satisfies

$$|x(t) - x^*| + \sum_{i=1}^{p} |w_i(t) - y^*| + |\hat{\theta}(t) - \theta| \leq$$
$$M \exp(-\sigma t) \left(|x(0) - x^*| + \sum_{i=1}^{p} |w_i(0) - y^*| + |\hat{\theta}(0) - \theta| \right) \tag{5}$$

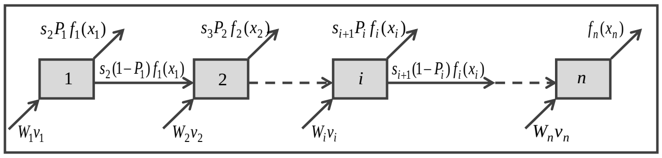

Fig. 1. Scheme of the freeway model.

for all $t > 0$.

2) For every sequence $\{d(i) \in D\}_{i=0}^{\infty}$ and for every $(x_0, w_0, \hat{\theta}_0) \in S \times Y^p \times \Theta$ the solution $(x(t), w(t), \hat{\theta}(t))$ of the closed-loop system (2), (3) with (4), initial condition $(x(0), w(0), \hat{\theta}(0)) = (x_0, w_0, \hat{\theta}_0)$ corresponding to input $\{d(i) \in D\}_{i=0}^{\infty}$ satisfies $\hat{\theta}(t) = \theta$, for all $t \geq m + p + 1$.

Remark: The dynamic feedback stabilizer (4) achieves dead-beat estimation of the vector of unknown parameters $\theta \in \Theta$. More specifically, the variable $\hat{\theta}$ provides an estimation of the vector of unknown parameters $\theta \in \Theta$. Due to the dead-beat estimation, the exponential convergence property for the closed-loop system is preserved, as estimate (5) shows.

III. Application To Freeway Traffic Control

This section focuses on the application of the proposed adaptive control scheme to freeway traffic control. The objective of the control strategy is to stabilize the corresponding closed-loop system to the desired UEP.

A. The freeway model

We consider a freeway which consists of $n \geq 3$ components or cells; typical cell lengths may be 200-500 m. Each cell may have an external inflow (e.g., from corresponding on-ramps), located near the cell's upstream boundary; and an external outflow (e.g., via corresponding off-ramps), located near the cell's downstream boundary (Fig.1). The number of vehicles at time $t \geq 0$ in component $i \in \{1, ..., n\}$ is denoted by $x_i(t)$. The total outflow and the total inflow of vehicles of the component $i \in \{1, ..., n\}$ at time $t \geq 0$ are denoted by $F_{i,out}(t) \geq 0$ and $F_{i,in}(t) \geq 0$, respectively. All flows during a time interval are measured in [veh]. Consequently, the balance of vehicles (conservation equation) for each component $i \in \{1, ..., n\}$ gives:

$$x_i(t+1) = x_i(t) - F_{i,out}(t) + F_{i,in}(t), \ \ t \geq 0. \quad (6)$$

Each component of the network has storage capacity $a_i > 0$ (i.e. $x_i \in [0, a_i]$ for each $i = 1, ..., n$). Based on (6) and the assumption that the outflows of every cell are constant percentages of the total outflow from the same cell as proposed in [14], we obtain the freeway model:

$$\begin{aligned} x_1^+ &= x_1 - s_2 f_1(x_1) + \min(q_1, c_1(a_1 - x_1), v_1) \\ &= x_1 - s_2 f_1(x_1) + W_1 v_1, \end{aligned} \quad (7)$$

$$\begin{aligned} x_i^+ &= x_i - s_{i+1} f_i(x_i) \\ &\quad + \min(q_i, c_i(a_i - x_i), v_i + (1 - P_{i-1}) f_{i-1}(x_{i-1})) \\ &\quad = x_i - s_{i+1} f_i(x_i) + W_i v_i + s_i (1 - P_{i-1}) f_{i-1}(x_{i-1}), \\ i &= 2, ..., n-1, \end{aligned} \quad (8)$$

$$\begin{aligned} x_n^+ &= x_n - f_n(x_n) \\ &\quad + \min(q_n, c_n(a_n - x_n), v_n + (1 - P_{n-1}) f_{n-1}(x_{n-1})) \\ &\quad = x_n - f_n(x_n) + W_n v_n + s_n (1 - P_{n-1}) f_{n-1}(x_{n-1}), \end{aligned} \quad (9)$$

where f_i, denote the attempted outflow from cell i to cell $i+1$, illustrating what in the specialized literature of Traffic Engineering (see, e.g., [15]) is called the demand part of the fundamental diagram of the i^{th} cell. Moreover, $q_i \in (0, +\infty)$ denotes the maximum flow that the i^{th} cell can receive (or the capacity flow of the i^{th} cell) and $c_i \in (0, 1]$ $(i = 1, ..., n)$ is the jam velocity of the i^{th} cell. The variables $v_i(t) \geq 0$ denote the attempted external inflow to cell $i \in \{1, ..., n\}$ from regions out of the freeway and the variables $W_i(t) \in [0, 1]$ indicate the percentage of the attempted external inflow to cell $i \in \{1, ..., n\}$ that becomes actual inflow. The variables $s_i(t) \in [0, 1]$, for each $i = 2, ..., n$, indicate the percentage of the attempted outflow and they are given by the following formula:

$$\begin{aligned} s_i(t) &= (1 - d_i(t)) \\ &\min\left(1, \max\left(0, \frac{\min(q_i, c_i(a_i - x_i(t))) - v_i(t)}{(1 - P_{i-1}) f_{i-1}(x_{i-1}(t))}\right)\right) \\ &+ d_i(t) \min\left(1, \frac{\min(q_i, c_i(a_i - x_i(t)))}{(1 - P_{i-1}) f_{i-1}(x_{i-1}(t))}\right), \end{aligned} \quad (10)$$

where $d_i(t) \in [0, 1]$, $i = 2, ..., n$, $t \geq 0$ are time-varying parameters denoting all possible cases for the relative priorities of the inflows (see, [13]). The constants P_i are the well-known exit rates of the freeway, for which we assume that $P_n = 1$, $P_i < 1$ for $i = 1, ..., n-1$, and that all exits to regions out of the network can accommodate the respective exit flows. Furthermore, notice that $v_i(t)$, $i = 2, ..., n$, correspond to external on-ramp flows which may be determined by a ramp metering strategy. For the very first cell 1, we assume for convenience, that there is just one inflow, v_1. All the above are illustrated in Fig.1. For a more complete justification of the derivation of the above models, see [13].

We next make the following assumption for the functions $f_i : [0, a_i] \to \Re_+, (i = 1, ..., n)$:

(H) *There exist constants $\delta_i \in (0, a_i]$ and $r_i \in (0, 1)$ such that $f_i(z) = r_i z$ for $z \in [0, \delta_i]$. Moreover, there exists a positive constant $f_i^{min} > 0$ such that $f_i(\delta_i) = r_i \delta_i \geq f_i(z) \geq f_i^{min}$ for all $z \in [\delta_i, a_i]$.*

Assumption (H) is a technical assumption that allows a very general class of demand functions (which are also allowed to be discontinuous). A more general assumption than assumption (H) was used in [13] (the demand functions $f_i : [0, a_i] \to \Re_+, (i = 1, ..., n)$ was not necessarily linear on the corresponding intervals $[0, \delta_i]$), but therein, it was assumed that all parameters of the model were known.

B. Global Exponential Stabilization of Freeway Models

Define the vector field $\tilde{F} : D \times S \times (0, +\infty) \times \Re_+^{n-1} \to S$ for all $d = (d_2, ..., d_n) \in D = [0, 1]^{n-1}$, $x \in S =$

$\prod_{i=1}^{n}(0,a_i]$ and $v \in (0,+\infty)\times \Re_+^{n-1}$ such that:

$$\tilde{F}(d,x,v) = (\tilde{F}_1(d,x,v),...,\tilde{F}_n(d,x,v))' \in \Re^n$$

with

$$\tilde{F}_1(d,x,v) = x_1 - s_2 f_1(x_1) + \min(q_1, c_1(a_1 - x_1), v_1),$$

$$\begin{aligned}
&\tilde{F}_i(d,x,v) = x_i - s_{i+1} f_i(x_i) + \\
&\quad \min(q_i, c_i(a_i - x_i), v_i + (1-P_{i-1}) f_{i-1}(x_{i-1})) \\
&i = 2,...,n-1, \\
&\tilde{F}_n(d,x,v) = x_n - f_n(x_n) + \\
&\quad \min(q_n, c_n(a_n - x_n), v_n + (1-P_{n-1}) f_{n-1}(x_{n-1})), \\
&s_i = \\
&(1-d_i)\min\left(1, \max\left(0, \frac{\min(q_i, c_i(a_i - x_i)) - v_i}{(1-P_{i-1}) f_{i-1}(x_{i-1})}\right)\right) \\
&\quad + d_i \min\left(1, \frac{\min(q_i, c_i(a_i - x_i))}{(1-P_{i-1}) f_{i-1}(x_{i-1})}\right).
\end{aligned} \tag{11}$$

Notice that, using definition (11), the control system (7), (8), (9) can be written in the following vector form:

$$x^+ = \tilde{F}(d,x,v), x \in S, d \in D, v \in (0,+\infty)\times \Re_+^{n-1}. \tag{12}$$

Consider the freeway model (12) under assumption (H). Let $v^* = (v_1^*,...,v_n^*) \in (0,+\infty)\times\Re_+^{n-1}$ be a vector that satisfies:

$$\begin{aligned}
&v_1^* < \min(q_1, c_1(a_1-\delta_1), r_1\delta_1), \\
&v_i^* + \sum_{j=1}^{i-1} v_j^* \Big(\prod_{k=j}^{i-1}(1-P_k)\Big) < \min(q_i, c_i(a_i-\delta_i), r_i\delta_i).
\end{aligned} \tag{13}$$

Any inflow vector that satisfies (13), defines an UEP $x^* = (x_1^*,...,x_n^*) \in \prod_{i=1}^{n}(0,\delta_i)$ with the following form:

$$\begin{aligned}
&x_1^* = r_1^{-1} v_1^*, \\
&x_i^* = r_i^{-1}\left(v_i^* + \sum_{j=1}^{i-1} v_j^* \Big(\prod_{k=j}^{i-1}(1-P_k)\Big)\right), i = 2,...,n.
\end{aligned} \tag{14}$$

The UEP is not globally exponentially stable for arbitrary $v_1^* > 0$, $v_i^* \geq 0$ $(i = 2,...,n)$; indeed, for relatively large values of inflows v_i^*, $(i = 1,...,n)$, other equilibria for model (12) (congested equilibria) may appear, for which the cell densities are large and can attract the solution of (12).

The following (see, [13]) is the main result in feedback design that provides the nominal feedback for the adaptive control scheme that we intend to use. It shows that a continuous, robust, global exponential stabilizer exists for every model of the form (12) under assumption (H).

Theorem 3.1: *Consider system (12) with $n \geq 3$ under assumption (H) for each $i = 1,...,n$. Then there exist a subset $R \subseteq \{1,...,n\}$ of the set of all indices $i \in \{1,...,n\}$ with $v_i^* > 0$, constants $\sigma \in (0,1]$, $b_i \in (0,v_i^*)$ for $i \in R$ and a constant $\tau^* > 0$ such that for every $\tau \in (0,\tau^*)$ the feedback law $k : S \to \Re_+^n$ defined by:*

$$\begin{aligned}
&k(x) = (k_1(x),...,k_n(x))' \in \Re_n \text{ with} \\
&k_i(x) = \max\left(b_i, v_i^* - \tau^{-1}(v_i^* - b_i)\Xi(x)\right) \\
&\text{for all } x \in S, i \in R, \\
&k_i(x) = v_i^* \text{ for all } x \in S, i \notin R,
\end{aligned} \tag{15}$$

where

$$\Xi(x) := \sum_{i=1}^{n} \sigma^i \max(0, x_i - x_i^*), \text{ for all } x \in S, \tag{16}$$

achieves robust global exponential stabilization of the UEP x^* of system (12), i.e., x^* is RGES for the closed-loop system (12) with $v = k(x)$.

The result of Theorem 3.1 (see, [13]) is based on the construction of a Control Lyapunov function for system (12) under a more general assumption than assumption (H). The feedback law provides values for the controllable inflows v_i, $i \in R$, in the interval $[b_i, v_i^*]$ for all $i \in R$, where $b_i \in (0, v_i^*)$ for $i \in R$ are the minimum allowable inflows. Since the proof of Theorem 3.1 is constructive, criteria for the selection of the index set $R \subseteq \{1,...,n\}$ and the constants $\sigma \in (0,1]$, $b_i \in (0, v_i^*)$ for $i \in R$ and $\tau^* > 0$ are provided.

In what follows, we assume that $x^* = (x_1^*,...,x_n^*) \in \prod_{i=1}^{n}(0, \mu_i - \varepsilon]$ (where $\mu_i \in (0,\delta_i)$), $v_i^* \in [b_i + \varepsilon, v_{i,max}]$ (where $v_{i,max}$ is the maximum admissible inflow value for the i^{th} cell, see, also, [16]) for $i \in R$ and for some $\varepsilon \in (0, 1/2)$ and $v^* \in (0, v_{1,max}] \times \prod_{i=2}^{n}[0, v_{i,max}]$. Moreover, we assume that $P_i \in [0, 1-\varepsilon]$ for $i = 1,...,n-1$ and $r_i \in [\varepsilon, 1-\varepsilon]$ for $i = 1,...,n$.

Another feature of the present problem is that the selection of the UEP may be made in an implicit way. For example, we may want the UEP that guarantees the maximum outflow from the freeway. In such cases, the equilibrium position of the controllable inflows is determined as a function of the nominal values of the uncontrollable inflows and the parameters of the freeway, i.e., there exists a smooth function

$$g : D_g \to \prod_{i\in R}[b_i + \varepsilon, v_{i,max}],$$

where $D_g = [0, 1-\varepsilon]^{n-1} \times \prod_{i\notin R}[0, v_{i,max}] \times [\varepsilon, 1-\varepsilon]^n$, such that:

$$(v_i^*; i \in R) = g(P, v_i^*; i \notin R, r), \tag{17}$$

where $P = (P_1,...,P_{n-1})' \in [0, 1-\varepsilon]^{n-1}$ and $r = (r_1,...,r_n)' \in [\varepsilon, 1-\varepsilon]^n$.

C. Measurements and Unknown Parameters

Let $m \in \{1,...,n\}$ be the cardinal number of the set R and let $u \in U = \prod_{i\in R}[b_i, v_{i,max}] \subseteq (0,+\infty)^m$ be the vector of all controllable inflows v_i with $i \in R$.

The model parameters which are (usually) unknown or uncertain are: the exit rates $P_i \in [0,1)$ for $i = 1,...,n-1$, the uncontrollable inflows $v_i^* \in \Re_+$ for $i \notin R$ and the demand coefficients $r_i \in (0,1)$ for $i = 1,...,n$. All these parameters will be denoted by $\theta = (P, v_i^*; i \notin R, r)$ and are assumed to take values in a compact set $\Theta := [0, 1-\varepsilon]^{n-1} \times \prod_{i \notin R}[0, v_{i,max}] \times [\varepsilon, 1-\varepsilon]^n$, for some $\varepsilon \in (0, 1/2)$. Therefore, the control system (7), (8), (9) can be written in the following vector form:

$$\begin{gathered} x^+ = \bar{F}(d, \theta, x, u), \\ x \in S, d \in D, \theta \in \Theta, u \in U = \prod_{i \in R}[b_i, v_{i,max}]. \end{gathered} \tag{18}$$

Notice that the feedback law defined by (15) is a feedback law of the form $u = k(\theta, x)$: the feedback law depends on the unknown parameters through x^* and $(v_i^*; i \in R)$ (recall (14) and (17)). It follows that assumption (H1) holds for system (18). An explicit definition of the feedback law $k : \Theta \times S \to U$ is given by the following equations for all $\hat{\theta} = (\hat{P}, \hat{v}_i^*; i \notin R, \hat{r}) \in \Theta$, $x \in S$ with $\hat{r} = (\hat{r}_1, ..., \hat{r}_n)' \in [\varepsilon, 1-\varepsilon]^n$, $\hat{P} = (\hat{P}_1, ..., \hat{P}_{n-1})' \in [0, 1-\varepsilon]^{n-1}$:

$$(\hat{v}_i^*; i \in R) = g(\hat{P}, \hat{v}_i^*; i \notin R, \hat{r}), \tag{19}$$

$$\begin{gathered} \hat{x}_1^* = \min(\hat{r}_1^{-1}\hat{v}_1^*, \mu_1 - \varepsilon), \\ \hat{x}_i^* = \min\left(\hat{r}_i^{-1}\left(\hat{v}_i^* + \sum_{j=1}^{i-1}\hat{v}_j^*\left(\prod_{k=j}^{i-1}(1-\hat{P}_k)\right)\right), \mu_i - \varepsilon\right), \\ \textit{for } i = 2, ..., n, \end{gathered} \tag{20}$$

$$\begin{gathered} u = k(\hat{\theta}, x), \ \textit{with} \\ k_i(\hat{\theta}, x) = \max(b_i, \hat{v}_i^* - \tau^{-1}(\hat{v}_i^* - b_i)\Xi(\hat{\theta}, x)), \\ \textit{for all } x \in S, i \in R, \end{gathered} \tag{21}$$

$$\Xi(\hat{\theta}, x) := \sum_{i=1}^{n}\sigma^i \max(0, x_i - \hat{x}_i^*), \ \textit{for all } x \in S. \tag{22}$$

The measured quantities are the cell densities $x \in S$ and the outflows from each cell. We have two kinds of outflows from each cell: the outflow to regions out of the freeway

$$\begin{gathered} Q_{out} = (Q_{1,out}, ..., Q_{n,out})' \in \Re_+^n, \\ Q_{i,out} = P_i s_{i+1} f_i(x_i), i = 1, ..., n-1, \\ Q_{n,out} = f_n(x_n), \end{gathered} \tag{23}$$

and the outflows from one cell to the next cell

$$\begin{gathered} Q = (Q_1, ..., Q_{n-1})' \in \Re_+^{n-1}, \\ Q_i = (1-P_i)s_{i+1}f_i(x_i), i = 1, ..., n-1. \end{gathered} \tag{24}$$

Therefore, the measured output is given by:

$$y = h(d, \theta, x) = (x, Q_{out}, Q) \in S \times \Re_+^n \times \Re_+^{n-1}. \tag{25}$$

Assumption (H) guarantees that $h(D \times \Theta \times S) \subseteq Y$ where $Y := S \times \prod_{i=1}^n [0, a_i] \times \prod_{i=1}^n [0, a_i]$ is a bounded set.

Next, define a mapping $\Psi : h(D \times \Theta \times S) \times Y \to \Theta$ for which $\theta = (P_1, ..., P_{n-1}, v_i^*; i \notin R, r_1, ..., r_n)' = \Psi(y(t), y(t-1))$ for all $t \geq 1$ with $y(t-1) \in A$, where $A \subseteq Y$ is the set for which:

$$\begin{gathered} w = (w_1, w_2, w_3) \in A \Leftrightarrow \\ (w_1, w_2, w_3) \in Y, w_1 \in \Omega = \prod_{i=1}^{n}(0, \mu_i) \\ \textit{and } w_{2,i} + w_{3,i} > 0 \textit{ for } i = 1, ..., n-1. \end{gathered} \tag{26}$$

The mapping $\Psi : h(D \times \Theta \times S) \times Y \to \Theta$ is defined by

$$\hat{\theta} = (\hat{P}_1, ..., \hat{P}_{n-1}, \hat{v}_i^*; i \notin R, \hat{r}_1, ..., \hat{r}_n)' = \Psi(y, w), \tag{27}$$

with

$$\hat{P}_i = \min\left(1-\varepsilon, \frac{w_{2,i}}{w_{2,i} + w_{3,i}}\right), i = 1, ..., n-1, \tag{28}$$

$$\begin{gathered} \hat{v}_i^* = \max(0, \min(v_{i,max}, x_i - w_{1,i} \\ + w_{3,i} + w_{2,i} - w_{3,i-1})), \ i \in \{2, ..., n-1\} \backslash R, \end{gathered} \tag{29}$$

$$\begin{gathered} \hat{v}_n^* = \max(0, \min(v_{n,max}, x_n - w_{1,n} + w_{2,n} - w_{3,n-1})), \\ \textit{if } n \notin R, \end{gathered} \tag{30}$$

$$\begin{gathered} \hat{v}_1^* = \max(0, \min(v_{1,max}, x_1 - w_{1,1} + w_{3,1} + w_{2,1})), \\ \textit{if } 1 \notin R, \end{gathered} \tag{31}$$

$$\hat{r}_i = \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,i} + w_{3,i}}{w_{1,i}}\right)\right), i = 1, ..., n-1, \tag{32}$$

$$\hat{r}_n = \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,n}}{w_{1,n}}\right)\right). \tag{33}$$

Using assumption (H), (13), (14) and (25), it follows that there exists $y^* \in Y$ with $y^* = h(d, \theta, x^*)$ for all $d \in D$. By virtue of our assumption $x^* = (x_1^*, ..., x_n^*) \in \prod_{i=1}^n (0, \mu_i)$ and $v^* \in (0, v_{1,max}] \times \prod_{i=2}^n [0, v_{i,max}]$, (26), we conclude that A contains all $w \in Y$ in a neighborhood of y^*. It follows that (H2) holds with $p = 1$ for system (18) with output given by (23), (24), (25).

The fact that Assumption (H3) holds for the system (18) with output given by (23), (24), (25) is a direct consequence of the following proposition.

Proposition 3.2: *Suppose that $b_i > 0$ ($i \in R$) and $v_{i,max}$ ($i \notin R$) are sufficiently small and that $\tau > 0$ is sufficiently small ($\tau \leq \epsilon^2\sigma^n \min_{i \in R}\left((v_{i,max} - b_i)^{-1}\right)$). Then, there exists an integer $m \geq 1$ such that for every sequence $\{\left(d(t), \hat{\theta}(t)\right) \in D \times \Theta\}_{t=0}^{\infty}$ and for every $x_0 \in S$, the solution $x(t)$ of (18), (25) with $u = k(\hat{\theta}, x)$, initial condition $x(0) = x_0$ corresponding to inputs $\{\left(d(t), \hat{\theta}(t)\right) \in D \times \Theta\}_{t=0}^{\infty}$ satisfies $y(t-1-i(t)) \in A$ for some $i(t) \in \{0, 1, .., m\}$ and for all $t \geq m+1$.*

The main result for the freeway model is a consequence of Theorem 2.1 and the fact that all functions are sufficiently smooth in a neighborhood of the equilibrium.

Corollary 3.3: *Consider system (18) with output given by (23), (24), (25). Suppose that $b_i > 0$ ($i \in R$) and $v_{i,max}$($i \notin R$) are sufficiently small and that $\tau > 0$ is sufficiently small. Then the dynamic feedback law given by:*

$$w_1^+ = x, w_2^+ = Q_{out}, w_3^+ = Q, \tag{34}$$

$$\hat{P}_i^+ = \begin{cases} \hat{P}_i & \text{if } w \notin A \\ \min\left(1-\varepsilon, \frac{w_{2,i}}{w_{2,i}+w_{3,i}}\right) & \text{if } w \in A \end{cases}, \tag{35}$$
$i = 1, ..., n-1,$

$$(\hat{v}_i^*)^+ = \begin{cases} \hat{v}_i^* & \text{if } w \notin A \\ \max(0, \min(v_{i,max}, x_i - w_{1,i} \\ \quad +w_{3,i} + w_{2,i} - w_{3,i-1})) & \text{if } w \in A \end{cases},$$
$$if \quad i \in \{2, ..., n-1\} \setminus R, \tag{36}$$

$$(\hat{v}_n^*)^+ = \begin{cases} \hat{v}_n^* & \text{if } w \notin A \\ \max(0, \min(v_{n,max}, x_n - w_{1,n} \\ \quad + w_{2,n} - w_{3,n-1})) & \text{if } w \in A \end{cases},$$
$$if \quad n \notin R, \tag{37}$$

$$(\hat{v}_1^*)^+ = \begin{cases} \hat{v}_1^* & \text{if } w \notin A \\ \max(0, \min(v_{1,max}, x_1 - w_{1,1} \\ \quad + w_{3,1} - w_{2,1})) & \text{if } w \in A \end{cases},$$
$$if \quad 1 \notin R, \tag{38}$$

$$\hat{r}_i^+ = \begin{cases} \hat{r}_i & \text{if } w \notin A \\ \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,i}+w_{3,i}}{w_{1,i}}\right)\right) & \text{if } w \in A \end{cases},$$
$$i = 1, ..., n-1, \tag{39}$$

$$\hat{r}_n^+ = \begin{cases} \hat{r}_n & \text{if } w \notin A \\ \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,n}}{w_{1,n}}\right)\right) & \text{if } w \in A \end{cases}, \tag{40}$$

with (19)-(22), $\hat{P} = (\hat{P}_1, ..., \hat{P}_{n-1})$, $P = (P_1, ..., P_{n-1})$, $\hat{r} = (\hat{r}_1, ..., \hat{r}_n)$, $r = (r_1, ..., r_n)$, $w = (w_1, w_2, w_3)$, $\hat{v}^* = (\hat{v}_1^*, ..., \hat{v}_n^*)$, achieves the following:

1) There exist constants $M, \sigma > 0$ such that for every sequence $\{d(i) \in D\}_{i=0}^{\infty}$ and for every $(x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0) \in S \times Y \times \Theta$, the solution of the closed-loop system (18), (25) with (34)-(40), (19)-(22), initial condition $(x(0), w(0), \hat{P}(0), \hat{v}_j^*(0); j \notin R, \hat{r}(0)) = (x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0)$ corresponding to input $\{d(i) \in D\}_{i=0}^{\infty}$ satisfies for all $t \geq 0$:

$$|x(t) - x^*| + |w(t) - y^*| + |\hat{r}(t) - r| + |\hat{P}(t) - P| + |\hat{v}^*(t) - v^*| \leq M exp(-\sigma t)\Bigg(|x(0) - x^*| + |w(0) - y^*| + |\hat{r}(0) - r| + |\hat{P}(0) - P| + \sum_{i \notin R} |\hat{v}_i^*(0) - v_i^*|\Bigg). \tag{41}$$

2) There exists an integer $N \geq 1$ such that for every sequence $\{d(i) \in D\}_{i=0}^{\infty}$ and for every $(x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0) \in S \times Y \times \Theta$, the solution of the closed-loop system (18), (25) with (34)-(40), (19)-(22), initial condition $(x(0), w(0), \hat{P}(0), \hat{v}_j^*(0); j \notin R, \hat{r}(0)) = (x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0)$ corresponding to input $\{d(i) \in D\}_{i=0}^{\infty}$ satisfies $\hat{P}(t) = P$, $\hat{r}(t) = r$, $\hat{v}^*(t) = v^*$, for all $t \geq N$.

IV. An Illustrating Example

The following example illustrates the application of the results of the previous section to a specific freeway model.

Consider a freeway model of the form (7), (8), (9), (10) with $n = 5$ cells. Each cell is 0.5 km and it has 3 lanes, an on-ramp and an off-ramp. The first and the third on-ramp are assumed to be controllable, hence $R = \{1, 3\}$ and $u = (v_1, v_3)$. The inflows from the rest of the on-ramps are assumed to be unknown and therefore they will have to be estimated. Furthermore, we assume that $d_i(t) \equiv 0$, thus the on-ramp inflows have absolute priority over the internal inflows. The simulation time step is set to be $T = 15s$ and the cell capacities are $a_i = 170$ [veh] for $i = 1, ..., 5$. Note that, since all flows and densities are measured in [veh], the cell length, the simulation time step and the number of lanes do not appear explicitly, but they are only involved implicitly in the value of every variable and every constant (e.g. critical density, jam density, flow capacity, wave speed etc.) corresponding to density or flow.

The formulas of the demand functions are given by the following equations:

$$f_i(z) = \begin{cases} (\frac{5}{11})z & z \in [0, 55] \\ 25 - (\frac{7}{115})(z - 55) & z \in (55, 170] \end{cases} (i = 1, ..., 4),$$

$$f_5(z) = \begin{cases} (\frac{4}{11})z & z \in [0, 55] \\ 20 - (\frac{3}{115})(z - 55) & z \in (55, 170] \end{cases}. \tag{42}$$

Assumption (H) holds with $\delta_i = 55$ [veh] for $i = 1, ..., 5$, $r_i = 5/11$, $f_i^{min} = 18$ for $i = 1, ..., 4$, $r_5 = 4/11$ and $f_5^{min} = 17$. Thus, every cell has the same critical and jam density which correspond to 36.7 [veh/km/lane] and 113.3 [veh/km/lane], respectively, in common traffic units, with the above settings. Definitions (42) guarantee that the demand functions for $i = 1, ..., 4$ lead to 20% higher flow capacity than the flow capacity of the fifth cell ($f_i(\delta_i) = 25$ [veh], $i = 1, ..., 4$, and $f_5(\delta_5) = 20$ [veh], corresponding to 2000 and 1600 [veh/h/lane] respectively) and therefore the last cell is a bottleneck for the freeway. The capacity drop phenomenon has been taken into account by considering a linearly decreasing demand function for over-critical densities $x_i \in (55, 170]$ (similar to the one proposed in [20]). The congestion wave speeds are $c_i = 0.22$ for $i = 1, ..., 5$, corresponding to 26.4 [km/h], while the cell flow capacities q_i for $i = 1, ..., 5$ satisfy the inequalities $q_i \geq c_i a_i$ for $i = 1, ..., 5$ and therefore, they play no role in the model.

Our goal is to globally exponentially stabilize the system at an UEP which is as close as possible to the critical density (due to the fact that the flow value at the critical density is largest). Therefore, we selected as the upper bound for the equilibrium densities and for each cell to be the $\mu_i = \delta_i - \epsilon$ $(i = 1, ..., 5)$, where $\epsilon = 10^{-4}$. The vectors of the exit rates and the maximum admissible

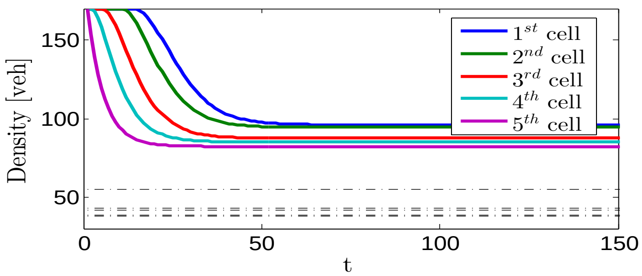

Fig. 2. Time evolution of the states of the open-loop system (dashed lines correspond to the UEP for constant inflows $v^* = [17.29316, 1, 4, 2, 2.5]$ [veh]) with fully congested initial condition $x_0 = (a_1, a_2, a_3, a_4, a_5)'$.

inflows are set to be $P = [0.04, 0.15, 0.08, 0.1]$ and $v_{max} = [25, 1.3, 4, 2.3, 2.8]$ respectively. The uncontrollable inflows are constant and equal to $v_2^* = 1$, $v_4^* = 2$ and $v_5^* = 2.5$. Summarizing, the vector of the parameters θ becomes $\theta = [P_1, ..., P_4, v_2^*, v_4^*, v_5^*, r_1, ..., r_5]$.

The function

$$g : D_g \to \prod_{i=1,3} [b_i + \epsilon, v_{i,max}],$$

with $D_g = [0, 1-\epsilon]^4 \times \prod_{i=2,4,5} [0, v_{i,max}] \times [\epsilon, 1-\epsilon]^5$ and $b_1 = b_3 = 0.2$, involved in (19) has been selected in such a way so that the 3^{rd} inflow takes its maximum admissible value ($v_3^* = v_{3,max} = 4$), while the inflow from the 1^{st} cell is formed in such a way so that the outflow from the last (5^{th}) cell is approximately maximized: Then, the UEP is $x^* = [38.045, 38.723, 41.715, 42.778, 54.9997]$ for $v^* = [17.29316, 1, 4, 2, 2.5]$, $P = [0.04, 0.15, 0.08, 0.1]$ and $r = [5/11, 5/11, 5/11, 5/11, 4/11]$.

The above UEP is not globally exponentially stable due to the existence of additional (congested) equilibria. This is shown in Fig.2, where the solution of the open-loop system, with constant inflows $v^* = [17.29316, 1, 4, 2, 2.5]$, is attracted by the congested equilibrium $[96.19, 94.6, 87.73, 85.22, 82.33]'$ leading to much lower outflow than the capacity flow of the last cell. Therefore, if the objective is the operation of the freeway with largest outflow, then a control strategy will be needed.

We are in a position to guarantee global exponential attractivity of the UEP for the freeway model that was described above by using Corollary 3.3. Indeed, Corollary 3.3 guarantees that there exist constants $\sigma \in (0, 1]$, $b_1, b_3 > 0$ and $\tau > 0$ such that, the feedback law $k : \Theta \times S \to U$ defined by:

$$\begin{gathered} k_1(\hat{\theta}, x) = \\ \max\left(b_1, \hat{v}_1^* - \tau^{-1}(\hat{v}_1^* - b_1) \sum_{i=1}^{5} \sigma_i \max(0, x_i - \hat{x}_i^*)\right), \\ k_3(\hat{\theta}, x) = \\ \max\left(b_3, \hat{v}_3^* - \tau^{-1}(\hat{v}_3^* - b_3) \sum_{i=1}^{5} \sigma_i \max(0, x_i - \hat{x}_i^*)\right), \end{gathered} \tag{43}$$

$$(v_1^*, v_3^*) = g(\hat{P}, \hat{v}_i^*; i \notin R, \hat{r}), \tag{44}$$

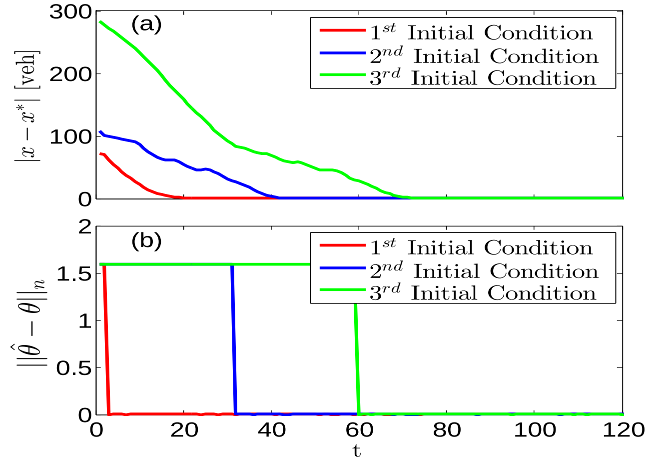

Fig. 3. (a) The Euclidean norm of the deviation $x(t) - x^*$ of the state from the UEP, i.e., $|x(t) - x^*|$, (b) The weighted norm $\|\hat{\theta}(t) - \theta\|_n$ of the deviation of the estimated parameters from the nominal parameters vector, for the closed-loop system (18), (25),(34)-(40), (43),(44),(20), (22) for $x_0 = (10, 15, 10, 15, 10)'$ (red line), $x_0 = (70, 85, 64, 120, 100)'$ (blue line) and $x_0 = (a_1, a_2, a_3, a_4, a_5)'$ (green line).

for the closed-loop system (18), (23), (24), (25) with (34)-(40), (43), (44), (20) and (22), achieves global exponential attractivity of the UEP $x^* = [38.045, 38.723, 41.715, 42.778, 54.9997]$.

We tested various values of the constants $\sigma \in (0, 1]$ and $\tau > 0$. Low values for $\sigma \in (0, 1]$ require small values for $\tau > 0$ in order to guarantee global exponential stability for the closed-loop system. All the following tests, were conducted with the same values $\sigma = 0.7$ and $\tau = 10$.

All the following simulation tests were conducted with the same initial conditions for the observer states $w_{1,i}(0) = 100$ [veh], $w_{2,i}(0) = 20$ [veh], $w_{3,i}(0) = 20$ [veh] for $i = 1, ..., 5$, $\hat{p}_i(0) = 0$ for $i = 1, ..., 4$, $\hat{v}_i^*(0) = 0$ for $i = 2, 4, 5$ and $\hat{r}_i(0) = 0.7$ for $i = 1, ..., 5$.

Fig.3(a) shows the evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the UEP, i.e., $|x(t) - x^*|$, for the closed-loop system with the proposed feedback regulator (34)-(40), (43), (44), (20) and (22) for three different initial conditions. The first initial condition corresponds to very low densities ($x_0 = (10, 15, 10, 15, 10)'$), the second initial condition corresponds to congested states with high deviations between each other ($x_0 = (70, 85, 65, 120, 100)'$), while the third initial condition corresponds to the state where the density of every cell has its maximum value, i.e. a_i ($i = 1, ..., 5$). For the same simulation test, Fig.3(b) depicts the evolution of the weighted norm $\|\hat{\theta}(t) - \theta\|_n$ of the deviation of the estimated parameters from the nominal parameters vector, which is defined by:

$$\begin{aligned} \|\hat{\theta}(t) - \theta\|_n = & \left| \left(\frac{1}{1-\epsilon}(\hat{P}(t) - P), \frac{\hat{v}_2^*(t) - v_2^*}{v_{2,max}}, \frac{\hat{v}_4^*(t) - v_4^*}{v_{4,max}}, \right.\right. \\ & \left.\left. \frac{\hat{v}_5^*(t) - v_5^*}{v_{5,max}}, \frac{1}{1-\epsilon}(\hat{r}(t) - r) \right) \right|. \end{aligned} \tag{45}$$

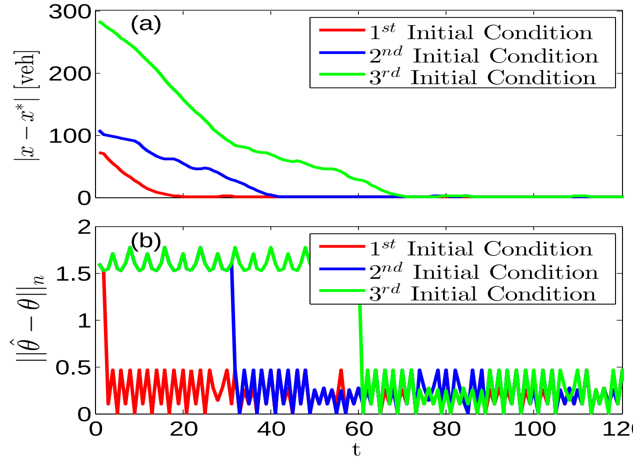

Fig. 4. (a) The Euclidean norm of the deviation $x(t)-x^*$ of the state vector from the UEP, (b) The weighted norm $\|\hat{\theta}(t)-\theta(t)\|_n$ of the deviation of the estimated parameters from the nominal parameters vector, for the closed-loop system (18), (25), (34)-(40), (43),(44),(46),(20), (22) for $x_0 = (10, 15, 10, 15, 10)'$ (red line), $x_0 = (70, 85, 64, 120, 100)'$ (blue line) and $x_0 = (a_1, a_2, a_3, a_4, a_5)'$ (green line).

Indeed, Fig.3 shows that the proposed feedback stabilizer achieves dead-beat estimation of the vector θ, preserving the exponential convergence property for the closed-loop system.

We also tested the performance of the proposed feedback stabilizer under the effect of periodic uncontrollable inflows with different frequencies and different amplitudes, given by:

$$v_2^* = 1 + 0.3\cos\left(\frac{3\pi t}{2}\right), v_4^* = 2 + 0.1\cos(\pi t) \text{ and } v_5^* = 2.5 + 0.2\cos\left(\frac{\pi t}{4}\right). \tag{46}$$

Figures 4(a) and 4(b), depict the evolution of the Euclidean norm of the deviation $x(t)-x^*$ and the evolution of the weighted norm $\|\cdot\|_n$ defined by (45) for the deviation $\hat{\theta}(t)-\theta(t)$, respectively, with respect to the unknown time-varying uncontrollable inflows (46) and under the proposed feedback regulator. The initial conditions were the same as in the previous case. Again, the proposed regulator achieved to lead the system to the equilibrium state.

V. Concluding Remarks

Novel results for adaptive control schemes for uncertain discrete-time systems, which guarantee robust, global, exponential convergence to the desired equilibrium point of the system, were provided in the present work. The proposed adaptive scheme did not require the knowledge of a Lyapunov function for the closed-loop system under the action of the nominal feedback stabilizer and is directly applicable to highly nonlinear, uncertain discrete-time systems with unknown constant parameters. The applicability of the obtained results to real control problems was demonstrated by the rigorous application of the proposed adaptive control scheme to uncertain, freeway models. Simulation results showed the efficacy of the proposed adaptive control scheme even under the presence of time-varying parameters.

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References

[1] T. Hayakawa, W. M. Haddad, and A. Leonessa, "A Lyapunov-Based Adaptive Control Framework for Discrete-Time Non-Linear Systems with Exogenous Disturbances," *International Journal of Control*, vol. 77, no. 3, pp. 250–263, 2004.

[2] Y. Zhang, C. Wen, and Y. C. Soh, "Robust Adaptive Control of Uncertain Discrete-Time Systems," *Automatica*, vol. 35, no. 2, pp. 321–329, 1999.

[3] ——, "Robust Adaptive Control of Nonlinear Discrete-Time Systems by Backstepping without Overparameterization," *Automatica*, vol. 37, no. 4, pp. 551–558, 2001.

[4] J. Zhao and I. Kanellakopoulos, "Adaptive Control of Discrete-Time Strict-Feedback Nonlinear Systems," in *Proceedings of the American Control Conference*, vol. 1, 1997, pp. 828–832.

[5] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*, 1st ed. New York, USA: John Wiley & Sons, Inc., 1995.

[6] L.-L. Xie and L. Guo, "Fundamental Limitations of Discrete-Time Adaptive Nonlinear Control," *IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1777–1782, 1999.

[7] I. Karafyllis and C. Kravaris, "On the Observer Problem for Discrete-Time Control Systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 1, pp. 12–25, 2007.

[8] M. Papageorgiou, H. Hadj-Salem, and J.-M. Blosseville, "ALINEA: A Local Feedback Control Law For On-Ramp Metering," *Transportation Research Record*, no. 1320, 1991.

[9] Y. Wang and M. Papageorgiou, "Local Ramp Metering in the Case of Distant Downstream Bottlenecks," in *Proceedings of the IEEE Intelligent Transportation Systems Conference*, 2006, pp. 426–431.

[10] Y. Wang, M. Papageorgiou, J. Gaffney, I. Papamichail, G. Rose, and W. Young, "Local Ramp Metering in Random-Location Bottlenecks Downstream of Metered On-Ramp," *Transportation Research Record: Journal of the Transportation Research Board*, vol. 2178, pp. 90–100, 2010.

[11] N. Shlayan and P. Kachroo, "Feedback Ramp Metering Using Godunov Method Based Hybrid Model," *Journal of Dynamic Systems, Measurement, and Control*, vol. 135, no. 5, pp. 051 010–051 010, 2013.

[12] X. Sun and R. Horowitz, "Set of New Traffic-Responsive Ramp-Metering Algorithms and Microscopic Simulation Results," *Transportation Research Record: Journal of the Transportation Research Board*, vol. 1959, pp. 9–18, 2006.

[13] I. Karafyllis, M. Kontorinaki, and M. Papageorgiou, "Global Exponential Stabilization of Freeway Models," *International Journal of Robust and Nonlinear Control*, vol. 26, pp. 1184–1210, 2015.

[14] C. F. Daganzo, "The Cell Transmission Model, Part II: Network Traffic," *Transportation Research Part B: Methodological*, vol. 29, no. 2, pp. 79–93, 1995.

[15] J. P. Lebacque, "The Godunov Scheme and What it Means for First-Order Traffic Flow Models," in *Proceedings of the International Symposium on Transportation and Traffic Theory*, 1996, pp. 647–677.

[16] I. Karafyllis, M. Kontorinaki, and M. Papageorgiou, "Robust Global Adaptive Exponential Stabilization of Discrete-Time Systems with Application to Freeway Traffic Control," *submitted to IEEE Transactions on Automatic Control*, 2015, (see also, arXiv:1509.00257 [math.OC]).

[17] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton: Princeton University Press, 2008.

[18] I. Karafyllis and M. Papageorgiou, "Global Exponential Stability for Discrete-Time Networks With Applications to Traffic Networks," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 1, pp. 68–77, 2015.

[19] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations Numerical Methods And Applications*. CRC Press, 2002.

[20] J. Lebacque, "Two-Phase Bounded-Acceleration Traffic Flow Model: Analytical Solutions and Applications," *Transportation Research Record: Journal of the Transportation Research Board*, vol. 1852, pp. 220–230, 2003.