Robust Global Adaptive Exponential Stabilization of Discrete-Time Systems with Application to Freeway Traffic Control

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***Abstract*—This paper is devoted to the development of adaptive control schemes for uncertain discrete-time systems, which guarantee robust global exponential convergence to the desired equilibrium point of the system. The proposed control scheme consists of a nominal feedback law, which achieves robust global exponential stability properties when the vector of the parameters is known, in conjunction with a nonlinear dead-beat observer. The proposed adaptive control scheme depends on certain parameter observability assumptions. The obtained results are applicable to highly nonlinear uncertain discrete-time systems with unknown constant parameters. The successful applicability of the obtained results to real control problems is demonstrated by the rigorous application of the proposed adaptive control scheme to uncertain freeway models. A provided example demonstrates the efficiency of the approach.**

***Index Terms*—Nonlinear systems, adaptive control, discrete-time systems, freeway models.**

I. Introduction

Adaptive control for discrete-time systems has been studied in many works (see for instance [1], [2], [3], [4] and in many cases it is a direct extension of adaptive control schemes for continuous-time systems (see [5]). The limitations of adaptive control schemes for discrete-time systems have been studied in [6]. The major shortcoming of many adaptive control methodologies is that the closed-loop system does not exhibit an exponential convergence rate to the desired equilibrium point of the system, even if the nominal feedback law achieves global exponential stability properties when the parameters are precisely known.

This work is devoted to the development of adaptive control schemes for uncertain discrete-time systems, with unknown constant parameters, which guarantee robust, global, exponential convergence to the desired equilibrium point of the system. The idea is simple: use a nominal feedback law, which achieves robust, global, exponential stability properties when the vector of the parameters is known, in conjunction with a nonlinear, dead-beat observer. The dead-beat observer (designed using an extension of the methodology described in [7]) achieves the precise knowledge of the vector of unknown parameters after a transient period; then the states of the closed-loop system are robustly led to the desired equilibrium point with an exponential rate by the nominal feedback law. The proposed adaptive scheme does not require the knowledge of a Lyapunov function for the closed-loop system under the action of the nominal feedback stabilizer.

The design as well as the successful application of the adaptive control scheme requires restrictive observability assumptions, which may not be fulfilled for a general nonlinear system. However, when the observability assumptions are met, then, the obtained results are applicable to highly nonlinear, uncertain discrete-time systems with unknown constant parameters. The applicability of the obtained results to real control problems is demonstrated by the rigorous application of the proposed adaptive control scheme to uncertain freeway models.

Traffic congestion in freeways leads to serious degradation of the infrastructure causing excessive delays, and impacting traffic safety and the environment. Extensive research has been conducted to investigate and develop traffic control measures which can tackle this phenomenon. However, measures such as ramp metering, variable speed limits or dynamic route guidance have to be driven by appropriate control strategies in order to achieve their target. Traffic control strategies such as nonlinear optimal control [8], [9] and Model Predictive Control [10], [11] have been extensively studied but they are highly demanding from the computational point of view. However, the efficiency of traffic operations can also be enhanced by explicit feedback control approaches such as the pioneering I-type regulator ALINEA [12] and its extensions [13], [14], as well as other proposed feedback control algorithms in [15], [16], [17], [18]. These explicit feedback control strategies should guarantee local stability properties for the desired uncongested equilibrium point (UEP) of the freeway model.

A Lyapunov approach was adopted in [19], which led to the robust, global exponential stabilization of the UEP of a nonlinear freeway model. However, the nonlinear feedback stabilizer demands the knowledge of several model parameters, which are usually unknown. The present work proposes an adaptive control scheme, which is based on a dead-beat nonlinear observer and guarantees the robust, global exponential convergence rate to the desired UEP of the freeway model. The nonlinear freeway model in [19] is a generalization of various freeway models (see [20], [9], [21]), which are special cases of the considered model.

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The structure of the present work is as follows. Section 2 is devoted to the development of the robust, global, exponential adaptive control scheme for nonlinear uncertain discrete-time systems. The obtained results are applied rigorously in Section 3 to uncertain freeway models for the robust, global, exponential attractivity of the (unknown) desired UEP of the freeway model. An illustrating example of a freeway model is presented in Section 4, where it is also shown that the proposed adaptive control scheme is robust, even if the vector of the unknown parameters is not constant, and even if modeling errors are present. The concluding remarks of the paper are given in Section 5.

Notation

- $\Re_+ := [0, +\infty)$. For every set S, $S^n = \overbrace{S \times ... \times S}^{\text{n times}}$ for every positive integer n. $\Re_+^n := (\Re_+)^n$. For every $x \in \Re$, $[x]$ denotes the integer part of $x \in \Re$. For certain sets $S_1, S_2, ..., S_n$, the set $S_1 \times S_2 \times ... \times S_n$ is denoted by $\prod_{i=1}^n S_i$.
- Let $x, y \in \Re^n$. By $|x|$ we denote the Euclidean norm of $x \in \Re^n$ and by x' we denote the transpose of $x \in \Re^n$.
- When R is an index set, then by $(x_i; i \in R)$ we denote a vector with components all $x_i \in \Re$ with $i \in R$, in increasing order. For example, if $R = \{2, 5, 10\}$, then $(x_i; i \in R) = (x_2, x_5, x_{10})'$.

II. Exponential Stabilization Of Systems With Unknown Parameters

Consider the uncertain discrete-time dynamical system:

$$z^+ = F(d, z), z \in X, d \in D \tag{1}$$

where $X \subseteq \Re^n$ is a non-empty closed set, $D \subseteq \Re^l$ is a non-empty set and $F : D \times X \to X$ is a locally bounded mapping. In this setting, $z \in X$ denotes the state of system (1) and $d \in D$ is an unknown, time-varying input. Let $z^* \in X$ be an equilibrium point of (1), i.e., $F(d, z^*) = z^*$ for all $d \in D$. Given $z_0 \in X$, $\{d(t) \in D\}_{t=0}^\infty$ we are in a position to determine the solution $z(t)$ of (1), with $z(0) = z_0$ corresponding to input $\{d(t) \in D\}_{t=0}^\infty$, by means of the recursive relation $z(t+1) = F(d(t), z(t))$, for all $t \geq 0$.

In this work we adopt the following robust exponential stability notion (see similar notions in [22], [23], [24]).

Definition 2.2: *We say that $z^* \in X$ is Robustly Globally Exponentially Stable (RGES) for system (1) if there exist constants $M, \sigma > 0$ such that for every $z_0 \in X$, $\{d(t) \in D\}_{t=0}^\infty$, the solution $z(t)$ of (1) with $z(0) = z_0$ corresponding to input $\{d(t) \in D\}_{t=0}^\infty$ satisfies $|z(t) - z^*| \leq M \exp(-\sigma t)|z_0 - z^*|$ for all $t \geq 0$.*

We next consider discrete-time systems with uncertain constant parameters and outputs. Consider the discrete-time system:

$$x^+ = f(d, \theta^*, x, u), x \in S, d \in D, u \in U, \theta^* \in \Theta \tag{2}$$

where $S \subseteq \Re^n$, $D \subseteq \Re^l$, $U \subseteq \Re^m$, $\Theta \subseteq \Re^q$ are non-empty sets and $f : D \times \Theta \times S \times U \to S$ is a locally bounded mapping. In this setting, $x \in S$ denotes the state of the system (2), $d \in D$ is an unknown, time-varying input, $u \in U$ is the control input and $\theta^* \in \Theta$ denotes the vector of unknown, constant parameters. The measured output of the system is given by

$$y(t) = h(d(t), \theta^*, x(t)) \tag{3}$$

where $h : D \times \Theta \times S \to \Re^k$ is a locally bounded mapping. Let $Y \subseteq \Re^k$ be a set with $h(D \times \Theta \times S) \subseteq Y$. We assume that $x^* \in S$ is an equilibrium point for system (2) and $d \in D$ is a vanishing perturbation, i.e., there exist vectors $y^* \in h(D \times \{\theta^*\} \times S)$ and $u^* \in U$ such that $f(d, \theta^*, x^*, u^*) = x^*$, $y^* = h(d, \theta^*, x^*)$ for all $d \in D$. Notice that $y^* \in Y$.

The main result of this section provides sufficient conditions for dynamic, robust, global, exponential stabilization of the equilibrium point $x^* \in S$. The stabilizer is constructed under the following assumptions. By $y^{(p)}(t) = (y(t-1), y(t-2), ..., y(t-p))$ for certain positive integer $p > 0$, we denote the "p-history" of the signal $y(t)$ (defined for all $t \geq p$). By $(y^*, ..., y^*)$ we mean the vector in $\Re^{kp}$ which is formed by combining the vector $y^* \in \Re^k$ p times. Since $y^* \in Y$, it follows that $(y^*, ..., y^*) \in Y^p$.

(H1) *Suppose that there exists a mapping $K : \Theta \times Y \to U$ such that $x^* \in S$ is RGES for the closed-loop system (2), (3) with $u = K(\theta^*, y)$.*

(H2) *Suppose that there exist a positive integer $p > 0$, a set $A \subseteq Y^p$ which contains all $w \in Y^p$ in a neighborhood of $(y^*, ..., y^*)$ and a mapping $\Psi : Y \times A \to \Theta$, such that for every sequence $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^\infty$ and for every $x_0 \in S$, the solution $x(t)$ of (2), (3) with $u = K(\hat{\theta}, y)$, initial condition $x(0) = x_0$ corresponding to inputs $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^\infty$ satisfies $\theta^* = \Psi(y(t), y^{(p)}(t))$ for all $t \geq p$ with $y^{(p)}(t) \in A$.*

(H3) *There exists a positive integer $m > 0$, such that for every sequence $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^\infty$ and for every $x_0 \in S$, the solution $x(t)$ of (2), (3) with $u = K(\hat{\theta}, y)$, initial condition $x(0) = x_0$ corresponding to inputs $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^\infty$ satisfies $y^{(p)}(t - i(t)) \in A$ for some $i(t) \in \{0, 1, ..., m\}$ and for all $t \geq m + p$.*

Assumption (H1) is a standard assumption, which guarantees the existence of a robust global exponential stabilizer when the vector of the parameters $\theta^* \in \Theta$ is known. Assumptions (H2)-(H3) are equivalent to complete, robust observability of θ^* from the output given by (3) (see, also [7]). More specifically, Assumption (H2) guarantees the existence of a function Ψ (the reconstruction map, see [7]), which gives the exact value of θ^*, provided that the p-history of the output signal belongs to a specific set A. Assumption (H3) guarantees that the p-history of the output signal is bound to enter the set A, every m time units.

The following result combines a certainty equivalence type controller with a finite-time identifier and guarantees exponential convergence both of the state $x(t)$ and the estimate $\hat{\theta}(t)$ to x^* and θ^*, respectively, for every disturbance $d(t)$.

Theorem 2.1: *Consider system (2) with output given by (3) under assumptions (H1), (H2), (H3). Moreover, suppose that the sets $f(D\times\Theta\times S\times U)$, Y, Θ are bounded. Finally, assume that there exist a constant $L\geq 0$, neighborhoods $N_1\subseteq\Re^n$ of x^*, $N_2\subseteq\Re^k$ of y^*, $N_3\subseteq\Re^q$ of θ^*, such that the inequalities $|f(d,\theta^*,x,K(\hat{\theta},h(d,\theta^*,x)))-x^*|+|h(d,\theta^*,x)-y^*|\leq L|x-x^*|+L|\hat{\theta}-\theta^*|$ and $|\Psi(h(d,\theta^*,x),w)-\theta^*|\leq L|x-x^*|+L\sum_{i=1}^{p}|w_i-y^*|$ hold for all $x\in N_1\cap S$, $d\in D$, $\hat{\theta}\in N_3\cap\Theta$, $w_i\in N_2\cap Y(i=1,...,p)$ with $w=(w_1,...,w_p)$.*
Then, the dynamic feedback stabilizer

$$\begin{aligned} w_1^+ &= y\\ w_2^+ &= w_1\\ &\vdots\\ w_p^+ &= w_{p-1}\\ \hat{\theta}^+ &= \begin{cases}\hat{\theta} & \text{if } w\notin A\\ \Psi(y,w) & \text{if } w\in A\end{cases}\\ u &= K(\hat{\theta},y)\end{aligned} \tag{4}$$

where $w=(w_1,...,w_p)\in Y^p$, $\hat{\theta}\in\Theta$ achieves the following:
1) There exist constants $M,\sigma>0$ such that for every sequence $\{d(t)\in D\}_{t=0}^{\infty}$ and for every $(x_0,w_0,\hat{\theta}_0)\in S\times Y^p\times\Theta$, the solution $(x(t),w(t),\hat{\theta}(t))$ of the closed-loop system (2), (3) with (4), initial condition $(x(0),w(0),\hat{\theta}(0))=(x_0,w_0,\hat{\theta}_0)$ corresponding to input $\{d(t)\in D\}_{t=0}^{\infty}$ satisfies

$$\begin{aligned}&|x(t)-x^*|+\sum_{i=1}^{p}|w_i(t)-y^*|+|\hat{\theta}(t)-\theta^*|\leq\\ &M\exp(-\sigma t)\left(|x(0)-x^*|+\sum_{i=1}^{p}|w_i(0)-y^*|+|\hat{\theta}(0)-\theta^*|\right)\end{aligned} \tag{5}$$

for all $t\geq 0$.
2) For every sequence $\{d(t)\in D\}_{t=0}^{\infty}$ and for every $(x_0,w_0,\hat{\theta}_0)\in S\times Y^p\times\Theta$ the solution $(x(t),w(t),\hat{\theta}(t))$ of the closed-loop system (2), (3) with (4), initial condition $(x(0),w(0),\hat{\theta}(0))=(x_0,w_0,\hat{\theta}_0)$ corresponding to input $\{d(t)\in D\}_{t=0}^{\infty}$ satisfies $\hat{\theta}(t)=\theta^*$, for all $t\geq m+p+1$.

Remark 2.1: The dynamic feedback stabilizer (4) achieves dead-beat estimation (provided by the variable $\hat{\theta}\in\Theta$) of the vector of unknown constant parameters $\theta^*\in\Theta$. Due to the dead-beat estimation, the exponential convergence property for the closed-loop system is preserved, as estimate (5) shows.

The proof of Theorem 2.1 relies on the following technical lemma. Its proof is provided in the Appendix.

Lemma 2.2: *Consider system (1) and let $\Omega\subseteq X$ be a given set. Suppose that $F(D\times X)$ is bounded. Moreover, suppose that the following hold:*
i) There exist constants $M,\sigma>0$ such that for every $z_0\in\Omega$, $\{d(t)\in D\}_{t=0}^{\infty}$ the solution $z(t)$ of (1) with initial condition $z(0)=z_0$ corresponding to input $\{d(t)\in D\}_{t=0}^{\infty}$ satisfies $|z(t)-z^*|\leq M|z_0-z^*|\exp(-\sigma t)$, for all $t\geq 0$.
ii) There exists an integer $N\geq 1$ such that for every $z_0\in X$, $\{d(t)\in D\}_{t=0}^{\infty}$ and $t\geq N$ there exists $i(t)\in\{0,1,...,N\}$ for which the solution $z(t)$ of (1) with initial condition $z(0)=z_0$ corresponding to input $\{d(t)\in D\}_{t=0}^{\infty}$ satisfies $z(t-i(t))\in\Omega$.
iii) There exists a constant $L\geq 1$, such that the inequality $|F(d,z)-z^*|\leq L|z-z^*|$ holds for all $d\in D$ and for all $z\in X$ in a neighborhood of z^*.
Then, $z^*\in X$ is RGES for the uncertain system (1).

Remark 2.2: It should be noticed that Lemma 2.2 requires that the exponential stability estimate $|z(t)-z^*|\leq M|z_0-z^*|\exp(-\sigma t)$ holds only for initial conditions z_0 that belong to the set Ω. Therefore, one can exploit this fact by selecting the set $\Omega\subseteq X$ in a convenient way. As always, there is a price to pay for this relaxation of requirements for RGES: one has to show that assumptions ii), iii) of Lemma 2.2 hold as well.

We are now ready to provide the proof of Theorem 2.1.

Proof of Theorem 2.1: Let $\Phi(x)$ be the (possibly empty) set of all $w=(w_1,...,w_p)\in Y^p$ for which there exist $\xi\in S$, $(d(i),\hat{\theta}(i))\in D\times\Theta$, $i=0,...,p-1$ such that the vectors $\bar{x}(i)$, $i=0,...,p$, defined by the recursive formula

$$\begin{aligned}\bar{x}(0)&=\xi\\ \bar{x}(i+1)&=f(d(i),\theta^*,\bar{x}(i),K(\hat{\theta}(i),h(d(i),\theta^*,\bar{x}(i))))\end{aligned} \tag{6}$$

for $i=0,...,p-1$, satisfy $\bar{x}(p)=x$ and $w_{p-i}=h(d(i),\theta^*,\bar{x}(i))$ for $i=0,...,p-1$. Notice that $\Phi(x^*)\neq\emptyset$ since by selecting $\xi=x^*\in S$, $\hat{\theta}(i)=\theta^*\in\Theta$ and arbitrary $d(i)\in D$ for $i=0,...,p-1$, the recursive formula (6) gives $\bar{x}(p)=x^*$ and $w_{p-i}=y^*$ for $i=0,...,p-1$.

All assumptions of Lemma 2.2 hold with $X=S\times Y^p\times\Theta$, $z=(x,w,\hat{\theta})$, $\Omega=\cup_{x\in S}\{(x,w,\theta^*):w\in\Phi(x)\}$, $N=m+p+1$, $z^*=(x^*,y^*,...,y^*,\theta^*)$ and

$$F(d,z):=\begin{bmatrix}f(d,\theta^*,x,K(\hat{\theta},h(d,\theta^*,x)))\\ h(d,\theta^*,x)\\ w_1\\ \vdots\\ w_{p-1}\\ g(h(d,\theta^*,x),w,\hat{\theta})\end{bmatrix},$$

where

$$g(h(d,\theta^*,x),w,\hat{\theta}):=\begin{cases}\hat{\theta} & \text{if } w\notin A\\ \Psi(h(d,\theta^*,x),w) & \text{if } w\in A\end{cases}.$$

Notice again that $\Omega\neq\emptyset$ since $\Phi(x^*)\neq\emptyset$. We show next that assumptions (i), (ii) of Lemma 2.2 are direct consequences of assumptions (H1), (H2), (H3).

Let $\{d(t)\in D\}_{t=0}^{\infty}$ be an arbitrary sequence and let $(x_0,w_0,\hat{\theta}_0)\in\Omega$ be an arbitrary vector with $\hat{\theta}_0=\theta^*$. Consider the solution $(x(t),w(t),\hat{\theta}(t))$ of the closed-loop system (2), (3) with (4), initial condition $(x(0),w(0),\hat{\theta}(0))=(x_0,w_0,\hat{\theta}_0)$ corresponding to input $\{d(t)\in D\}_{t=0}^{\infty}$. By virtue of (6), the component $x(t)$ of the solution satisfies $x(t)=\bar{x}(t+p)$ for all $t\geq 0$, for certain solution $\bar{x}(i)$ of the system $\bar{x}^+=f(\delta,\theta^*,\bar{x},K(\nu,h(d,\theta^*,\bar{x})))$ (that corresponds to certain inputs $\{(\delta(t),\nu(t))\in D\times\Theta\}_{i=0}^{\infty}$ with $\delta(t+p)=d(t)$, $\nu(t+p)=\hat{\theta}(t)$ for all $t\geq 0$ and appropriate initial condition $\xi\in S$). Moreover, $w(t)=\bar{y}^{(p)}(t+p)\in\Phi(x(t))$ for all $t\geq 0$, where $\bar{y}(t)=h(\delta(t),\theta^*,\bar{x}(t))$. Notice that if $w(0)=w_0\in A$ then $\bar{y}^{(p)}(p)\in A$, and, consequently, assumption (H2) guarantees that $\hat{\theta}(1) = \theta^*$. If $w(0) = w_0 \notin A$ then $\hat{\theta}(1) = \hat{\theta}(0) = \theta^*$. Using induction and the previous argument, it follows that $\hat{\theta}(t) = \theta^*$ for all $t \geq 0$. Therefore, assumption (i) of Lemma 2.2 is a consequence of assumption (H1).

Assumption (ii) of Lemma 2.2 follows from the fact that $w(t) = y^{(p)}(t) \in \Phi(x(t))$ for all $t \geq p$. Assumption (H3) guarantees that $w(t - i(t)) = y^{(p)}(t - i(t)) \in A$ for some $i(t) \in \{0, 1, ..., m\}$ and for all $t \geq m + p$. It follows from (4) that $\hat{\theta}(t - i(t) + 1) = \theta^*$. Since $t - i(t) + 1 \geq p + 1$, we also get $w(t - i(t) + 1) \in \Phi(x(t))$ and thus $z(t - i(t) + 1) \in \Omega$. Therefore, assumption (ii) of Lemma 2.2 holds with $N = m + p + 1$.

Since $A \subseteq Y^p$ contains all $w \in Y^p$ in a neighborhood of $(y^*, ..., y^*)$ and since there exist neighborhoods $N_1 \subseteq \Re^n$ of x^*, $N_2 \subseteq \Re^k$ of y^*, $N_3 \subseteq \Re^q$ of θ^*, such that the inequalities

$$|f(d, \theta^*, x, K(\hat{\theta}, x)) - x^*| + |h(d, \theta^*, x) - y^*| \leq \\ L|x - x^*| - L|\hat{\theta} - \theta^*|,$$

$$|\Psi(h(d, \theta^*, x), w) - \theta^*| \leq L|x - x^*| + L \sum_{i=1}^{p} |w_i - y^*|$$

hold for all $x \in N_1 \cap S$, $d \in D$, $\hat{\theta} \in N_3 \cap \Theta$, $w_i \in N_2 \cap Y (i = 1, ..., p)$ with $w = (w_1, ..., w_p)$, it follows that assumption (iii) of Lemma 2.2 holds. $\triangleleft$

III. Application To Freeway Traffic Control

A. The freeway model

We consider a freeway which consists of $n \geq 3$ components or cells; typical cell lengths may be 200-500 m. Each cell may have an external inflow (e.g. from corresponding on-ramps), located near the cell's upstream boundary; and an external outflow (e.g. via corresponding off-ramps), located near the cell's downstream boundary (Fig.1). The number of vehicles at time $t \geq 0$ in component $i \in \{1, ..., n\}$ is denoted by $x_i(t)$. The total outflow and the total inflow of vehicles of the component $i \in \{1, ..., n\}$ at time $t \geq 0$ are denoted by $F_{i,out}(t) \geq 0$ and $F_{i,in}(t) \geq 0$, respectively. All flows during a time interval are measured in [veh]. Consequently, the balance of vehicles (conservation equation) for each component $i \in \{1, ..., n\}$ gives:

$$x_i(t + 1) = x_i(t) - F_{i,out}(t) + F_{i,in}(t), \ \ t \geq 0. \tag{7}$$

Each component of the network has storage capacity $a_i > 0$ $(i = 1, ..., n)$. Our first assumption states that the external (off-ramp) flows from each cell are constant percentages of the total exit flow, i.e., there exist constants $P_i \in [0, 1)$ $(i = 1, ..., n)$, such that:

$$\begin{pmatrix} \textit{flow of vehicles} \\ \textit{from cell } i \textit{ to cell } i+1 \end{pmatrix} = (1 - P_i)F_{i,out}(t) \tag{8}$$

for $i = 1, ..., n - 1$,

$$\begin{pmatrix} \textit{flow of vehicles} \\ \textit{from cell } i \textit{ to regions out} \\ \textit{of the freeway} \end{pmatrix} = P_i F_{i,out}(t) \tag{9}$$

for $i = 1, ..., n$.

The constants P_i are known as exit rates. Since the n-th cell is the last downstream cell of the considered freeway, we may assume that $P_n = 1$. We also assume that $P_i < 1$ for $i = 1, ..., n-1$, and that all exits to regions out of the network can accommodate the respective exit flows.

Our second assumption is dealing with the attempted outflows $f_i(x_i)$, i.e. the flows that will exit the cell if there is sufficient space in the downstream cell. We assume that there exist functions $f_i : [0, a_i] \to \Re_+$ with $0 < f_i(x_i) < x_i$ for $x_i \in (0, a_i]$, variables $s_i(t) \in [0, 1]$, $i = 2, ..., n$, so that:

$$\begin{gathered} F_{i-1,out}(t) = s_i(t) f_{i-1}(x_{i-1}(t)), \ \ i = 2, ..., n, t \geq 0 \\ \text{and} \ \ F_{n,out}(t) = f_n(x_n(t)) \end{gathered} \tag{10}$$

The variable $s_i(t) \in [0, 1]$, for each $i = 2, ..., n$, indicates the percentage of the attempted outflow from cell $i - 1$ that becomes actual outflow from the same cell. The function $f_i : [0, a_i] \to \Re_+$ is called, in the specialized literature of Traffic Engineering (see, e.g., [20], [9], [25], [21], [26], [27]), the demand-part of the fundamental diagram of the i-th cell, i.e. the flow that will exit the cell i if there is sufficient space in the downstream cell $i + 1$. Notice that equation (10) for $F_{n,out}(t)$ follows from our assumption that all exits to regions out of the network can accommodate the exit flows.

Let $v_i \geq 0$ $(i = 1, ..., n)$ denote the attempted external inflow to component $i \in \{1, ..., n\}$ from the region out of the freeway. Typically, v_i, $i = 2, ..., n$, correspond to external on-ramp flows which may be determined by a ramp metering control strategy. For the very first cell 1, we assume, for convenience, that there is just one external inflow, $v_1 > 0$. Let the variables $W_i(t) \in [0, 1]$, $i = 1, ..., n$, indicate the percentage of the attempted external inflow to component $i \in \{1, ..., n\}$ that becomes actual inflow. Then, we obtain from (8) and (10):

$$\begin{gathered} F_{1,in}(t) = W_1(t) v_1(t) \ \ \text{and} \\ F_{i,in}(t) = W_i(t) v_i(t) + s_i(t)(1 - P_{i-1}) f_{i-1}(x_{i-1}(t)), \\ \text{for } i = 2, ..., n \end{gathered} \tag{11}$$

Our next assumption requires that the inflow of vehicles at the cell $i \in \{1, ..., n\}$ at time $t \geq 0$, denoted by $F_{i,in}(t) \geq 0$, cannot exceed the supply function of cell $i \in \{1, ..., n\}$ at time $t \geq 0$, i.e.,

$$F_{i,in}(t) \leq \min(q_i, c_i(a_i - x_i(t))) \tag{12}$$

where $q_i \in (0, +\infty)$ denotes the maximum flow that the i-th cell can receive (or the capacity flow of the i-th cell) and $c_i \in (0, 1]$ $(i = 1, ..., n)$ denotes the congestion wave speed of the i-th cell.

Following [20], we assume that, when the total demand flow of a cell is lower than the supply of the downstream cell, i.e.

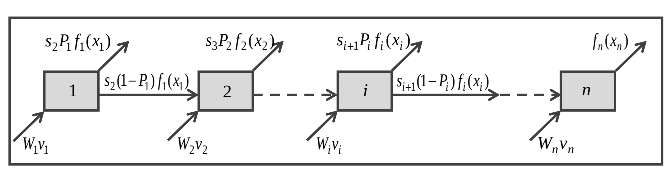

Fig. 1. Scheme of the freeway model.

when $v_i(t)+(1-P_{i-1})f_{i-1}(x_{i-1}(t)) \leq \min(q_i, c_i(a_i-x_i(t)))$ for some $i \in \{2,...,n\}$, then the demand flow can be fully accommodated by the downstream cell, and hence we have $s_i(t) = W_i(t) = 1$. Similarly, when $v_1(t) \leq \min(q_1, c_1(a_1 - x_1(t)))$, then we have $W_1(t) = 1$. In contrast, when the total demand flow of a cell is higher than the supply of the downstream cell, i.e. when $v_i(t)+(1-P_{i-1})f_{i-1}(x_{i-1}(t)) > \min(q_i, c_i(a_i - x_i(t)))$ for some $i \in \{2,...,n\}$ (or when $v_1(t) > \min(q_1, c_1(a_1 - x_1(t))))$, then the demand flow cannot be fully accommodated by the downstream cell, and the actual flow is determined by the supply function, i.e. we have $F_{i,in}(t) = \min(q_i, c_i(a_i - x_i(t)))$ (or $F_{1,in}(t) = \min(q_1, c_1(a_1 - x_1(t))))$. Therefore, for $i = 2,...,n$ and $t \geq 0$ we get:

$$F_{1,in}(t) = \min(q_1, c_1(a_1 - x_1(t)), v_1(t)) \tag{13}$$

$$\begin{aligned} s_i(t) &= (1-d_i(t))\times \\ &\min\left(1, \max\left(0, \frac{\min(q_i, c_i(a_i - x_i(t))) - v_i(t)}{(1-P_{i-1})f_{i-1}(x_{i-1}(t))}\right)\right) \\ &+d_i(t)\min\left(1, \frac{\min(q_i, c_i(a_i - x_i(t)))}{(1-P_{i-1})f_{i-1}(x_{i-1}(t))}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} &F_{i,in}(t) = \\ &\min(q_i, c_i(a_i - x_i(t)), v_i(t) + (1-P_{i-1})f_{i-1}(x_{i-1}(t))) \end{aligned} \tag{15}$$

where

$$d_i(t) \in [0,1] \tag{16}$$

are time-varying parameters. Note that, if the supply is higher than the total demand, then (14) yields $s_i = 1$, irrespective of the value of d_i, since the total demand flow can be accommodated by the downstream cell. Thus, the parameter d_i determines the relative inflow priorities, when the downstream supply prevails. Specifically, when $d_i(t) = 0$, then the on-ramp inflow has absolute priority over the internal inflow; on the other hand, when $d_i(t) = 1$, then the internal inflow has absolute priority over the on-ramp inflow; while intermediate values of d_i reflect intermediate priority cases. The parameters $d_i(t) \in [0,1]$ are treated as unknown parameters (disturbances). Notice that by introducing the parameters $d_i(t) \in [0,1]$ (and by allowing them to be time-varying), we have taken into account all possible cases for the relative priorities of the inflows (and we also allow the priority rules to be time-varying); see [28], [20] for freeway models with specific priority rules, which are special cases of our general approach.

All the above are illustrated in Fig.1. Combining equations (7), (8), (9), (10), (13) and (15) we obtain the following discrete-time dynamical system:

$$\begin{aligned} x_1^+ &= x_1 - s_2 f_1(x_1) + \min(q_1, c_1(a_1 - x_1), v_1) \\ &= x_1 - s_2 f_1(x_1) + W_1 v_1 \end{aligned} \tag{17}$$

$$\begin{aligned} x_i^+ &= x_i - s_{i+1} f_i(x_i) \\ &+ \min(q_i, c_i(a_i - x_i), v_i + (1-P_{i-1})f_{i-1}(x_{i-1})) \\ &= x_i - s_{i+1} f_i(x_i) \\ &+ W_i v_i + s_i(1-P_{i-1})f_{i-1}(x_{i-1}) \end{aligned} \tag{18}$$

for $i = 2,...,n-1$,

$$\begin{aligned} x_n^+ &= x_n - f_n(x_n) \\ &+ \min(q_n, c_n(a_n - x_n), v_n + (1-P_{n-1})f_{n-1}(x_{n-1})) \\ &= x_n - f_n(x_n) + W_n v_n + s_n(1-P_{n-1})f_{n-1}(x_{n-1}) \end{aligned} \tag{19}$$

where $s_i \in [0,1]$, $i = 2,...,n$ are given by (14). The values of $W_i \in [0,1]$, $i = 1,...,n$, may also be similarly derived from (15) when $v_i > 0$ but they are not needed in what follows. Define $S = \prod_{i=1}^{n}(0, a_i]$. Since the functions $f_i : [0, a_i] \to \Re_+$ satisfy $0 < f_i(x_i) < x_i$ for $x_i \in (0, a_i]$, it follows that (17), (18), (19) is an uncertain control system on S (i.e., $x = (x_1,...,x_n)' \in S$) with inputs $v = (v_1,...,v_n)' \in (0,+\infty) \times \Re_+^{n-1}$ and disturbances $d = (d_2,...,d_n)' \in [0,1]^{n-1}$. We emphasize again that the uncertainty $d \in [0,1]^{n-1}$ appears in the equations (17), (18) and (19) only when the supply function prevails, i.e., only when $v_i(t)+(1-P_{i-1})f_{i-1}(x_{i-1}(t)) > \min(q_i, c_i(a_i-x_i(t)))$ for some $i \in \{2,...,n\}$.

We make the following assumption for the functions $f_i : [0, a_i] \to \Re_+, (i = 1,...,n)$:

(H) *There exist constants* $\delta_i \in (0, a_i]$ *and* $r_i \in (0,1)$ *such that* $f_i(z) = r_i z$ *for* $z \in [0, \delta_i]$. *Moreover, there exists a positive constant* $f_i^{min} > 0$ *such that* $f_i(\delta_i) = r_i\delta_i \geq f_i(z) \geq f_i^{min}$ *for all* $z \in [\delta_i, a_i]$.

Assumption (H) reflects some of the basic properties of the so-called demand function [21] in the Godunov discretization; whereby δ_i is the critical density, where $f_i(x_i)$ achieves a maximum value. The implications of Assumption (H) for the demand function are illustrated in Fig. 2. The linearity of the demand functions on the interval $[0, \delta_i]$ is a consequence of the consideration of constant free flow speed for under-critical densities (here, represented by the dimensionless variable $r_i \in (0,1)$), which is suggested in many studies in the literature (see, for example, [20]). Notice also, that Assumption (H) includes the possibility of reduced demand flow for overcritical densities (i.e., when $x_i(t) \geq \delta_i$), since $f_i(x_i)$ is allowed to be any arbitrary function (e.g. discontinuous or decreasing or, even, increasing), taking any values within the bounds mentioned in (H) (corresponding to the grey area in Fig. 2), for $x_i \in [\delta_i, a_i]$; this could be used to reflect the capacity drop phenomenon, as it is treated in some recent works [29], [30]. Fig. 2 presents, within the grey area of overcritical densities, three examples of demand functions, which satisfy assumption (H).

A more general assumption than assumption (H) was used in [19], but in [19] it was assumed that all parameters of the model were known. More specifically, in [19], it was not necessary the demand functions $f_i : [0, a_i] \to \Re_+, (i = 1,...,n)$ to be linear on the corresponding intervals $[0, \delta_i]$.

B. Global Exponential Stabilization of Freeway Models

Define the vector field $\tilde{F} : D \times S \times (0,+\infty) \times \Re_+^{n-1} \to S$ for all $x \in S = \prod_{i=1}^{n}(0, a_i]$, $d = (d_2,...,d_n) \in D = [0,1]^{n-1}$ and $v \in (0,+\infty)\times\Re_+^{n-1}$, with $\tilde{F}_i$ being the right hand sides of (17)-(19), for $i = 1,...,n$, and s_i given by (14), for $i = 2,...,n$.

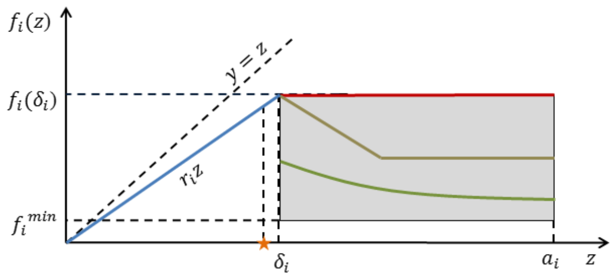

Fig. 2. Implications of Assumption (H): The demand functions f_i should be linear on the interval $[0, \delta_i]$, while they can take any value on the interval $(\delta_i, a_i]$ within the depicted grey area.

Then, the control system (17), (18), (19) can be written in the following vector form:

$$x^+ = \tilde{F}(d, x, v), x \in S, d \in D, v \in (0, +\infty) \times \Re_+^{n-1} \quad (20)$$

Consider the freeway model (20) under assumption (H). Let $v^* = (v_1^*, ..., v_n^*)' \in (0, +\infty) \times \Re_+^{n-1}$ be a vector that satisfies:

$$\begin{gathered} v_1^* < \min(q_1, c_1(a_1 - \delta_1), r_1\delta_1) \\ v_i^* + \sum_{j=1}^{i-1} v_j^* \left(\prod_{k=j}^{i-1} (1 - P_k) \right) < \min(q_i, c_i(a_i - \delta_i), r_i\delta_i) \end{gathered} \quad (21)$$

Any inflow vector that satisfies (21), defines an UEP $x^* = (x_1^*, ..., x_n^*) \in \prod_{i=1}^n (0, \delta_i)$ for the freeway model:

$$\begin{gathered} x_1^* = r_1^{-1} v_1^* \\ x_i^* = r_i^{-1} \left(v_i^* + \sum_{j=1}^{i-1} v_j^* \prod_{k=j}^{i-1} (1 - P_k) \right), i = 2, ..., n \end{gathered} \quad (22)$$

The UEP is not globally exponentially stable for arbitrary $v_1^* > 0$, $v_i^* \geq 0$ $(i = 2, ..., n)$. Indeed, simulations show that there are critical values of inflows, so that if the inflows $v_i^* \geq 0$ $(i = 1, ..., n)$ are larger than the critical values, then other equilibria for model (20) (congested equilibria) appear. These congested equilibria have large cell densities and attract the solution of (20).

The following result (see [19]) is the main result in feedback design that provides the nominal feedback for the adaptive control scheme that we intend to use. The result shows that a continuous, robust, global exponential stabilizer exists for every freeway model of the form (20) under assumption (H).

Theorem 3.1: *Consider system (20) with $n \geq 3$ under assumption (H) for each $i = 1, ..., n$. Then there exist a subset $R \subseteq \{1, ..., n\}$ of the set of all indices $i \in \{1, ..., n\}$ with $v_i^* > 0$, constants $\sigma \in (0, 1]$, $b_i \in (0, v_i^*)$ for $i \in R$ and a constant $\tau^* > 0$, such that for every $\tau \in (0, \tau^*)$ the feedback law $K : S \to \Re_+^n$ defined by:*

$$\begin{gathered} K(x) = (K_1(x), ..., K_n(x))' \in \Re_n \text{ with} \\ K_i(x) = \max\left(b_i, v_i^* - \tau^{-1}(v_i^* - b_i)\Xi(x) \right) \\ \text{for all } x \in S, i \in R, \\ K_i(x) = v_i^* \text{ for all } x \in S, i \notin R, \end{gathered} \quad (23)$$

where

$$\Xi(x) := \sum_{i=1}^n \sigma^i \max(0, x_i - x_i^*), \text{ for all } x \in S, \quad (24)$$

achieves robust global exponential stabilization of the UEP x^* of system (20), i.e., x^* is RGES for the closed-loop system (20) with $v = K(x)$.

The result of Theorem 3.1 (see [19]) is based on the construction of a Control Lyapunov function for system (20) under a more general assumption than assumption (H). The feedback law provides values for the controllable inflows v_i, $i \in R$, in the interval $[b_i, v_i^*]$ for all $i \in R$, where $b_i \in (0, v_i^*)$ for $i \in R$ are the minimum allowable inflows. Since the proof of Theorem 3.1 is constructive, criteria for the selection of the index set $R \subseteq \{1, ..., n\}$ and the constants $\sigma \in (0, 1]$, $b_i \in (0, v_i^*)$ for $i \in R$ and $\tau^* > 0$ are provided.

Without loss of generality, we will assume, in what follows, that $R \neq \emptyset$ (because otherwise the UEP is open-loop RGES).

Let $\mu_i \in (0, \delta_i)$, $v_{i,max} < (0, +\infty)$ $(i = 1, ..., n)$ be constants such that:

$$\begin{gathered} v_{1,max} < \min(q_1, c_1(a_1 - \mu_1)), \\ v_{i,max} + (1 - P_{i-1}) r_{i-1} \mu_{i-1} < \min(q_i, c_i(a_i - \mu_i)), \\ i = 2, ..., n. \end{gathered} \quad (25)$$

It follows that if $x \in \Omega = \prod_{i=1}^n (0, \mu_i)$ and $v \in (0, v_{1,max}] \times \prod_{i=1}^n [0, v_{i,max}]$:

$$W_i = 1, \text{ for } i = 1, ..., n \text{ and } s_i = 1 \text{ for } i = 2, ..., n \quad (26)$$

$$\begin{gathered} x_1^+ = x_1 - f_1(x_1) + v_1 \\ x_i^+ = x_i - f_i(x_i) + v_i + (1 - P_{i-1}) f_{i-1}(x_{i-1}) \\ \text{for } i = 2, ..., n \end{gathered} \quad (27)$$

In what follows, we assume that $x^* = (x_1^*, ..., x_n^*) \in \prod_{i=1}^n (0, \mu_i - \varepsilon]$, $v_i^* \in [b_i + \varepsilon, v_{i,max}]$ for $i \in R$ and for some $\varepsilon \in (0, 1/2)$ and $v^* \in (0, v_{1,max}] \times \prod_{i=2}^n [0, v_{i,max}]$. Moreover, we assume that $P_i \in [0, 1 - \varepsilon]$ for $i = 1, ..., n - 1$ and $r_i \in [\varepsilon, 1 - \varepsilon]$ for $i = 1, ..., n$.

Another feature of the present problem is that the selection of the UEP may be made in an implicit way. For example, we may want the UEP that guarantees the maximum outflow from the freeway. In such cases, the equilibrium position of the controllable inflows is determined as a function of the nominal values of the uncontrollable inflows and the parameters of the freeway, i.e., there exists a smooth function

$$g : [0, 1-\varepsilon]^{n-1} \times \prod_{i \notin R} [0, v_{i,max}] \times [\varepsilon, 1-\varepsilon]^n \to \prod_{i \in R} [b_i + \varepsilon, v_{i,max}]$$

such that

$$(v_i^*; i \in R) = g(P, v_i^*; i \notin R, r) \quad (28)$$

where $P = (P_1, ..., P_{n-1})' \in [0, 1 - \varepsilon]^{n-1}$ and $r = (r_1, ..., r_n)' \in [\varepsilon, 1 - \varepsilon]^n$.

C. Measurements and Unknown Parameters

Let $m \in \{1, ..., n\}$ be the cardinal number of the set R and let $u \in U = \prod_{i \in R}[b_i, v_{i,max}] \subseteq (0, +\infty)^m$ be the vector of all controllable inflows v_i with $i \in R$.

The model parameters which are (usually) unknown or uncertain are: the exit rates $P_i \in [0,1)$ for $i = 1, ..., n-1$, the uncontrollable inflows $v_i^* \in \Re_+$ for $i \notin R$ and the demand coefficients $r_i \in (0,1)$ for $i = 1, ..., n$. All these parameters will be denoted by $\theta^* = (P, v_i^*; i \notin R, r)$ and are assumed to take values in a compact set $\Theta := [0, 1-\varepsilon]^{n-1} \times \prod_{i \notin R}[0, v_{i,max}] \times [\varepsilon, 1-\varepsilon]^n$, for some $\varepsilon \in (0, 1/2)$. Therefore, the control system (17), (18), (19) can be written in the following vector form:

$$
\begin{gathered}
x^+ = \bar{F}(d, \theta^*, x, u) \\
x \in S, d \in D, \theta^* \in \Theta, u \in U = \prod_{i \in R}[b_i, v_{i,max}]
\end{gathered}
\tag{29}
$$

Notice that the feedback law defined by (23) is a feedback law of the form $u = K(\theta^*, x)$: the feedback law depends on the unknown parameters through x^* and $(v_i^*; i \in R)$ (recall (22) and (28)). It follows that assumption (H1) holds for system (29). An explicit definition of the feedback law $K : \Theta \times S \to U$ is given by the following equations for all $\hat{\theta} = (\hat{P}, \hat{v}_i^*; i \notin R, \hat{r}) \in \Theta$, $x \in S$ with $\hat{r} = (\hat{r}_1, ..., \hat{r}_n)' \in [\varepsilon, 1-\varepsilon]^n$, $\hat{P} = (\hat{P}_1, ..., \hat{P}_{n-1})' \in [0, 1-\varepsilon]^{n-1}$:

$$
(\hat{v}_i^*; i \in R) = g(\hat{P}, \hat{v}_i^*; i \notin R, \hat{r}),
\tag{30}
$$

$$
\begin{gathered}
\hat{x}_1^* = \min(\hat{r}_1^{-1}\hat{v}_1^*, \mu_1 - \varepsilon), \\
\hat{x}_i^* = \min\left(\hat{r}_i^{-1}\left(\hat{v}_i^* + \sum_{j=1}^{i-1}\hat{v}_j^*\prod_{k=j}^{i-1}(1-\hat{P}_k)\right), \mu_i - \varepsilon\right) \\
\textit{for } i = 2, ..., n,
\end{gathered}
\tag{31}
$$

$$
\begin{gathered}
u = K(\hat{\theta}, x) \textit{ with} \\
K_i(\hat{\theta}, x) = \max(b_i, \hat{v}_i^* - \tau^{-1}(\hat{v}_i^* - b_i)\Xi(\hat{\theta}, x)) \\
\textit{for all } x \in S, i \in R,
\end{gathered}
\tag{32}
$$

$$
\Xi(\hat{\theta}, x) := \sum_{i=1}^{n}\sigma^i \max(0, x_i - \hat{x}_i^*), \textit{ for all } x \in S.
\tag{33}
$$

The measured quantities are the cell densities $x \in S$ and the outflows from each cell. We have two kinds of outflows from each cell: the outflow to regions out of the freeway

$$
\begin{gathered}
Q_{out} = (Q_{1,out}, ..., Q_{n,out})' \in \Re_+^n \\
Q_{i,out} = P_i s_{i+1} f_i(x_i), i = 1, ..., n-1 \\
Q_{n,out} = f_n(x_n)
\end{gathered}
\tag{34}
$$

and the outflows from one cell to the next cell

$$
\begin{gathered}
Q = (Q_1, ..., Q_{n-1})' \in \Re_+^{n-1} \\
Q_i = (1-P_i)s_{i+1}f_i(x_i), i = 1, ..., n-1
\end{gathered}
\tag{35}
$$

Therefore, the measured output is given by:

$$
y = h(d, \theta^*, x) = (x, Q_{out}, Q) \in S \times \Re_+^n \times \Re_+^{n-1}
\tag{36}
$$

Assumption (H) guarantees that $h(D \times \Theta \times S) \subseteq Y$ where $Y := S \times \prod_{i=1}^{n}[0, a_i] \times \prod_{i=1}^{n-1}[0, a_i]$ is a bounded set.

It follows from (26), (27), (34), (35), assumption (H) and the fact that $\mu_i \in (0, \delta_i)$ $(i = 1, ...n)$, that:

if $x(t-1) \in \Omega = \prod_{i=1}^{n}(0, \mu_i)$, $t \geq 1$, then the following equations hold:

$$
P_i = \frac{Q_{i,out}(t-1)}{Q_{i,out}(t-1) + Q_i(t-1)}, i = 1, ..., n-1
\tag{37}
$$

$$
\begin{gathered}
v_i^* = x_i(t) - x_i(t-1) + Q_i(t-1) + \\
Q_{i,out}(t-1) - Q_{i-1}(t-1), i \in \{2, ..., n\} \backslash R
\end{gathered}
\tag{38}
$$

$$
\begin{gathered}
v_1^* = x_1(t) - x_1(t-1) + Q_1(t-1) + \\
Q_{1,out}(t-1), \textit{if } i \notin R
\end{gathered}
\tag{39}
$$

$$
r_i = \frac{Q_{i,out}(t-1) + Q_i(t-1)}{x_i(t-1)}, i = 1, ..., n
\tag{40}
$$

Equations (37), (38), (39), (40), (36) allow us to define a mapping $\Psi : h(D \times \Theta \times S) \times Y \to \Theta$ for which $\theta^* = (P_1, ..., P_{n-1}, v_i^*; i \notin R, r_1, ..., r_n)' = \Psi(y(t), y(t-1))$ for all $t \geq 1$ with $y(t-1) \in A$, where $A \subseteq Y$ is the set for which:

$$
\begin{gathered}
w = (w_1, w_2, w_3) \in A \Leftrightarrow \\
(w_1, w_2, w_3) \in Y, w_1 \in \Omega = \prod_{i=1}^{n}(0, \mu_i) \\
\textit{and } w_{2,i} + w_{3,i} > 0 \textit{ for } i = 1, ..., n-1
\end{gathered}
\tag{41}
$$

The mapping $\Psi : h(D \times \Theta \times S) \times Y \to \Theta$ is defined by

$$
\hat{\theta} = (\hat{P}_1, ..., \hat{P}_{n-1}, \hat{v}_i^*; i \notin R, \hat{r}_1, ..., \hat{r}_n)' = \Psi(y, w)
\tag{42}
$$

with

$$
\hat{P}_i = \min\left(1-\varepsilon, \frac{w_{2,i}}{w_{2,i} + w_{3,i}}\right), i = 1, ..., n-1
\tag{43}
$$

$$
\begin{gathered}
\hat{v}_i^* = \max(0, \min(v_{i,max}, x_i - w_{1,i} + w_{3,i} + w_{2,i} - w_{3,i-1})), \\
i \in \{2, ..., n\} \backslash R \textit{ and } i \neq n
\end{gathered}
\tag{44}
$$

$$
\begin{gathered}
\hat{v}_n^* = \max(0, \min(v_{n,max}, x_n - w_{1,n} + w_{2,n} - w_{3,n-1})), \\
\textit{if } n \notin R
\end{gathered}
\tag{45}
$$

$$
\begin{gathered}
\hat{v}_1^* = \max(0, \min(v_{1,max}, x_1 - w_{1,1} + w_{3,1} + w_{2,1})), \\
\textit{if } 1 \notin R
\end{gathered}
\tag{46}
$$

$$
\hat{r}_i = \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,i} + w_{3,i}}{w_{1,i}}\right)\right), i = 1, ..., n-1
\tag{47}
$$

$$
\hat{r}_n = \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,n}}{w_{1,n}}\right)\right)
\tag{48}
$$

Using assumption (H), (3.16), (22) and (36), it follows that there exists $y^* \in Y$ with $y^* = h(d, \theta^*, x^*)$ for all $d \in D$. By virtue of our assumption $x^* = (x_1^*, ..., x_n^*) \in \prod_{i=1}^{n}(0, \mu_i)$ and $v^* \in (0, v_{1,max}] \times \prod_{i=2}^{n}[0, v_{i,max}]$, (41), we conclude that A contains all $w \in Y$ in a neighborhood of y^*. It follows that (H2) holds with $p = 1$ for system (29) with output given by (34), (35), (36).

In order to prove that assumption (H3) holds for system (29) with output given by (34), (35), (36), we need the following fact, which is a consequence of property (C5) shown in [19] and (25).

Fact: *Define $I_j(x) := \sum_{i=1}^{j} x_i$ for $j = 1, ..., n$. There exists a constant $C \in (0,1)$ such that the following inequality holds:*

$$\sum_{i=1}^{n} I_i(x^+) \leq (1-C)\sum_{i=1}^{n} I_i(x) + \sum_{i=1}^{n}(n+1-i)v_i,$$

for all $(x,v,d) \in S \times (0, v_{1,max}] \times \prod_{i=2}^{n}[0, v_{i,max}] \times [0,1]^{n-1}$ (49)

where x^+ is given by (29).

The following proposition guarantees that assumption (H3) holds for system (29) with output (34), (35), (36).

Proposition 3.2: *Suppose that $b_i > 0$ $(i \in R)$ and $v_{i,max}$ $(i \notin R)$ are sufficiently small and that $\tau > 0$ is sufficiently small $\left(\tau \leq \epsilon^2 \sigma^n \min_{i\in R}((v_{i,max} - b_i)^{-1})\right)$. Then there exists an integer $m \geq 1$ such that for every sequence $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^{\infty}$ and for every $x_0 \in S$, the solution $x(t)$ of (29), (36) with $u = K(\hat{\theta}, x)$, initial condition $x(0) = x_0$ corresponding to inputs $\{(d(t), \hat{\theta}(t)) \in D \times \Theta\}_{t=0}^{\infty}$ satisfies $y(t-1-i(t)) \in A$ for some $i(t) \in \{0, 1, ..., m\}$ and for all $t \geq m+1$.*

The main result for the freeway model is a consequence of Theorem 2.1 and the fact that all functions are sufficiently smooth in a neighborhood of the equilibrium.

Corollary 3.3: *Consider system (29) with output given by (34), (35), (36). Suppose that $b_i > 0$ $(i \in R)$ and $v_{i,max} (i \notin R)$ are sufficiently small and that $\tau > 0$ is sufficiently small. Then the dynamic feedback law given by:*

$$w_1^+ = x, w_2^+ = Q_{out}, w_3^+ = Q \tag{50}$$

$$\hat{P}_i^+ = \begin{cases} \hat{P}_i & \text{if } w \notin A \\ \min\left(1-\varepsilon, \frac{w_{2,i}}{w_{2,i}+w_{3,i}}\right) & \text{if } w \in A \end{cases} \tag{51}$$

$$(\hat{v}_i^*)^+ = \begin{cases} \hat{v}_i^* & \text{if } w \notin A \\ \max(0, \min(v_{i,max}, x_i - w_{1,i} \\ \quad +w_{3,i} + w_{2,i} - w_{3,i-1})) & \text{if } w \in A \end{cases} \tag{52}$$

$$(\hat{v}_n^*)^+ = \begin{cases} \hat{v}_n^* & \text{if } w \notin A \\ \max(0, \min(v_{n,max}, x_n - w_{1,n} \\ \quad + w_{2,n} - w_{3,n-1})) & \text{if } w \in A \end{cases} \tag{53}$$

$$(\hat{v}_1^*)^+ = \begin{cases} \hat{v}_1^* & \text{if } w \notin A \\ \max(0, \min(v_{1,max}, x_n - w_{1,1} \\ \quad + w_{3,1} + w_{2,1})) & \text{if } w \in A \end{cases} \tag{54}$$

$$\hat{r}_i^+ = \begin{cases} \hat{r}_i & \text{if } w \notin A \\ \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,i}+w_{3,i}}{w_{1,i}}\right)\right) & \text{if } w \in A \end{cases} \tag{55}$$

$$\hat{r}_n^+ = \begin{cases} \hat{r}_n & \text{if } w \notin A \\ \max\left(\varepsilon, \min\left(1-\varepsilon, \frac{w_{2,n}}{w_{1,n}}\right)\right) & \text{if } w \in A \end{cases}. \tag{56}$$

with (30), (31), (32), (33), $\hat{P} = (\hat{P}_1, ..., \hat{P}_{n-1})$, $P = (P_1, ..., P_{n-1})$, $\hat{r} = (\hat{r}_1, ..., \hat{r}_n)$, $r = (r_1, ..., r_n)$, $w = (w_1, w_2, w_3)$, $\hat{v}^* = (\hat{v}_1^*, ..., \hat{v}_n^*)$, achieves the following:

1) There exist constants $M, \sigma > 0$ such that for every sequence $\{d(t) \in D\}_{t=0}^{\infty}$ and for every $(x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0) \in S \times Y \times \Theta$, the solution of the closed-loop system (29), (36) with (50)-(56), (30)-(33), initial condition $(x(0), w(0), \hat{P}(0), \hat{v}_j^*(0); j \notin R, \hat{r}(0)) = (x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0)$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ satisfies:

$$\begin{aligned} &|x(t) - x^*| + |w(t) - y^*| + |\hat{r}(t) - r| + |\hat{P}(t) - P| + \\ &|\hat{v}^*(t) - v^*| \leq Mexp(-\sigma t)\big(|x(0) - x^*| + |w(0) - y^*| \\ &+ |\hat{r}(0) - r| + |\hat{P}(0) - P| + \sum_{i\notin R} |\hat{v}_i^*(0) - v_i^*|\big) \end{aligned} \tag{57}$$

for all $t \geq 0$.

2) There exists an integer $N \geq 1$ such that for every sequence $\{d(t) \in D\}_{t=0}^{\infty}$ and for every $(x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0) \in S \times Y \times \Theta$, the solution of the closed-loop system (29), (36) with (50)-(56), (30)-(33), initial condition $(x(0), w(0), \hat{P}(0), \hat{v}_j^*(0); j \notin R, \hat{r}(0)) = (x_0, w_0, \hat{P}_0, \hat{v}_{j,0}^*; j \notin R, \hat{r}_0)$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ satisfies $\hat{P}(t) = P$, $\hat{r}(t) = r$, $\hat{v}^*(t) = v^*$, for all $t \geq N$.

It is important to notice, that the work in [19] provides a state feedback law, which guarantees the robust, global, exponential stabilization of the freeway model (29) when the parameters of the freeway model are known.On the other hand, Corollary 3.3 provides a dynamic feedback law, under which the states of the freeway model (29) converge to the UEP, even when the vector of parameters is unknown. Below, we provide the proof of Corollary 3.3.

Proof of the Corollary 3.3: Let $N_1 \subseteq \Omega$ be a neighborhood of x^*, $N_2 \subseteq A$ be a neighborhood of y^*, and let $N_3 \subseteq \Re^{3n-1-m}$ be a neighborhood of θ^*. Since $\Omega = \prod_{i=1}^{n}(0, \mu_i)$, it follows from Assumption (H) and the fact that $\mu_i \in (0, \delta_i)$ for $i = 1, ..., n$ that $f_i(x_i) = r_i x_i$ for $i = 1, ..., n$. Definitions (34), (35), (36) in conjunction with (26) and the fact that $P_i \in [0,1)$ for $i = 1, ..., n-1$, $r_i \in (0,1)$ for $i = 1, ..., n-1$, imply that the following inequality holds for all $x \in \Omega$ and $d = (d_2, ..., d_n) \in D = [0,1]^{n-1}$:

$$\begin{aligned} &|h(d, \theta^*, x) - y^*| \leq |x - x^*| + |Q_{out} - Q_{out}^*| + |Q - Q^*| \\ &\leq |x - x^*| + \sum_{i=1}^{n-1} |P_i f_i(x_i) - P_i f_i(x_i^*)| + |f_n(x_n) - f_n(x_n^*)| \\ &\quad + \sum_{i=1}^{n-1} |(1-P_i) f_i(x_i) - (1-P_i) f_i(x_i^*)| \\ &\quad \leq |x - x^*| + \sum_{i=1}^{n} r_i |x_i - x_i^*| \leq \left(1 + \sum_{i=1}^{n} r_i\right)|x - x^*| \end{aligned} \tag{58}$$

Next, we notice that by virtue of (27) and the facts that $P_i \in [0,1)$ for $i = 1, ..., n-1$, $r_i \in (0,1)$ for $i = 1, ..., n$, $f_i(x_i^*) = v_i^* + (1-P_{i-1}) f_{i-1}(x_{i-1}^*)$ for $i = 2, ..., n$, $f_1(x_1^*) = v_1^*$, it follows that the following holds for all $x \in \Omega$, $d \in D$ and $u \in \Re^m$:

$$
\begin{aligned}
&|\bar{F}(d,\theta^*,x,u)-x^*| \le |x_1 - f_1(x_1) + v_1 - x_1^*| \\
&+\sum_{i=2}^{n}|x_i - f_i(x_i) + v_i + (1-P_{i-1})f_{i-1}(x_{i-1}) - x_i^*| \le \\
&\sum_{i=2}^{n}|x_i - f_i(x_i) + f_i(x_i^*) + (1-P_{i-1})f_{i-1}(x_{i-1}) \\
&-(1-P_{i-1})f_{i-1}(x_{i-1}^*) - x_i^*| \\
&+ m|u-u^*| + |x_1 - f_1(x_1) + v_1 - x_1^*| \le \\
&(1-r_1)|x_1 - x_1^*| + \sum_{i=2}^{n}(1-r_i)|x_i - x_i^*| + \\
&\sum_{i=2}^{n}(1-P_{i-1})r_{i-1}|x_{i-1}-x_{i-1}^*| + m|u-u^*| \le \\
&\left(n - \sum_{i=1}^{n} r_i + \sum_{i=2}^{n}(1-P_{i-1})r_{i-1}\right)|x-x^*| + m|u-u^*|
\end{aligned} \tag{59}
$$

where $u^* = (v_i^*; i \in R)$. Using (32) and (33), it is straightforward to show that there exists a constant $\tilde{L} > 0$ such that the following inequality holds for all $x, \hat{x}^* \in S$ and $\hat{v}^* \in \prod_{i=1}^{n}[0, v_{i,max}]$:

$$|u-u^*| \le \tilde{L}|x-x^*| + \tilde{L}|\hat{x}^* - x^*| + \tilde{L}|\hat{v}^* - v^*| \tag{60}$$

Using (30), (31) and the fact that the function $g : [0,1-\varepsilon]^{n-1} \times \prod_{i\notin R}[0, v_{i,max}] \times [\varepsilon, 1-\varepsilon]^n \to \prod_{i\in R}[b_i+\varepsilon, v_{i,max}]$ is a smooth function, it follows that the following inequality holds for all $\hat{\theta} \in N_3 \cap \Theta$:

$$|\hat{x}^* - x^*| + |\hat{v}^* - v^*| \le M|\hat{\theta} - \theta^*| \tag{61}$$

Finally, using definitions (42)-(48) in conjunction with the fact that $N_2 \subseteq A$, it follows that there exists a constant $\bar{L} > 0$ such that:

$$
\begin{aligned}
|\Psi(h(d,\theta^*,x),w) - \theta^*| \le &\bar{L}|x-x^*| + \bar{L}\sum_{i=1}^{p}|w_i - y^*| \\
&\textit{for all } x \in N_1 \cap S, d \in D, \hat{\theta} \in N_3 \cap \Theta, \\
w_i \in N_2 \cap Y (i = 1, ..., p) &\textit{ with } w = (w_1, ..., w_p).
\end{aligned} \tag{62}
$$

Since, we have already proved that assumptions (H1), (H2), (H3) hold for the closed-loop system (29), (36) with (50)-(56), (30)-(33), it follows from (58), (59), (60), (61) and (62) that all assumptions of Theorem 2.1 hold. Therefore, Corollary 3.3 is a direct application of Theorem 2.1 to the closed-loop system (29), (36) with (50)-(56), (30)-(33). The proof is complete. ◃

IV. An Illustrating Example

The following example illustrates the application of the results of the previous section to a specific freeway model. The selected values for the parameters have physical interpretation and the example demonstrates the efficiency of the proposed control scheme, even in the case of modeling errors.

Consider a freeway model of the form (14), (17), (18), (19) with $n = 5$ cells. The freeway stretch considered for the simulation test is 2.5 km long and has three lanes. Each cell is 0.5 km long and has an on-ramp and off-ramp. The first and the third on-ramp are assumed to be controllable, hence $R = \{1, 3\}$, and the vector of the controllable inflows is $u = (v_1, v_3)$. The inflows from the rest of the on-ramps are assumed to be unknown and therefore they will have to be estimated. Regarding the priority rules, we assume that $d_i(t) \equiv 0$ for the whole simulation horizon, which means that the on-ramp inflows have absolute priority over the internal inflows. The simulation time step is set to be $T = 15s$ and the cell capacities are $a_i = 170$ [veh], for $1 = 1, ..., 5$. Note that, since all flows and densities are measured in [veh], the cell length, the time step and the number of lanes do not appear explicitly, but they are only involved implicitly in the value of every variable and every constant (e.g. critical density, jam density, flow capacity, wave speed etc.) corresponding to density or flow.

The formulas of the demand functions are given by the following equations:

$$
f_i(z) = \begin{cases} (\frac{5}{11})z & z \in [0, 55] \\ 25 - (\frac{7}{115})(z-55) & z \in (55, 170] \end{cases} (i = 1, ..., 4),
$$

$$
f_5(z) = \begin{cases} (\frac{4}{11})z & z \in [0, 55] \\ 20 - (\frac{3}{115})(z-55) & z \in (55, 170] \end{cases}. \tag{63}
$$

Assumption (H) holds with $\delta_i = 55$ [veh], $a_i = 170$ [veh] for $i = 1, ..., 5$, $r_i = 5/11$, $f_i^{min} = 18$ for $i = 1, ..., 4$, $r_5 = 4/11$ and $f_5^{min} = 17$. Thus, every cell has the same critical and jam density which correspond to 36.7 [veh/km/lane] and 113.3 [veh/km/lane], respectively, in common traffic units with the above settings. Definitions (63) guarantee that the demand functions for $i = 1, ..., 4$ lead to 20% higher flow capacity ($f_i(\delta_i) = 25$ [veh] for $i = 1, ..., 4$, corresponding to 2000 [veh/h/lane]) than the flow capacity of the fifth cell ($f_5(\delta_5) = 20$ [veh], corresponding to 1600 [veh/h/lane]) and therefore the last cell is a strong bottleneck for the freeway (e.g. due to grade or curvature or tunnel or bridge etc.). Notice also, that the capacity drop phenomenon has been taken into account by considering a linearly decreasing demand function for over-critical densities $x_i \in (55, 170]$ (similar to the one proposed in [26]). Furthermore, the congestion wave speeds are $c_i = 0.22$ for $i = 1, ..., 5$ corresponding to 26.4 [km/h]. Finally, we suppose that the cell flow capacities q_i for $i = 1, ..., 5$ satisfy the inequalities $q_i \ge c_i a_i$ for $i = 1, ..., 5$ and therefore, they play no role in the model (14), (17), (18), (19).

Our goal is to globally exponentially stabilize the system at an UEP which is as close as possible to the critical density (due to the fact that the flow value at the critical density is largest). Therefore, we selected as the upper bound for the equilibrium densities and for each cell to be the $\mu_i = \delta_i - \epsilon$ $(i = 1, ..., 5)$, where $\epsilon = 10^{-4}$. The exit rates are set to be $P_1 = 0.04$, $P_2 = 0.15$, $P_3 = 0.08$, $P_4 = 0.1$ and we selected $v_{1,max} = 25$, $v_{2,max} = 1.3$, $v_{3,max} = 4$, $v_{4,max} = 2.3$ and $v_{5,max} = 2.8$, so that inequalities (25) hold. The uncontrollable inflows are $v_2^* = 1$, $v_4^* = 2$ and $v_5^* = 2.5$. Summarizing, the vector of the parameters θ^* consists of the exit rates $P = [P_1, P_2, P_3, P_4]'$, the unknown external uncontrollable inflows v_i^* $(i = 2, 4, 5)$ and the slopes $r = [r_1, ..., r_5]'$ of the demand functions. Hence, $\theta^* = [P_1, ..., P_4, v_2^*, v_4^*, v_5^*, r_1, ..., r_5]$.

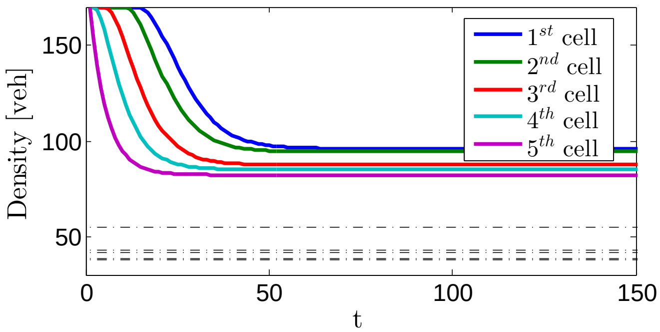

Fig. 3. Time evolution of the states of the open-loop system (dashed lines correspond to the UEP for constant inflows $v^* = [17.29316, 1, 4, 2, 2.5]$ [veh]) with fully congested initial condition $x_0 = (a_1, a_2, a_3, a_4, a_5)'$.

The function

$$g : [0, 1-\epsilon]^4 \times \prod_{i=2,4,5} [0, v_{i,max}] \times [\epsilon, 1-\epsilon]^5 \to \prod_{i=1,3} [b_i + \epsilon, v_{i,max}]$$

with $b_1 = b_3 = 0.2$ involved in (28) has been selected in such a way that the outflow from the last (fifth) cell is approximately maximized:

$$g(P, v_2^*, v_4^*, v_5^*, r) = (v_1^*, v_3^*) = (\hat{g}(P, v_2^*, v_4^*, v_5^*, r), 4) \quad (64)$$

where

$$\hat{g}(z) = \begin{cases} b_1 + \epsilon & z \in (-\infty, b_1] \\ 50^2 z^2 - 10^3 z + 100.2001 & z \in (b_1, b_1 + 2\epsilon] \\ z & z \in (b_1 + 2\epsilon, v_{1,max} - 1] \\ \frac{-1}{4} z^2 + 13z - 144 & z \in (v_{1,max} - 1, v_{1,max} + 1] \\ v_{1,max} & z \in (v_{1,max}, \infty) \end{cases} \quad (65)$$

where $z = \big(r_5 x_5^* - (v_5^* + (1-P_4)v_4^* + (1-P_3)(1-P_4)v_3^* + (1-P_2)(1-P_3)(1-P_4)v_2^*)\big) / \big((1-P_1)(1-P_2)(1-P_3)(1-P_4)\big)$ and $x_5^* = \mu_5 - 2\epsilon$.

The UEP is $x^* = [38.045, 38.723, 41.715, 42.778, 54.9997]$ for $v^* = [17.29316, 1, 4, 2, 2.5]$, $P = [0.04, 0.15, 0.08, 0.1]$ and $r = [5/11, 5/11, 5/11, 5/11, 4/11]$. The above UEP is not globally exponentially stable due to the existence of additional (congested) equilibria. This is shown in Fig.3, where the solution of the open-loop system, with constant inflows $v^* = [17.29316, 1, 4, 2, 2.5]$, is attracted by the congested equilibrium $[96.19, 94.6, 87.73, 85.22, 82.33]'$ leading to outflow, which is 0.72 [veh] lower than the capacity flow of the last cell. Therefore, if the objective is the operation of the freeway with largest outflow, then a control strategy will be needed.

We are in a position to guarantee global exponential attractivity of the UEP for the freeway model that was described above by using Corollary 3.3. Indeed, Corollary 3.3 guarantees that there exist constants $\sigma \in (0, 1]$, $b_1, b_3 > 0$ and $\tau > 0$ such that, the feedback law $K : \Theta \times S \to U$ defined by:

$$\begin{gathered} K_1(\hat{\theta}, x) = \\ \max\Big(b_1, \hat{v}_1^* - \tau^{-1}(\hat{v}_1^* - b_1) \sum_{i=1}^{5} \sigma^i \max(0, x_i - \hat{x}_i^*)\Big) \\ K_3(\hat{\theta}, x) = \\ \max\Big(b_3, \hat{v}_3^* - \tau^{-1}(\hat{v}_3^* - b_3) \sum_{i=1}^{5} \sigma^i \max(0, x_i - \hat{x}_i^*)\Big) \end{gathered} \quad (66)$$

$$(\hat{v}_1^*, \hat{v}_3^*) = g(\hat{P}, \hat{v}_i^*; i \notin R, \hat{r}), \quad (67)$$

for the closed-loop system (29), (34), (35), (36) with (50)-(56), (66), (67), (31) and (33), achieves global exponential attractivity of the UEP $x^* = [38.045, 38.723, 41.715, 42.778, 54.9997]$.

It is important here to note that the feedback law (66) aims to maximize the outflow from the fifth cell without assuming knowledge of the cell's capacity flow. The maximization is achieved by implicitly estimating the capacity flow of the fifth cell in real time, using the estimation of the slope of the demand function ($\hat{r}_5(t)$) and the (given) critical density of the same cell. Empirical traffic engineering investigations have shown that the capacity is stochastic, in the sense that traffic breakdown on different days may occur at different flow values. In contrast, the critical density, at which capacity flow occurs, is deemed more stable from day to day. This is the very practical reason why it is assumed in this work that the critical density is constant and known, while capacity flow is estimated in real time. Note that, this is in full accordance with simpler but proven (in many field installations) control laws like ALINEA [12], which also considers a given density set-point.

Selecting $b_1 = b_3 = 0.2$, we tested various values of the constants $\sigma \in (0, 1]$ and $\tau > 0$ by performing a simulation study with respect to many initial conditions. Low values for $\sigma \in (0, 1]$ require small values for $\tau > 0$ in order to guarantee global exponential stability for the closed-loop system. All the following tests of the proposed regulator were conducted with the same values $\sigma = 0.7$ and $\tau = 10$.

All the following simulation tests were conducted with the same initial conditions for the observer states $w_{1,i}(0) = 100$ [veh], $w_{2,i}(0) = 20$ [veh], $w_{3,i}(0) = 20$ [veh] for $i = 1, ..., 5$, $\hat{P}_i(0) = 0$ for $i = 1, ..., 4$, $\hat{v}_i^*(0) = 0$ for $i = 2, 4, 5$ and $\hat{r}_i(0) = 0.7$ for $i = 1, ..., 5$.

Fig.4 shows the evolution of the density of every cell and Fig.5(a) shows the evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the UEP, i.e., $|x(t) - x^*|$, for the closed-loop system with the proposed feedback regulator (50)-(56), (66), (67), (31) and (33) for three different initial conditions. The first condition corresponds to very low densities ($x_0 = (10, 15, 10, 15, 10)'$), the second initial condition corresponds to congested states with high deviations between each other ($x_0 = (70, 85, 65, 120, 100)'$), while the third initial condition corresponds to the state where the density of every cell has its maximum value, i.e. a_i ($i = 1, ..., 5$), which also corresponds to the initial condition

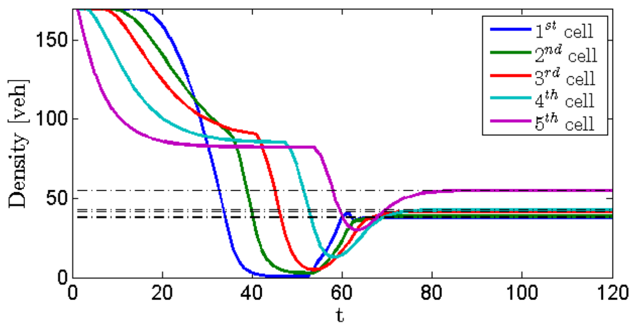

Fig. 4. Time evolution of the states of the closed-loop system (29), (36) with (50)-(56), (66), (67), (31) and (33) with fully congested initial condition $x_0 = (170, 170, 170, 170, 170)'$.

for Fig.4. Indeed, both Fig.4 and Fig.5 show that the proposed feedback stabilizer (50)-(56), (66), (67), (31) and (33) achieves dead-beat estimation of the vector θ^*, preserving the exponential convergence property for the closed-loop system.

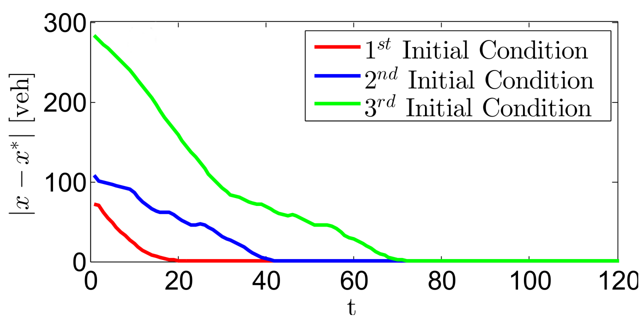

Fig. 5. The Euclidean norm of the deviation $x(t) - x^*$ of the state from the UEP, i.e., $|x(t) - x^*|$ for the closed-loop system (29), (36) with (50)-(56), (66), (67), (31) and (33) and three different initial conditions $x_0 = (10, 15, 10, 15, 10)'$ (red line), $x_0 = (70, 85, 64, 120, 100)'$ (blue line) and $x_0 = (170, 170, 170, 170, 170)'$ (green line).

We also tested the performance of the feedback law (50)-(56), (66), (67), (31) and (33) under the effect of periodic uncontrollable inflows with different frequencies and different amplitudes, given by:

$$v_2^* = 1 + 0.3\cos\left(\frac{3\pi t}{2}\right), v_4^* = 2 + 0.1\cos(\pi t) \text{ and } v_5^* = 2.5 + 0.2\cos\left(\frac{\pi t}{4}\right). \tag{68}$$

Fig.6(a) and Fig.6(b), depict the evolution of the Euclidean norm of the deviation $x(t) - x^*$ and the evolution of the weighted norm $\|\cdot\|_n$ of the deviation of the estimated parameters from the nominal parameters vector, defined by

$$\|\hat{\theta}(t) - \theta^*\|_n = \left|\left(\frac{1}{1-\epsilon}(\hat{P}(t) - P), \frac{\hat{v}_2^*(t) - v_2^*}{v_{2,max}}, \frac{\hat{v}_4^*(t) - v_4^*}{v_{4,max}}, \frac{\hat{v}_5^*(t) - v_5^*}{v_{5,max}}, \frac{1}{1-\epsilon}(\hat{r}(t) - r)\right)\right|, \tag{69}$$

with respect to the unknown time-varying uncontrollable inflows (68) and under the proposed feedback regulator (50)-(56), (66), (67), (31) and (33). The initial conditions were the same as in the previous case. Again, the proposed regulator achieved to lead the system to the equilibrium state by performing only small deviations for the estimated parameters. Fig.6 shows that the proposed feedback stabilizer (50)-(56), (66), (67), (31) and (33) achieves the exponential convergence property of the densities to the desired UEP.

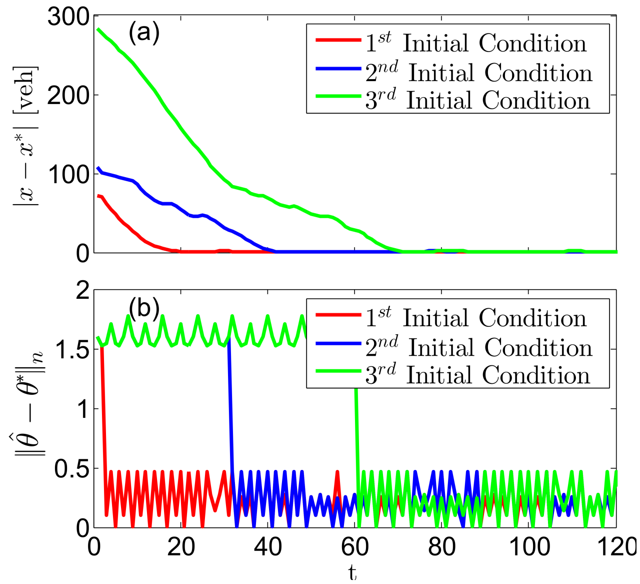

Fig. 6. (a) The Euclidean norm of the deviation $x(t) - x^*$ of the state vector from the UEP, (b) The weighted norm $\|\hat{\theta}(t) - \theta^*\|$ of the deviation of the estimated parameters from the nominal parameters vector, for periodic uncontrollable inflows given by (68) for the closed-loop system (29), (36) with (50)-(56), (66),(67),(31) and (33) and three different initial conditions $x_0 = (10, 15, 10, 15, 10)'$ (red line), $x_0 = (70, 85, 64, 120, 100)'$ (blue line) and $x_0 = (170, 170, 170, 170, 170)'$ (green line).

Furthermore, in order to illustrate the performance of the proposed feedback law under the presence of modeling errors, we considered the case where the demand functions do not satisfy assumption (H). More specifically, we considered the piecewise quadratic demand functions:

$$f_i(z) = \begin{cases} 0.7z - (\frac{0.49}{110})z^2 & , z \in [0, 55] \\ 25.025 - (\frac{7.025}{115})(z - 55) & , z \in (55, 170] \end{cases} \quad \text{for}(i = 1, ..., 4),$$

$$f_5(z) = \begin{cases} 0.56z - (\frac{0.392}{110})z^2 & , z \in [0, 55] \\ 20.02 - (\frac{3.02}{115})(z - 55) & , z \in (55, 170] \end{cases}. \tag{70}$$

In this case the UEP is $x^* = [30.77, 31.5, 34.85, 36.1, 54.9997]$. Fig.7, shows the evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the UEP and for the closed-loop system with the proposed feedback regulator (50)-(56), (66), (67), (31) and (33) and three different initial conditions. Again, Fig.7 shows that the proposed feedback stabilizer (50)-(56), (66), (67), (31) and (33) achieves the exponential convergence property of the densities to the desired UEP, even under the presence of modeling errors.

In the same vein, Fig. 8 shows the time evolution of the densities of every cell for the closed-loop system (29), (36)

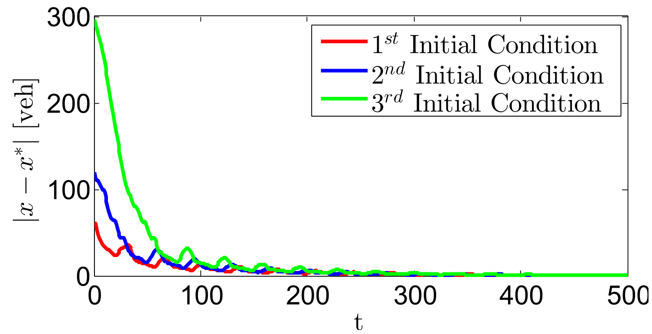

Fig. 7. (a) The Euclidean norm of the deviation $x(t) - x^*$ of the state from the uncongested equilibrium, i.e., $|x(t) - x^*|$, for the closed-loop system (29), (36) with (50)-(56), (66), (67), (31) and (33), piecewise quadratic demand functions and three different initial conditions $x_0 = (10, 15, 10, 15, 10)'$ (red line), $x_0 = (70, 85, 64, 120, 100)'$ (blue line) and $x_0 = (170, 170, 170, 170, 170)'$ (green line).

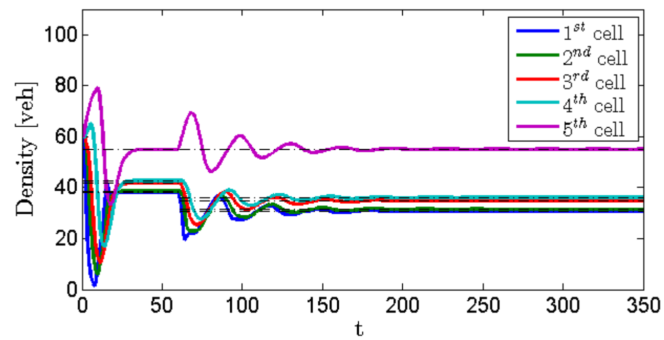

Fig. 8. Time-evolution of the states of the closed-loop system (29), (36) with (50)-(56), (66), (67), (31) and (33), for initial conditions $x_0 = (60, 60, 60, 60, 60)'$, and for piecewise linear demand functions for $t < 60$ and piecewise quadratic demand functions for $t \geq 60$.

with (50)-(56), (66), (67), (31) and (33) with initial condition $x_0 = (60, 60, 60, 60, 60)'$ and under the presence of the same modeling errors. More specifically, in this figure the demand functions are given by (63), which satisfy Assumption (H), for $t < 60$, while after that time modeling errors appear. This means that for $t \geq 60$ the demand functions are given by (70), which do not satisfy Assumption (H). Fig. 8 shows that the exponential convergence property to the desired uncongested equilibrium point is preserved even when modeling errors appear after an initial transient period.

V. Concluding Remarks

Novel results for adaptive control schemes for uncertain discrete-time systems, which guarantee robust, global, exponential convergence to the desired equilibrium point of the system, were provided in the present work. The proposed control scheme consists of a nominal feedback law, which achieves robust, global, exponential stability properties when the vector of the parameters is known, in conjunction with a nonlinear, dead-beat observer. The proposed adaptive scheme did not require the knowledge of a Lyapunov function for the closed-loop system under the action of the nominal feedback stabilizer and is directly applicable to highly nonlinear, uncertain discrete-time systems with unknown constant parameters. The applicability of the obtained results to real control problems was demonstrated by the rigorous application of the proposed adaptive control scheme to uncertain, freeway models. Simulation results showed the efficacy of the proposed adaptive control scheme even under the presence of modeling errors and/or time-varying parameters.

It is well-known that dead-beat observers may present limited robustness properties with respect to measurement errors. Therefore, additional work is needed for the robustification of the proposed dead-beat identifier or its replacement by different type of identifiers (e.g., Luenberger-type observers, Lyapunov-based identifiers). Alternative methods for the robustification are to be tested: least-squares methods, filtering approaches, or dead-zone techniques.

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Appendix

Proof of Lemma 2.2: By virtue of assumption (iii), there exists $\delta > 0$ such that the inequality $|F(d,z) - z^*| \leq L|z - z^*|$ holds for all $d \in D$ and $z \in A := \{y \in X : |y - z^*| < \delta\}$. Since $F : D \times X \to X$ is a bounded mapping, there exists a constant $R > 0$ which satisfies

$$\sup\{|F(d,z)| : z \in X, d \in D\} \leq R. \tag{71}$$

It follows from (71) and the triangle inequality that the following inequality holds:

$$\begin{aligned} &\sup\left\{\frac{|F(d,z) - z^*|}{|z - z^*|} : d \in D, z \in X \setminus A\right\} \leq \\ &\delta^{-1}\sup\{|F(d,z) - z^*| : z \in X, d \in D\} \leq \\ &\delta^{-1}(R + |z^*|) \end{aligned} \tag{72}$$

Combining (72) and the fact that $|F(d,z) - z^*| \leq L|z - z^*|$ holds for all $d \in D$ and for all $z \in A$, we get:

$$|F(d,z) - z^*| \leq \max\left(L, \delta^{-1}(R + |z^*|)\right)|z - z^*| \tag{73}$$

for all $(d,z) \in D \times X$.

Let $z_0 \in X$ be an arbitrary vector and let $\{d(t) \in D\}_{t=0}^{\infty}$ be an arbitrary sequence. Consider the solution $z(t)$ of $z^+ = F(d,z)$ with initial condition $z(0) = z_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$. By virtue of assumption (ii), there exists $i(N) \in \{0, 1, ..., N\}$ with $z(N - i(N)) \in \Omega$. By virtue of assumption (i), we get:

$$|z(t) - z^*| \leq M|z(k) - z^*|\exp(-\sigma(t - k)), \tag{74}$$

for all $t \geq k$, where $k = N - i(N)$.

Notice that $k \in \{0, 1, ..., N\}$. Using induction and (73), we get

$$|z(t) - z^*| \leq \tilde{L}^t|z_0 - z^*|, \text{for all } t \geq 0, \tag{75}$$

where $\tilde{L} := \max\left(L, \delta^{-1}(R + |z^*|)\right) \geq 1$. Combining (74), (75) and the fact that $k \in \{0, 1, ..., N\}$, we obtain:

$$|z(t) - z^*| \leq M\tilde{L}^N \exp(\sigma N)|z_0 - z^*| \exp(-\sigma t) \quad (76)$$

for all $t \geq 0$. Noticing that assumption (iii) guarantees that $z^* = F(d, z^*)$, we conclude that estimate (76) implies that $z^* \in X$ is RGES for the uncertain system (1). The proof is complete. $\triangleleft$

Proof of Proposition 3.2: Assume that $b_i > 0$ ($i \in R$) and $v_{i,max}$ ($i \notin R$) are sufficiently small so that

$$\sum_{i\in R}(n+1-i)b_i+\sum_{i\notin R}(n+1-i)v_{i,max} < C \min_{i=1,...,n}\left((n+1-i)\mu_i\right). \quad (77)$$

Since $\tau \leq \epsilon^2\sigma^n \min_{i\in R}\left((v_{i,max} - b_i)^{-1}\right)$ and $\hat{v}_i^* \in [b_i + \epsilon, v_{i,max}]$ for $i \in R$, it follows that

$$\tau^{-1}(\hat{v}_i^* - b_i) \geq \epsilon^{-1}(v_{i,max} - b_i)\sigma^{-n} \text{ for all } i \in R. \quad (78)$$

Let $m \geq 1$ be an integer that satisfies:

$$m \geq 2 + [\Lambda] \quad (79)$$

where $\Lambda = \left(\ln(\min_{i=1,...,n}((n+1-i)\mu_i) - C^{-1}\kappa) - \ln(\sum_{i=1}^{n}(n+1-i)a_i)\right)/\ln(1-C)$. Next, we show the following claim.

Claim: if $x \notin \Omega$ then for every $(\hat{\theta}, d) \in \Theta \times [0,1]^{n-1}$ it holds that:

$$\sum_{i=1}^{n} I_i(x^+) \leq (1-C)\sum_{i=1}^{n} I_i(x) + \kappa \quad (80)$$

where $C \in (0,1)$ is the constant involved in (49), $\kappa := \sum_{i\in R}(n+1-i)b_i + \sum_{i\notin R}(n+1-i)v_{i,max}$ and x^+ is given by (29) with $u = K(\hat{\theta}, x)$.

Proof of Claim: if $x \notin \Omega = \prod_{i=1}^{n}(0, \mu_i)$, then there exists $i \in \{1, ..., n\}$ such that $x_i \geq \mu_i$. Since $\hat{x}^* = (\hat{x}_1^*, ..., \hat{x}_n^*) \in \prod_{i=1}^{n}[0, \mu_i - \varepsilon]$ (recall (31)), it follows from (33) and the fact that $\sigma \in (0,1]$ that $\Xi(\hat{\theta}, x) \geq \sigma^n(x_i - \hat{x}_i^*) \geq \varepsilon\sigma^n$. Since (78) holds, it follows from (32) that $v_i = b_i$ for all $i \in R$. Inequality (80) is a consequence of (49) and the fact that $v_i^* \in [0, v_{i,max}]$ for all $i \notin R$. The proof of the claim is complete. We show next, by means of a contradiction, that for every sequence $\{d(t), \hat{\theta}(t) \in D \times \Theta\}_{t=0}^{\infty}$ and for every $x_0 \in S$, the solution $x(t)$ of (29), (36) with $u = K(\hat{\theta}, y)$, initial condition $x(0) = x_0$ corresponding to inputs $\{d(t), \hat{\theta}(t) \in D \times \Theta\}_{t=0}^{\infty}$ satisfies $y(t-1-i(t)) \in A$ for some $i(t) \in \{0, 1, ..., m\}$ and for all $t \geq m+1$. Suppose that, on the contrary, there exists a sequence $\{d(t), \hat{\theta}(t) \in D \times \Theta\}_{t=0}^{\infty}$, a vector $x_0 \in S$ and an integer $t \geq m+1$, such that the solution $x(t)$ of (29), (36) with $u = K(\hat{\theta}, y)$, initial condition $x(0) = x_0$ corresponding to inputs $\{d(t), \hat{\theta}(t) \in D \times \Theta\}_{t=0}^{\infty}$ satisfies $y(t-1-i(t)) \notin A$ for all $i(t) \in \{0, 1, ..., m\}$. By virtue of (41), this implies that $x(t-1-i(t)) \notin \Omega$ for all $i(t) \in \{0, 1, ..., m\}$ (notice that (26), (27), (34), (35), (36) and (41) guarantee that $x \in \Omega$ implies that $y \in A$). It follows from the Claim, that:

$$\sum_{i=1}^{n} I_i(x(l+1)) \leq (1-C)\sum_{i=1}^{n} I_i(x(l)) + \kappa \quad \textit{for } l = t-1-m, ..., t-1 \quad (81)$$

Using (81) repeatedly, we get:

$$\sum_{i=1}^{n} I_i(x(t-1)) \leq (1-C)^m\sum_{i=1}^{n} I_i(x(t-1-m)) + \kappa\frac{1-(1-C)^m}{C} \quad (82)$$

Using the definition $I_j(x) := \sum_{i=1}^{j} x_i$ for $j = 1, ..., n$ and the fact that $x \in S = \prod_{i=1}^{n}(0, a_i]$, we get from (82):

$$(n+1-j)x_j(t-1) \leq (1-C)^m\sum_{i=1}^{n}(n+1-i)a_i + C^{-1}\kappa \quad \textit{for all } j = 1, ..., n \quad (83)$$

Using (83), (76) and (79), we get:

$$(n+1-j)x_j(t-1) \leq \min_{i=1,...,n}((n+1-i)\mu_i)$$

for all $j = 1, ..., n$ which implies that $x(t-1) \in \Omega = \prod_{i=1}^{n}(0, \mu_i)$, a contradiction. The proof is complete. $\triangleleft$

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