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Greedy Algorithms for Reconstruction of High-Dimensional Sparse Vectors

by

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ABSTRACT

Reconstruction of signals from measured data is often encountered in various fields of science. However, the dimension of the target signal is often much larger than the number of the collected measurements. In these cases, signal reconstruction is practically impossible in general. Luckily, by assuming that the signal we wish to reconstruct has certain structure, the reconstruction becomes feasible.

In Compressed Sensing, we deal with the system $\mathbf{y} = \mathbf{A}\mathbf{x}$, where the so-called measurement matrix $\mathbf{A}$ has dimensions $(m \times n)$, with $m \ll n$. In this area, the notion of sparsity is used as a constraint on the target signal $\mathbf{x}$. In this thesis, we concentrate on greedy algorithms, studied extensively in the literature, and the conditions that guarantee successful reconstruction. First, we provide a theoretical background of Compressed Sensing and, afterwards, we proceed with the presentation and analysis of greedy algorithms, such as Orthogonal Matching Pursuit (OMP) and Compressive Sampling Matching Pursuit (CoSaMP). We complement our presentation with numerical experiments, using as performance metric the relative signal reconstruction error.

Then, we investigate the extension of sparse vector reconstruction in non-linear scenarios. For this purpose, we consider a greedy algorithm, the Gradient Support Pursuit (GraSP), which is an extension of CoSaMP. We present the conditions that must be satisfied in this framework for successful reconstruction, and compare the performance of GraSP to LASSO, of the GLMnet package, for the logistic model.

Finally, we propose a method for non-linear scenarios inspired by GraSP and OMP, test it for the logistic model, and compare the results to those of GraSP and GLMnet.

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Chapter 1

Introduction

In this thesis, we consider the problem of reconstructing high dimensional signals from measured data. This type of problem is encountered in many applications of statistics, biomedical, and signal processing. Unfortunately, in the majority of these applications, acquiring the number of measurements, indicated by the Shannon - Nyquist sampling theorem, may be too costly, infeasible (due to hardware limitations) or even dangerous. For example, in Magnetic Resonance Imaging (MRI), the measurement time could be so high that necessitates extended exposure of patients to radiation. These limitations can be overcome by assuming a structure on the underlying high dimensional signal. This structure is sparsity.

In Chapter 2, we consider the concept of Compressed Sensing, where we wish to reconstruct a sparse signal, having its measurements acquired in a linear manner. Through extensive work, a number of efficient algorithms are available for this setting. In this chapter, we will refer to the properties that must be satisfied to guarantee sparse signal recovery. Furthermore, we will mention the types of recovery emerged in Compressed Sensing as well as the limitations imposed on the number of measurements.

The main results concerning greedy algorithms are presented in Chapter 3. At first, we refer to the core ideas behind each algorithm and then we proceed with their theoretical analysis (also in Appendix A). Then, we present the sufficient conditions for sparse recovery, emerged from the analysis presented in Chapter 3. Finally, based on these results, we test the performance of the aforementioned greedy algorithms in terms of relative signal estimation error and illustrate the results.

While the linear setting is widely encountered in numerous areas of science, there are many applications that deal with high-dimensional signals that are measured in a non-linear manner. For example, in binary classification, the relation between the measurements and the underlying target signal is determined by a non-linear function. Consequently, it is essential to consider this topic as well. In Chapter 4, we present GraSP, a greedy algorithm that deals with this task. The GraSP algorithm shares ideas with the methods used in Compressed sensing area, thus, the transition becomes easier. In addition, we refer to the properties that must be met in this setting as well as to the main results. The proofs of the main theorems can be found in Appendix A. We conclude this chapter with simulations of GraSP and GLMnet for the logistic model.

In Chapter 5, we investigate a slightly different approach for the non-linear setting, by modifying GraSP based on ideas of OMP, a greedy algorithm presented in Chapter 3. We test the performance of this method in terms of relative signal estimation error and, finally, we conclude with some ideas for future work concerning the extension to other Generalized Linear Models (GLMs).

Proofs of the main theorems mentioned in Chapter 3 and 4 can be found in Appendix A. Moreover, Appendix B contains basic results, necessary for the analysis of the presented algorithms.

List of Symbols

$\mathcal{C}(\mathbf{A})$	Column space of matrix $\mathbf{A}$
$\mathcal{N}(\mathbf{A})$	Null space of matrix $\mathbf{A}$
$\overline{\mathcal{Z}}$	Complement of a set
$\mathbf{A}$	Bold upper case letters denote matrices
$\mathbf{A}^\dagger$	pseudoinverse of matrix $\mathbf{A}$
$\mathbf{A}^H$	conjugate transpose or Hermitian transpose of matrix $\mathbf{A}$
$\mathbf{A}^T$	transpose of matrix $\mathbf{A}$
$\mathbf{A}_\mathcal{S}$	restriction of matrix $\mathbf{A}_\mathcal{S}$ to columns indicated by the set $\mathcal{S}$
$\mathbf{Id}$	Identity matrix
$\mathbf{P}_S$	Restriction of the identity matrix to the columns indicated by the set set $\mathcal{S}$
$\mathbf{x}$	Bold lower case letters denote vectors
$\mathbf{x}_\mathcal{S}$	restriction of vector $\mathbf{x}_\mathcal{S}$ to the elements indicated by the indices in the set $\mathcal{S}$

Chapter 2

Compressed Sensing

2.1 Formulation of the problem

Reconstruction of signals from their measurements is a common task in various fields of science such statistics, machine learning, signal and, image processing. In cases where the measurements are obtained linearly, the problem can be formulated as

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{2.1}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the measured data vector associated with the vector (or signal) of interest $\mathbf{x} \in \mathbb{R}^n$. The matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is called the measurement or sensing matrix. In many applications in the aforementioned areas, we have $m < n$ [1], a fact that renders problem (2.1) underdetermined. Fortunately, by imposing some constraints to the vector $\mathbf{x}$, the reconstruction of $\mathbf{x}$ becomes possible. *Compressed Sensing* exploits the notion of sparsity for this cause. A vector $\mathbf{x}$ is called sparse if most of its elements are zero. Also, the support of a vector $\mathbf{x}$, denoted supp($\mathbf{x}$)), is the index set of its nonzero elements.

Definition 1. [1, p. 41] *A vector* $\mathbf{x}$ *is called s-sparse if at most s of its entries are nonzero, i.e. if*

$$\|\mathbf{x}\|_0 := \mathrm{card}(\mathrm{supp}(\mathbf{x})) \leq s. \tag{2.2}$$

Note that the support in most cases is not known. Otherwise, the problem of signal recovery from linear measurements would be trivial.

In compressed sensing, significantly large amount of work has been published concerning the adequacy of matrix $\mathbf{A}$ and the efficiency of algorithms. Actually, the design of the measurement matrix $\mathbf{A}$ is a tricky task with a huge impact on the success of reconstruction. Meanwhile, a variety of algorithms have been introduced in this area. It seems natural, at first, to perform ℓ_0-minimization, namely

$$(P_0): \qquad \min_{\mathbf{z}\in\mathbb{R}^{\mathbf{n}}} \|\mathbf{z}\|_0, \quad \text{s.t.,} \quad \mathbf{y} = \mathbf{A}\mathbf{z}. \tag{2.3}$$

[1] A possible measurement matrix encountered in statistical inference applications could hold measurements acquired by sensors during a period of time. It is obvious that the number of sensors would be much smaller than the number of measurements for all the days of the experiment

However, this problem is NP-hard. A tractable method, that is a convex relaxation of (P_0), is basis pursuit or ℓ_1-minimization (P_1)

$$(P_1): \qquad \min_{\mathbf{z}\in\mathbb{R}^{\mathbf{n}}} \|\mathbf{z}\|_1, \quad \text{s.t.,} \quad \mathbf{y} = \mathbf{A}\mathbf{z}. \tag{2.4}$$

An alternative approach to optimization are the so-called greedy algorithms. These strategies find the support of the solution iteratively and are sometimes preferable due to their simplicity. In this work, we will reflect on greedy methods for sparse recovery such as Orthogonal Matching Pursuit (OMP) and Compressive Sampling Matching Pursuit (CoSaMP). Various applications of compressed sensing can be found in [1, Chapter 1], [2, Chapter 1].

2.1.1 Metrics on measurement matrix

Compressed sensing relies on the appropriate design of the measurement matrix $\mathbf{A}$. In fact, the algorithms demand strong properties on matrix $\mathbf{A}$ in order to deduce sufficient conditions for sparse recovery. Specifically, we wish to recover all sparse vectors, so, every set of $2s$ columns of $\mathbf{A}$ must be linearly independent [1, p. 49]. There is an extensive literature on the conditions that the measurement matrix must satisfy. In the sequel, we shall introduce several metrics which will be used for the characterization of the measurement matrix $\mathbf{A}$.

The *mutual coherence* is a simple metric to assess the quality of matrix $\mathbf{A}$. Its computation is an easy task, therefore it is widely used in the analysis of algorithms of sparse recovery. The concept of mutual coherence was first introduced by David Donoho and Michael Elad [3]. The definition of mutual coherence follows.

Definition 2. [1, p.110] *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *be a matrix with* ℓ_2*-normalized columns* $\mathbf{a}_1,\ldots,\mathbf{a}_n$*, i.e.* $\|\mathbf{a}_i\|_2 = 1$ *for all* $i \in [n]$*. The mutual coherence of the matrix is defined as*

$$\mu(\mathbf{A}) \coloneqq \max_{1\le i\ne j\le n} |\langle \mathbf{a}_i \mathbf{a}_j\rangle|. \tag{2.5}$$

We notice that mutual coherence $\mu(\mathbf{A})$ is a measure of the correlation between the columns of $\mathbf{A}$. An orthogonal (or unitary) matrix has pairwise orthogonal columns, thus $\mu(\mathbf{A}) = 0$. Moreover, it is obvious that mutual coherence is strictly positive for general matrices. In compressed sensing area, where $m \ll n$, in order to design an appropriate measurement matrix $\mathbf{A}$, we have to keep $\mu(\mathbf{A})$ small. It is proven than mutual coherence for general matrices is bounded from below by

$$\mu(\mathbf{A}) \ge \sqrt{\frac{n-m}{m(n-1)}}. \tag{2.6}$$

The lower bound (2.6) is known as Welch bound [4], and it is achieved by special structure matrices called Grassmannian Frames [5, p. 29]. In cases where $m \ll n$, the lower bound for mutual coherence is approximately $\mu(\mathbf{A}) \geq 1/\sqrt{m}$.

An other metric concerning the matrix $\mathbf{A}$ is *spark* [3]. We proceed with the definition of spark.

Definition 3. [5, p. 23] *The spark of a matrix* $\mathbf{A}$ *is the smallest number of columns from* $\mathbf{A}$ *that are linearly-dependent.*

The definition of spark resembles to the definition of rank, however, spark is NP-hard to compute, as it demands combinatorial evaluation over all possible subsets of columns from $\mathbf{A}$. In some cases though, evaluation of spark becomes easier. For instance, if the entries of $\mathbf{A}$ are independent and identically distributed random variables, then, with probability one, spark $= m+1$, where m is the rank of matrix $\mathbf{A}$ [5, pp. 23-24]. The spark gives us an intuition about the vectors in the null-space of $\mathbf{A}$, $\mathcal{N}(\mathbf{A})$ [6, p. 17]. Vectors in $\mathcal{N}(\mathbf{A})$ must have at least spark nonzero entries. Also, large values of spark are suitable for good measurement matrices [5, pp. 23-24]. Combining these two findings, we conclude that the vectors in $\mathcal{N}(\mathbf{A})$ of a matrix $\mathbf{A}$ with large spark, are far from being sparse. A relation between spark and mutual coherence is provided by the next lemma.

Lemma 1. *For any matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$*, it holds true that*

$$\operatorname{spark}(\mathbf{A}) \geq 1 + \frac{1}{\mu(\mathbf{A})}. \tag{2.7}$$

Mutual coherence is a very useful quantity and is used frequently in the analysis of algorithms of sparse recovery. However, small mutual coherence means small sparsity level, a result that emerges from the analysis of various algorithms [1, p. 125]. To counter this problem, the *Restricted Isometry Property* (RIP) was introduced by Emmanuel Candes and Terence Tao [7].

Definition 4. *The s-th restricted isometry constant* $\delta_s = \delta_s(\mathbf{A})$ *of matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *is the smallest* $\delta \geq 0$ *such that*

$$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1+\delta)\|\mathbf{x}\|_2^2, \tag{2.8}$$

for all s-sparse vectors $\mathbf{x} \in \mathbb{R}^n$*.*

The relation (2.8) states that, for sufficiently small δ_s, every set of s columns of $\mathbf{A}$ approximates an orthonormal system. An implication of the relation (2.8) is that if a matrix $\mathbf{A}$ satisfies the RIP of order $2s$, then $\mathbf{A}$ approximately preserves the distance between every pair of s-sparse vectors

[6, p. 19]. It can be proved that it ensures stability over the reconstruction error when the signal is not exactly sparse or when measurements are contaminated with noise [6, p. 20].

The definition of the RIP is equivalent to:

$$\delta_s = \max_{S \subset [N], \mathrm{card}(S)=s} \|\mathbf{A}_S^H \mathbf{A}_S - \mathbf{Id}\|_{2\to 2}, \tag{2.9}$$

where $\mathbf{A}_S$ is a submatrix of $\mathbf{A}$ which is restricted to the columns indicated by $\mathcal{S}$ and $\mathbf{Id}$ is the identity matrix. $\|\mathbf{X}\|_{2\to 2}$ is the spectral norm. By (2.9), we can conclude that each submatrix $\mathbf{A}_S$ has its singular values in the interval $\left[\sqrt{1-\delta_s}, \sqrt{1+\delta_s}\right]$ or, equivalently, the eigenvalues of $\mathbf{A}_S^H \mathbf{A}_S$ are in the interval $[1-\delta_s, 1+\delta_s]$.

Finally, all submatrices $\mathbf{A}_{\mathcal{S}}$ are well conditioned and injective when $\delta_s < 1$ (in the opposite case, the null space of $\mathbf{A}_{\mathcal{S}}$ would be nontrivial). More formally, due to the fact that we want to recover all s-sparse vectors from their measurements as well as the fact that distinct s-sparse vectors have different measurement vectors, the definition $\delta_{2s} < 1$ is more appropriate.

In the sequel, we present some useful results that are essential for the comprehension of RIP. At first, we note that the sequence of restricted isometry constants behaves as follows [1, p. 134]

$$\delta_1 \leq \delta_2 \leq .... \leq \delta_s \leq \delta_{s+1} \leq ... \leq \delta_n. \tag{2.10}$$

In addition, the following theorem states the relationship between RIP and mutual coherence.

Proposition 1. [1, p. 134] *If the matrix* $\mathbf{A}$ *has* ℓ_2*-normalized columns* $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ *,...* $\mathbf{a}_n$ *i.e* $\|\mathbf{a}_j\|_2 = 1$ *for all* $j \in [n]$*, then*

$$\delta_1 = 0, \tag{2.11}$$

$$\delta_2 = \mu(\mathbf{A}), \tag{2.12}$$

$$\delta_s \leq (s-1)\mu(\mathbf{A}), \quad s \geq 2. \tag{2.13}$$

The following theorem yields a lower bound for RIP constant δ_s.

Theorem 1. [1, p. 139] *For a matrix* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $2 \leq s \leq n$*, one has*

$$m \geq c\,\frac{s}{\delta_s^2}, \tag{2.14}$$

provided $n \geq Cm$ *and* $\delta_s \leq \delta_*$*, where the constants* C*,* c*,* δ_* *depend only on each other and* $c \ll 1$*.*

For the special case of matrices with mutual coherence of optimal order, $\mu(\mathbf{A}) \leq c/\sqrt{m}$, and, by (2.13), we deduce that

$$\delta_s \leq (s-1)\mu(\mathbf{A}) \leq cs/\sqrt{m}. \tag{2.15}$$

Thus,

$$m \geq c' s^2, \tag{2.16}$$

in virtue of (2.15), with the quadratic scaling, keeps δ_s small.

Using Theorem 1, it can be shown that

$$\delta_s \geq \sqrt{cs/m}. \tag{2.17}$$

Now, the result deduced from Theorem 1 yields

$$m \geq c's. \tag{2.18}$$

The bound (2.18) has not been achieved up to this day in the deterministic framework. For random matrices, drawn from a subgaussian distribution, the restricted isometry constants $\delta_s \leq \delta$ are satisfied with high probability, by requiring [1, p. 271]

$$m \geq C\delta^{-2} s \ln(eN/s). \tag{2.19}$$

2.1.2 Choosing measurement matrix

Based on the previous properties, a suitable measurement matrix should have small mutual coherence, large spark and, finally, small RIP constant. In [1, Chapter 5], various constructions of matrices with small mutual coherence are presented. Using the mutual coherence, sufficient conditions are obtained for the recovery of all s-sparse vectors from $\mathbf{y} = \mathbf{Ax}$. Precisely, for ℓ_1-minimization and OMP, we require that $(2s-1)\mu(\mathbf{A}) < 1$. This yields that $m \geq Cs^2$ measurements are necessary for recovery of all s-sparse vectors [1, p. 125]. Moreover, the available estimation methods for RIP constants of explicit matrices are based on (2.9) and Gershgorin disk theorem. However, these tools, combined with (2.13), also lead to a quadratic scale of measurements. In the majority of compressed sensing applications, m should be kept small enough, thus, finding matrices that surpass this quadratic bottleneck becomes essential.

Random matrices seem to overcome this issue. It has been shown that random matrices satisfy RIP, with high probability, when m is of order $s\ln(n/s)$ [1, Chapter 9]. This result concerns matrices with entries drawn independently from a subgaussian distribution. Gaussian and Bernoulli random variables are included in this scheme. In [1, Chapter 9], one can find interesting results concerning ℓ_1-minimization.

The same type of guarantees are deduced for greedy algorithms that ensure successful recovery under RIP such as OMP and CoSaMP as we will see in Chapter 3. Especially, OMP, under RIP-based analysis, recovers exact sparse vectors with $m = \mathcal{O}(s \ln(n))$. For matrices with random entries that satisfy certain type of concentration inequalities, it can be proved that such recovery guarantees apply [1, Chapter 9].

2.1.3 Uniform and Nonuniform Recovery

In compressed sensing, one can encounter two different kinds of recovery results. Uniform recovery is the recovery, with high probability, of *all* s-sparse vector using a *single* random measurement matrix. On the other hand, nonuniform recovery concerns a single s-sparse vector and its recovery, with high probability, using a random measurement matrix. To this day, numerous papers have been investigating this issue, especially for greedy algorithms.

Briefly, for ℓ_1 - minimization, uniform and nonuniform recovery results are available for different kinds of random matrices with number of measurements of the optimal order. For example, for a Gaussian measurement matrix and for large n, having $m > 2s \ln(n/s)$, the recovery of a fixed s-sparse vector will succeed with high probability. More on this issue in [1, p. 281].

For greedy algorithms, things are more intriguing. It is showed that, for some matrices (either deterministic or random), OMP will fail to recover all s-sparse vectors [1, p. 156], [8]. In [9], an analysis of OMP exploiting RIP property is used, stating uniform recovery. However, the number of measurements scale quadratically in sparsity level. Uniform guarantees with $m = \mathcal{O}(s \ln(n))$ for OMP are proved in [10], although a number of iterations proportional to the sparsity level is required. In [11], new guarantees for uniform recovery are proposed. CoSaMP was the first greedy algorithm with similar guarantees with ℓ_1-minimization. The respective theorems of recovery are presented in Chapter 3.

Chapter 3

Algorithms for sparse recovery

3.1 Greedy Algorithms

In this chapter, we present an overview of the greedy algorithms used for sparse recovery reconstruction. An algorithm is characterized as greedy if, in each step, it makes a selection among predictors (columns of the measurement matrix), usually according to an optimization criterion. Greedy algorithms are well known to signal processing and statistics communities. We consider the most representative methods, broadly used in the compressed sensing area.

3.1.1 Matching Pursuit

Matching Pursuit (MP) is a simple greedy algorithm widely used in signal processing since the '60s. At each iteration, one predictor (column of $\mathbf{A}$) is chosen and the associated coefficient is updated. The full algorithm is listed in Algorithm 1. In Approximation theory, MP is often called Pure Greedy Algorithm (PGA).

Algorithm 1: Matching Pursuit Algorithm

Input: measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^m$

1 initialization $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \mathbf{0}$

2 **for** *iteration* $i = 1$; $i = i + 1$ *till the halting criterion is met* **do**

3     $\mathbf{g}^i = \mathbf{A}^T \mathbf{r}^{i-1}$

4     $j^i = \underset{j}{\operatorname{argmax}} |\mathbf{g}^i_j| / \|\mathbf{a}_j\|_2$

5     $x^i_{j^i} = x^{i-1}_{j^i} + \mathbf{g}^i_{j^i} / \|\mathbf{a}_{j^i}\|_2^2$

6     $\mathbf{r}^i = \mathbf{r}^{i-1} - \mathbf{g}^i_{j^i} \mathbf{a}_{j^i} / \|\mathbf{a}_{j^i}\|_2^2$

Output: $\mathbf{r}^i$, $\mathbf{x}^i$

At i-th iteration, the selected predictor is denoted by $\mathbf{a}_{j^i}$. The coefficient update in step 5 minimizes the cost function $\|\mathbf{y} - \mathbf{A}\mathbf{x}^i\|_2^2$ with respect to the coefficient associated with the selected predictor.

3.1.2 Orthogonal Matching Pursuit

An interesting variation of MP is the Orthogonal Matching Pursuit (OMP), known also as Orthogonal Greedy Algorithm. The main difference between these two strategies is the way the coefficients are updated at each iteration. In OMP, the elements of $\mathbf{x}$ at each iteration are updated by projecting $\mathbf{y}$ onto the space spanned by the predictors already in the support. Thus, it is obvious that OMP minimizes the $\|\mathbf{y} - \mathbf{A}\mathbf{x}^i\|_2^2$ over the current support $\mathcal{T}^i$. As a result, the residual becomes orthogonal to the active predictors. Consequently, a variable already in the support set can not be selected again. The steps of the OMP algorithm are summarised in Algorithm 2.

Algorithm 2: Orthogonal Matching Pursuit Algorithm

Input: measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^m$

initialization $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \mathbf{0}$

for *iteration* $i = 1$; $i = i + 1$ *till the halting criterion is met* **do**

- $\mathbf{g}^i = \mathbf{A}^T \mathbf{r}^{i-1}$
- $j^i = \underset{j}{\operatorname{argmax}} |\mathbf{g}_j^i| / \|\mathbf{a}_j\|_2$
- $\mathcal{T}^i = \mathcal{T}^{i-1} \cup j^i$
- $\mathbf{x}_{\mathcal{T}^i}^i = \mathbf{A}^\dagger{}_{\mathcal{T}^i} \mathbf{y}, \quad \mathbf{x}_{\overline{\mathcal{T}^i}}^i = \mathbf{0}$
- $\mathbf{r}^i = \mathbf{y} - \mathbf{A}\mathbf{x}^i$

Output: $\mathbf{r}^i$, $\mathbf{x}^i$

A stopping criterion for OMP could be $\|\mathbf{r}^i\|_2 \leq \epsilon$, for a small positive tolerance ϵ. In case where there is no measurement error, $\mathbf{y} = \mathbf{A}\mathbf{x}^i$ is valid. Moreover, if the sparsity level s of the target vector $\mathbf{x}$ is known, then we can perform s iterations of OMP. Thus, after s iterations, $\mathbf{x}^s$ will have s nonzero entries. However, if a wrong index is chosen, it will remain in the support, and eventually it will be included in the solution. This weakness is surpassed by the next algorithm, Compressive Sampling Matching Pursuit.

3.1.3 Compressive Sampling Matching Pursuit

Compressive Sampling Matching Pursuit (CoSaMP) algorithm has been introduced by Needell and Tropp [12]. CoSaMP is an iterative algorithm used for the recovery of sparse signals when the sparsity level s is known. Meanwhile, CoSaMP provides uniform recovery guarantees (like ℓ_1- minimization)[12]. The pseudocode of CoSaMP follows in Algorithm 3.

In step 8, the operator H_s keeps the s largest absolute values of the input vector and sets the others to zero.

Algorithm 3: Compressive Sampling Matching Pursuit Algorithm

Input: measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^m$, sparsity level s

initialization $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \mathbf{0}$

for *iteration* $i = 1$; $i = i + 1$ *till the halting criterion is met* **do**

- $\mathbf{g}^i = \mathbf{A}^T(\mathbf{y} - \mathbf{A}\mathbf{x}^{i-1})$
- $\mathcal{T} = \text{support}(\mathbf{x}^{i-1})$
- $\mathcal{L} = \text{support}(\mathbf{g}^i_{2s})$: index set of $2s$ largest absolute values
- $\mathcal{S}^i = \mathcal{T} \cup \mathcal{L}$
- $\mathbf{b}^i = \underset{\mathbf{z} \in \mathbb{R}^{\mathbf{n}}}{\text{argmin}} \|\mathbf{y} - \mathbf{A}\mathbf{z}\|_2$, $\text{supp}(\mathbf{z}) \subseteq \mathcal{S}^i$
- $\mathbf{x}^i = H_s(\mathbf{b}^i)$

Output: $\mathbf{x}^i$

3.1.4 Iterative Hard Thresholding Algorithm

The Iterative Hard Thresholding (IHT) algorithm is based on the non-linear operator H_s, which keeps the s largest absolute values of the input vector and sets the others to zero. IHT is presented in Algorithm 4.[1]

Algorithm 4: Iterative Hard Thresholding Algorithm

Input: measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^m$, sparsity level s

initialization $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \mathbf{0}$

for *iteration* $i = 1$; $i = i + 1$ *till the halting criterion is met* **do**

- $\mathbf{g}^{i-1} = \mathbf{A}^T \mathbf{r}^{i-1}$
- $\mathbf{x}^i = H_s\left(\mathbf{x}^{i-1} + \mu^i \mathbf{g}^{i-1}\right)$: H_s keeps its s largest absolute values and sets the other ones to zero.
- $\mathbf{r}^i = \mathbf{y} - \mathbf{A}\mathbf{x}^i$

Output: $\mathbf{x}^i$

3.2 Greedy Algorithms - Analysis

The characteristics of greedy algorithms have been investigated extensively. The key to their appeal is surely their simple implementation and their speed. For example, OMP is used in the Single Pixel Camera of Rice University [14]. In this section, we will present a number of results for the aforementioned algorithms, based on the properties on the measurement matrix $\mathbf{A}$.

[1] In the literature, IHT algorithm is encountered without the constant μ. However, as pointed out in [13], μ is essential for better performance.

3.2.1 Orthogonal Matching Pursuit

The Two-Ortho Case

For the analysis of OMP we are based on [15]. First, we discuss the case where $\mathbf{A}$ is a concatenation of two orthogonal (unitary) matrices, $\mathbf{A} = [\mathbf{\Psi}, \mathbf{\Phi}]$. The result that is derived from this initial assumption is over-pessimistic as it demands strong constraints on $\mathbf{A}$.

Some notation is essential to make the analysis easier to the reader. First, we assume that a solution $\mathbf{x}$ exists with s nonzero elements. Second, the vector $\mathbf{y}$ is a linear combination of the first s_p columns of $\mathbf{\Psi}$ and the first s_q columns of $\mathbf{\Phi}$, with $s = s_p + s_q$. Finally, we define the respective supports $\mathcal{S}_p$ and $\mathcal{S}_q$, with $\mathrm{card}(\mathcal{S}_p) = s_p$ and $\mathrm{card}(\mathcal{S}_q) = s_q$. So

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\phi}_i, \tag{3.1}$$

where x_i^{ψ} and x_i^{ϕ} are the coefficients of the linear combination of the active columns of $\mathbf{\Psi}$ and $\mathbf{\Phi}$, respectively. At the first iteration of the algorithm, we have the residual $\mathbf{r}^0$ set to $\mathbf{y}$. The algorithm chooses one of the s elements in the proper support if its associated column has the maximum correlation with the residual. Supposing that, without loss of generality, x_1^{ψ} is the largest in magnitude nonzero element in $\mathbf{x}$, for all $j \notin \mathcal{S}_p$ and $j \notin \mathcal{S}_q$ we should require that

$$|\boldsymbol{\psi}_1^T \mathbf{y}| > |\boldsymbol{\psi}_j^T \mathbf{y}|, \tag{3.2}$$

$$|\boldsymbol{\psi}_1^T \mathbf{y}| > |\boldsymbol{\phi}_j^T \mathbf{y}|. \tag{3.3}$$

In the sequel, we substitute in (3.2) the relation (3.1). So, we conclude that

$$\left| \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_1^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i \right| > \left| \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_j^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_j^T \boldsymbol{\phi}_i \right|. \tag{3.4}$$

We proceed with estimating lower and upper bounds for (3.4) to obtain the worst-case scenario. At first, we consider the left-hand side of the inequality (3.4). Using the triangle inequality, we have

$$\left| \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_1^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i \right| \geq \left| \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_1^T \boldsymbol{\psi}_i \right| - \left| \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i \right|. \tag{3.5}$$

Then, we exploit the orthogonality of $\mathbf{\Psi}$. So, (3.5) gives

$$\left| \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_1^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i \right| \geq |x_1^{\psi}| - \left| \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i \right|. \tag{3.6}$$

Also, we have

$$\left|\sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i\right| \leq \sum_{i=1}^{s_q} |x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i| \leq \sum_{i=1}^{s_q} |x_i^{\phi}| \mu(\mathbf{A}), \tag{3.7}$$

where we used the definition (2.5). Moreover, having that x_1^{ψ} has the largest absolute value of all the elements of $\mathbf{x}$, we obtain

$$s_q |x_1^{\psi}| > \sum_{i=1}^{s_q} |x_i^{\phi}|. \tag{3.8}$$

Finally, using the relations (3.7) and (3.8) in (3.6), we obtain

$$\left|\sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_1^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i\right| \geq |x_1^{\psi}| - \sum_{i=1}^{s_q} |x_i^{\phi}| \mu(\mathbf{A}) \geq |x_1^{\psi}|(1 - s_q \mu(\mathbf{A})). \tag{3.9}$$

In the same manner, we develop an upper bound for the right-hand side for the inequality (3.4)

$$\left|\sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_j^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_j^T \boldsymbol{\phi}_i\right| \leq |x_1^{\psi}| s_q \mu(\mathbf{A}), \tag{3.10}$$

where $\left|\sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_j^T \boldsymbol{\psi}_i\right| = 0$. Having these two tighter bounds for both sides of the inequality (3.4), we can write

$$|x_1^{\psi}|(1 - s_q \mu(\mathbf{A})) > |x_1^{\psi}| s_q \mu(\mathbf{A}), \tag{3.11}$$

which leads to

$$s_q < \frac{1}{2\mu(\mathbf{A})}. \tag{3.12}$$

If we change our initial assumption, and assume that x_1^{ϕ} is the largest in magnitude nonzero element of $\mathbf{x}$, then the same bound is valid for s_p. Then, following analogous steps we can prove that

$$s_q < \frac{1}{2\mu(\mathbf{A})}. \tag{3.13}$$

Now, we proceed with the second requirement (3.3). We use (3.1) in (3.3) to obtain

$$\left|\sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\psi}_1^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\psi}_1^T \boldsymbol{\phi}_i\right| > \left|\sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\phi}_j^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\phi}_j^T \boldsymbol{\phi}_i\right|. \tag{3.14}$$

The lower bound for the left-hand side of (3.14) is obviously the same. However, the upper bound for the right-hand side of (3.14) is slightly different. So

$$\left| \sum_{i=1}^{s_p} x_i^{\psi} \boldsymbol{\phi}_j^T \boldsymbol{\psi}_i + \sum_{i=1}^{s_q} x_i^{\phi} \boldsymbol{\phi}_j^T \boldsymbol{\phi}_i \right| \leq |x_1^{\psi}| s_p \mu(\mathbf{A}). \tag{3.15}$$

This leads to the requirement

$$|x_1^{\psi}|(1 - s_q \mu(\mathbf{A})) > |x_1^{\psi}| s_p \mu(\mathbf{A}). \tag{3.16}$$

Finally, we have

$$s_p + s_q < \frac{1}{\mu(\mathbf{A})}. \tag{3.17}$$

The relation (3.17) does not add anything to the analysis, as it is covered by $s_q, s_p < \dfrac{1}{2\mu(\mathbf{A})}$.

Having the requirements (3.2) and (3.3) satisfied, OMP selects a nonzero from the proper support at the first iteration.

Then, the residual $\mathbf{r}^1$ is updated by subtracting the column chosen in the first step of the algorithm (the column with the biggest correlation with the current residual) multiplied by a coefficient. Thus, the residual is still a linear combination of s columns at most. Using the same arguments presented above, OMP will perform well at the following iterations. Moreover, due to orthogonality, the OMP algorithm never chooses the same column twice. Finally, the same result holds for the s-th iteration of OMP, and the algorithm terminates successfully. The following theorem summarises these results.

Theorem 2. [15, p. 57] *For a system of linear equations* $\mathbf{Ax} = [\boldsymbol{\Psi}, \boldsymbol{\Phi}]\mathbf{x} = \mathbf{y}$ *with two ortho-matrices* $\boldsymbol{\Psi}, \boldsymbol{\Phi}$ *with dimensions* $n \times n$, *if a solution* $\mathbf{x}$ *exists such that it has* s_p *nonzeros in its first half and* s_q *nonzeros in the second, and the two obey*

$$\max(s_p, s_q) < \frac{1}{2\mu(\mathbf{A})}, \tag{3.18}$$

then *OMP run with threshold parameter* $e_0 = 0$ *is guaranteed to find* $\mathbf{x}$ *exactly in* $s = s_p + s_q$ *iterations.*

The General Case

The result of the two-ortho case provides us with intuition to proceed with a generalisation of the Theorem 2 concerning general matrices $\mathbf{A}$. The analysis of this scenario is very close to the analysis of the previous section but the provided result is weaker. In fact, in the two-ortho case, we could drop some inner products due to the strict structure that was assumed. Here, we have to do with arbitrary matrices so this is not the case.

Theorem 3. [15, p. 65] *For a system of linear equations* $\mathbf{Ax} = \mathbf{y}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ *full-rank with* $m < n$, *if a solution* $\mathbf{x}$ *exists obeying*

$$\|\mathbf{x}\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{A})}\right), \tag{3.19}$$

then OMP run with threshold parameter $e_0 = 0$ *is guaranteed to find it exactly.*

Theorem 3 implies that, for arbitrary matrices, the solution $\mathbf{x}$ must have a small sparsity level. The proof is given in the Appendix A. Using Lemma (2.7), the relation of Theorem 3 becomes

$$\|\mathbf{x}\|_0 < \frac{\operatorname{spark}(\mathbf{A})}{2}. \tag{3.20}$$

The above inequality gives the relation between spark($\mathbf{A}$) and the sparsity s of the solution.

Number of measurements

The results that have been considered so far exploit the property of mutual coherence of the measurement matrix. For both of the two cases presented, we obtain sufficient conditions that guarantee the exact recovery of every s-sparse vector from its linear measurements $\mathbf{Ax} = \mathbf{y}$. In Chapter 2, we have mentioned that mutual coherence is bounded from below as follows

$$\mu\left(\mathbf{A}\right) \geq \sqrt{\frac{n-m}{m(n-1)}}. \tag{3.21}$$

For large n, the right-hand side of the inequality scales like $\dfrac{1}{\sqrt{m}}$. Also, matrices with small coherence are favorable for the problem.

Now, considering the result for general matrices, Theorem 3, with mutual coherence of optimal order, namely $\mu\left(\mathbf{A}\right) \geq c\,\dfrac{1}{\sqrt{m}}$, we derive the bound

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{A})}\right) < \frac{1}{2}\left(1 + \sqrt{m}/c\right) \Rightarrow m \geq Cs^2. \tag{3.22}$$

This bound implies that the number of measurements scales quadratically in the sparsity level. More precisely, (3.22) states that for a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with near optimal coherence, in order to recover sparse vectors, m of order s^2 are required.

Exact Recovery Condition

The Exact Recovery Condition (ERC) was introduced by J.A. Tropp in his work concerning sparse approximation [16]. By satisfying ERC, sufficient conditions for sparse recovery are deduced, following a slightly different path for the analysis of OMP algorithm. We follow the analysis presented in [15].

Definition 5. [15] *For a given support $\mathcal{S}$ and for a matrix $\mathbf{A}$, the Exact Recovery Condition (ERC) is given by*

$$ERC\,(\mathbf{A}, \mathcal{S}) := \max_{i \notin S} \|\mathbf{A}_{\mathcal{S}}^{\dagger} \mathbf{a}_i\|_1 < 1, \tag{3.23}$$

where $\mathbf{A}_{\mathcal{S}}$ is the submatrix of $\mathbf{A}$ constructed from the columns that correspond to the given support $\mathcal{S}$.

Using the definition of the pseudo-inverse matrix, the above definition considers linear systems of the form $\mathbf{A}_{\mathcal{S}}\mathbf{x} = \mathbf{a}_i$, with $\mathbf{a}_i$ a column of $\mathbf{A}$ outside of the support. Moreover, by the relation between least-squares and pseudoinverse we can deduce

$$\|\mathbf{A}_{\mathcal{S}}\mathbf{x} - \mathbf{a}_i\|_2 \geq \|\mathbf{A}_{\mathcal{S}}\mathbf{x}^* - \mathbf{a}_i\|_2, \tag{3.24}$$

where $\mathbf{x}^* = \mathbf{A}_{\mathcal{S}}^{\dagger}\mathbf{a}_i$.

We conclude that ERC states that the minimum ℓ_2-norm solutions to *all* systems of the form $\mathbf{A}_{\mathcal{S}}\mathbf{x} = \mathbf{a}_i$, must *all* have ℓ_1-norm smaller than 1. In the following theorem, the connection between ERC and successful OMP performance is given.

Theorem 4. [15] *For a sparse vector $\mathbf{x}$ over the support $\mathcal{S}$, which is the solution of the linear system $\mathbf{A}\mathbf{x} = \mathbf{y}$, if the ERC is met, then OMP is guaranteed to succeed in recovering $\mathbf{x}$.*

This is a stronger condition than those we have mentioned before concerning the mutual coherence. However, Theorem 4 requires that the support $\mathcal{S}$ is given. As a consequence, if we only know the cardinality of the support, the usage of ERC to guarantee successful recovery is not appropriate as it demands $\binom{n}{s}$ tests to verify. In the sequel, we proceed with the proof of Theorem 4.

Proof. We assume that we are in the first step of the algorithm. In order to have a successful outcome out of this step, we require that

$$\|\mathbf{A}_{\mathcal{S}}^T\mathbf{y}\|_\infty > \|\mathbf{A}_{\bar{\mathcal{S}}}^T\mathbf{y}\|_\infty. \tag{3.25}$$

The relation (3.25) claims that the maximal correlation between the active predictors (or atoms) of $\mathbf{A}$ and $\mathbf{y}$ must be larger than the maximal correlation between predictors outside the support. Thus, we require that

$$\rho = \frac{\|\mathbf{A}_{\overline{\mathcal{S}}}^T \mathbf{y}\|_\infty}{\|\mathbf{A}_{\mathcal{S}}^T \mathbf{y}\|_\infty} < 1. \tag{3.26}$$

Also, we have $\mathbf{A}_{\mathcal{S}} \mathbf{x}_{\mathcal{S}} = \mathbf{y}$, thus $\mathbf{y}$ belongs to the column space of $\mathbf{A}_{\mathcal{S}}$. Moreover, $\mathbf{P}\mathbf{y} = \mathbf{y} \in C\left(\mathbf{A}_{\mathcal{S}}\right)$, where $\mathbf{P}$ is the projection matrix onto the column span of $\mathbf{A}_{\mathcal{S}}$. Consequently,

$$\mathbf{y} = \mathbf{P}\mathbf{y} = \left(\mathbf{A}_{\mathcal{S}} \left(\mathbf{A}_{\mathcal{S}}^T \mathbf{A}_{\mathcal{S}}\right)^{-1} \mathbf{A}_{\mathcal{S}}^T\right) \mathbf{y} = \left(\mathbf{A}_{\mathcal{S}}^T\right)^\dagger \mathbf{A}_{\mathcal{S}}^T \mathbf{y}. \tag{3.27}$$

Substituting the relation (3.27) in (3.26), we have

$$\begin{aligned} \rho &= \frac{\|\mathbf{A}_{\overline{\mathcal{S}}}^T \mathbf{y}\|_\infty}{\|\mathbf{A}_{\mathcal{S}}^T \mathbf{y}\|_\infty} \\ &= \frac{\|\mathbf{A}_{\overline{\mathcal{S}}}^T \left(\mathbf{A}_{\mathcal{S}}^T\right)^\dagger \mathbf{A}_{\mathcal{S}}^T \mathbf{y}\|_\infty}{\|\mathbf{A}_{\mathcal{S}}^T \mathbf{y}\|_\infty} \\ &\leq \|\mathbf{A}_{\overline{\mathcal{S}}}^T \left(\mathbf{A}_{\mathcal{S}}^T\right)^\dagger \|_\infty, \end{aligned}$$

where, in the last inequality, we used the definition of ℓ_∞-induced norm. Using the properties of matrix norms (see Appendix B), we obtain

$$\|\mathbf{A}_{\overline{\mathcal{S}}}^T \left(\mathbf{A}_{\mathcal{S}}^T\right)^\dagger \|_\infty = \| \left(\mathbf{A}_{\mathcal{S}}\right)^\dagger \mathbf{A}_{\overline{\mathcal{S}}}\|_1. \tag{3.28}$$

So, by requiring $\| \left(\mathbf{A}_{\mathcal{S}}\right)^\dagger \mathbf{A}_{\overline{\mathcal{S}}}\|_1 < 1$ (ERC property), we guarantee that $\rho < 1$. Consequently, the first step of OMP is successful.

In the following steps, $\mathbf{y}$ is replaced by the residual which is in the span of $\mathbf{A}_{\mathcal{S}}$, so the analysis above remains the same. Thus, the OMP algorithm succeeds in recovering $\mathbf{x}$ in s steps. The original proof can be found in [15]. □

We need to find a way to guarantee that ERC property is satisfied. The next theorem is showed by Tropp in [16]. We proceed with the theorem.

Theorem 5. [15] *For a matrix* $\mathbf{A}$ *with mutual coherence* $\mu\left(\mathbf{A}\right)$*, if* $s < \frac{1}{2}\left(1 + \frac{1}{\mu\left(\mathbf{A}\right)}\right)$*, then, for all supports* $\mathcal{S}$ *with cardinality equal or smaller than* s*, the ERC is satisfied.*

Proof. For a column *outside* the support, suppose $\mathbf{a}_i$, we require $\|\mathbf{A}_{\mathcal{S}}^{\dagger}\mathbf{a}_i\|_1 = \|\left(\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}\right)^{-1}\mathbf{A}_{\mathcal{S}}^T\mathbf{a}_i\|_1 < 1$. If we substitute the previous relation with a more strict one, we obtain $\|\left(\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}\right)^{-1}\|_1\|\mathbf{A}_{\mathcal{S}}^T\mathbf{a}_i\|_1 < 1$. We notice that the vector $\mathbf{A}_{\mathcal{S}}^T\mathbf{a}_i$ has its elements bounded from below by the value $-\mu\left(\mathbf{A}\right)$ and from above by the value $\mu\left(\mathbf{A}\right)$ (by exploiting the definition of mutual coherence (2.5)). Thus, $\|\mathbf{A}_{\mathcal{S}}^T\mathbf{a}_i\|_1 \leq s\mu\left(\mathbf{A}\right)$.

To provide a bound for $\|\left(\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}\right)^{-1}\|_1$, we work as follows. At first, the matrix $\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}} = \mathbf{G}$ is a Gram matrix thus is positive-semidefinite. This means that its eigenvalues are nonnegative. The values of the elements in the diagonal of $\mathbf{G}$ are equal to $\mathbf{G}_{ii} = 1$ (due to the normalization of the columns of $\mathbf{A}$). Using the Gershgorin disk theorem, the radius is

$$r_i = \sum_{i \neq j} |\mathbf{G}_{ij}| < (s-1)\mu\left(\mathbf{A}\right). \tag{3.29}$$

Thus, the eigenvalues of $\mathbf{G}$ lie in the disk with center $\mathbf{G}_{ii} = 1$ and radius r_i. Formally, in the interval

$$\left[1-(s-1)\mu\left(\mathbf{A}\right), 1+(s-1)\mu\left(\mathbf{A}\right)\right]. \tag{3.30}$$

Since $\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}$ has nonnegative eigenvalues, $(s-1)\mu\left(\mathbf{A}\right) \leq 1$.Using the above result, we conclude that the matrix $\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}$ is strictly-diagonally dominant. Using this property of $\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}$, we can use the bound by J. M. Varah in his work "A Lower Bound for the Smallest Singular Value of a Matrix" [17]. Finally, we have

$$\|\left(\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}\right)^{-1}\|_1 \leq \frac{1}{1-(s-1)\mu\left(\mathbf{A}\right)}. \tag{3.31}$$

Combining the bounds obtained, we conclude in the final requirement

$$\|\left(\mathbf{A}_{\mathcal{S}}^T\mathbf{A}_{\mathcal{S}}\right)^{-1}\|_1\|\mathbf{A}_{\mathcal{S}}^T\mathbf{a}_i\|_1 \leq \frac{s\mu\left(\mathbf{A}\right)}{1-(s-1)\mu\left(\mathbf{A}\right)} < 1. \tag{3.32}$$

Working in the relation (3.32), we obtain $s < \frac{1}{2}\left(1+\frac{1}{\mu\left(\mathbf{A}\right)}\right)$. This condition is the same for all columns outside the support. The proof is complete. The proof follows the one given in [15]. □

The above results exploit the notion of mutual coherence as well as the ERC introduced by Tropp to analyse the performance of OMP. As we have already seen, OMP behaves very well for exact recovery of s-sparse vector from its measurements. However, this is not the case for uniform recovery of

s-sparse vectors (recovery of all s-sparse vectors from their linear measurements) using m of optimal order. To reflect more on this subject, we have to recall the definition of RIP (2.8).

The RIP for a matrix $\mathbf{A}$ holds, by definition, for all s-sparse vectors. However, for specific matrices with special structure, the RIP is not enough to guarantee the recovery of all s-sparse vectors via OMP, in at most s iterations [1, p. 156].

To bypass this issue, we can perform more iterations. In the sequel, we simply state a general result (in the complex setting) for successful recovery using the RIP.

Proposition 2. [1, p. 156] *Given $\mathbf{A} \in \mathbb{C}^{m\times n}$, let $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ for some $\mathbf{x}$ s-sparse $\in \mathbb{C}^n$ with $\mathcal{S} = supp(\mathbf{x})$ and $\mathbf{e} \in \mathbb{C}^m$. Let $(\mathbf{x}^i)$ denote the sequence defined by step 6 of Algorithm 2 started at an index set $\mathcal{S}^0$. With $s^0 = \mathrm{card}(\mathcal{S}^0)$ and $s' = \mathrm{card}(\mathcal{S} \setminus \mathcal{S}^0)$, if $\delta_{s+s^0+12s'} < 1/6$, then there is a constant $C > 0$ depending only on $\delta_{s+s^0+12s'}$ such that*

$$\|\mathbf{y} - \mathbf{A}\mathbf{x}^{\bar{i}}\|_2 \leq C\|\mathbf{e}\|_2, \tag{3.33}$$

where $\bar{i} = 12s'$.

For standard OMP with $\mathcal{S}^0 = \emptyset$ and $\mathbf{e} = \mathbf{0}$, the above implies that exact recovery via OMP is guaranteed in $12s$ iterations. Indeed, this is satisfied since $\delta_{13s} < 1$.

Signal recovery from random measurements

A serious improvement concerning the number of linear measurements required for signal recovery has been made by J. A. Tropp and A. C. Gilbert [18]. They showed both empirically and theoretically that OMP can reliably recover a s-sparse vector using $\mathcal{O}(s \ln n)$ random linear measurements. This result for the OMP algorithm is quite similar to those of Basis Pursuit approach.

The theorem, which is the main result of the paper mentioned before, states that OMP can recover a sparse vector with high probability.

Theorem 6. [18] *Let $\mathbf{A}$ be a $m \times n$ Gaussian matrix, and fix a s-sparse signal $\mathbf{x} \in \mathbb{R}^n$. Then OMP recovers $\mathbf{x}$ from its measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$ correctly with high probability, provided that the number of measurements is $m \sim s \ln n$.*

The authors state a list with the properties an admissible matrix should satisfy in their paper. The proof of the previous theorem exploits probabilistic tools. Finally, we have to mention that, even though this result gives a huge improvement, it does not provide OMP with uniform guarantees.

Next, we mention the results for CoSaMP and IHT, using the RIP property. Precisely, under the condition

$$\delta_{\kappa s} \leq \delta_*, \tag{3.34}$$

for some integer κ, and $\delta_* \leq 1$ both depending on the algorithm, uniform recovery is guaranteed. The condition 2 is of the same type.

3.2.2 Compressive Sampling Matching Pursuit

In virtue of the limitations of OMP in recovering all s-sparse vectors, D. Needell, J. A. Tropp presented CoSaMP algorithm that has guarantees analogous to ℓ_1-minimization. We proceed with a result concerning the CoSaMP (see Algorithm 3) using the RIP property.

Theorem 7. [1, p. 164] *Suppose that the* $4s$*-th restricted isometry constant of the matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *satisfies*

$$\delta_{4s} < \frac{\sqrt{\sqrt{11/3} - 1}}{2}. \tag{3.35}$$

Then, for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{e} \in \mathbb{R}^m$, $\mathcal{S} \subset [n]$ *with* $\mathrm{card}(\mathcal{S}) = s$, *the sequence* $(\mathbf{x}^i)$ *defined by the algorithm with* $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ *satisfies*

$$\|\mathbf{x}^i - \mathbf{x}_\mathcal{S}\|_2 \leq \rho^i \|\mathbf{x}^0 - \mathbf{x}_\mathcal{S}\|_2 + \tau \|\mathbf{A}\mathbf{x}_{\overline{\mathcal{S}}} + \mathbf{e}\|_2, \tag{3.36}$$

where constants $0 < \rho < 1$ *and* $\tau > 0$ *depend only on* δ_{4s}.
The proof of Theorem 7 can be found in Appendix A.

3.2.3 Iterative Hard thresholding

In this section, a result for s-sparse recovery via Iterative Hard thresholding is considered. Here, we will only refer to the result for approximately sparse vectors measured with some errors, as it demands a weaker condition than the one of exact sparse vectors (the sufficient condition for exact sparse recovery can be found in [19]). The success of this algorithm is based on the simple intuition that for small RIP constants the matrix $\mathbf{A}^T\mathbf{A}$ behaves as an identity matrix when its domain and range are restricted to small supports.

Theorem 8. [1, p. 148] *Suppose that the* $3s$*-th restricted isometry constant of the matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *satisfies:*

$$\delta_{3s} < \frac{1}{\sqrt{3}}. \tag{3.37}$$

Then, for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{e} \in \mathbb{R}^m$, $\mathcal{S} \subset [n]$ *with* $\mathrm{card}(\mathcal{S}) = s$, *the sequence* $(\mathbf{x}^i)$ *constructed by the algorithm with* $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ *satisfies*

$$\|\mathbf{x}^i - \mathbf{x}_{\mathcal{S}}\|_2 \leq \rho^i \|\mathbf{x}^0 - \mathbf{x}_{\mathcal{S}}\|_2 + \tau \|\mathbf{A}\mathbf{x}_{\overline{\mathcal{S}}} + \mathbf{e}\|_2, \tag{3.38}$$

where constants $\rho = \sqrt{3}\delta_{3s} < 1$ *and* $\tau \leq 2.18/(1-\rho)$.
The proof is similar to the proof of CoSaMP algorithm in the Appendix A, thus is ommited.

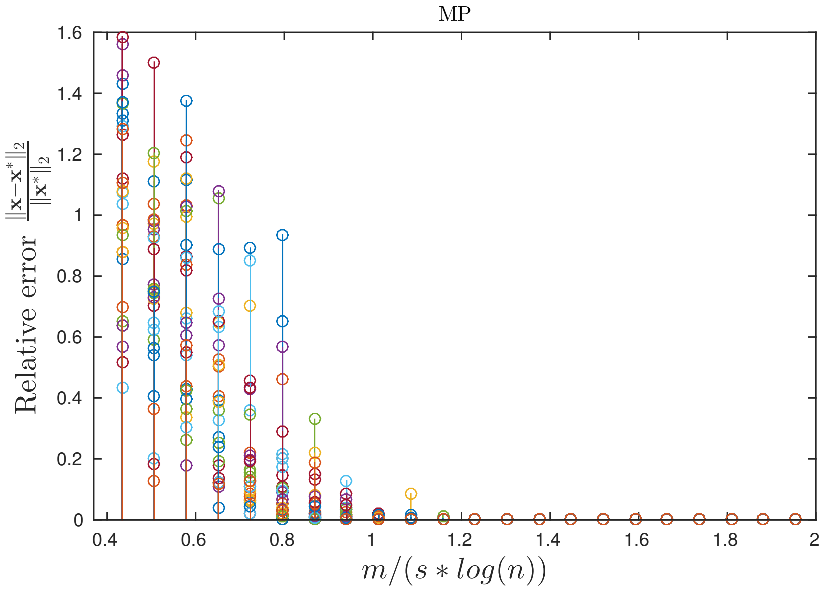
It is possible to construct deterministic matrices that satisfy the condition (3.34). However, these constructions demand m to be relatively large, for example $m = \mathcal{O}(s^2 \ln n)$. It is already mentioned that, by randomizing the measurement matrix, the quadratic bottleneck is overcome. In our simulations, we use random matrices.

3.3 Simulations

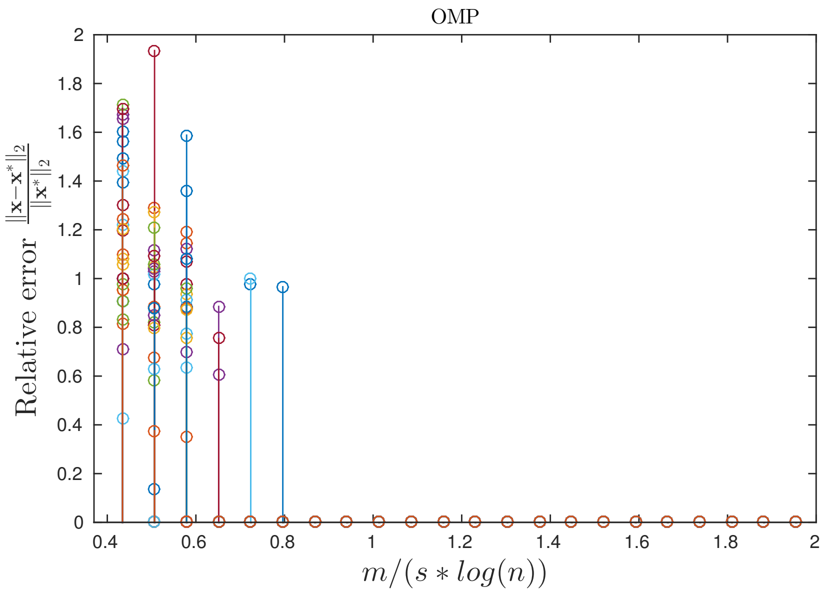
In our simulations, we conducted the following experiment. We generate a s-sparse vector ($s = 20$) and a $m \times n$, with $n = 1000$, measurement matrix with entries drawn from i.i.d. $\mathcal{N}(0, 1)$. We test the performance of OMP, MP, CoSaMP and IHT for the nonuniform recovery framework for different dimension m in the interval $m = [50; 10; 350]$. The parameter μ of IHT is set to 0.4. For each value of m, we run 30 trials.

At first, we point out that all algorithms require $m \geq Cs\ln(n)$ to recover s-sparse vectors, although, the constant $C > 0$, which depends on the algorithm, may be slightly different. In Figure 3.1, we plot the results in terms of relative signal estimation error for MP and OMP. We observe that OMP succeeds in recovering s-sparse vectors for smaller ratio $(m/s\ln(n))$ (actually, for smaller constant C) than MP. Next, we illustrate the performance of CoSaMP, and IHT in Figure 3.2. In general, we observe that CoSaMP outperforms IHT. Specifically, CoSaMP succeeds s-sparse recovery for smaller constant C than IHT.

From Figures 3.1 and 3.2, we observe that OMP and CoSaMP have better performance. However, CoSaMP attains large relative signal estimation error for $C = 0.5$. In Figure 3.1, a smoother decrease of relative signal estimation error is observed for OMP.



(a)

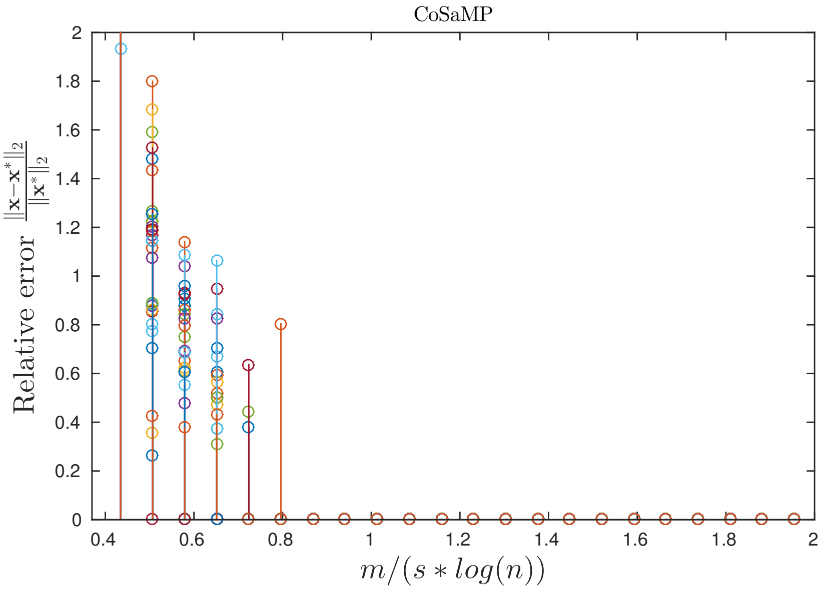


(b)

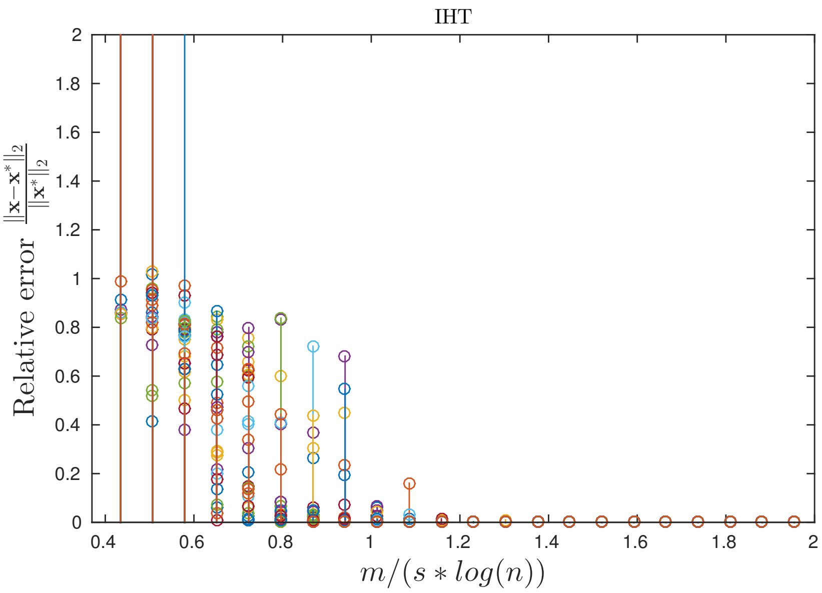
Figure 3.1: Relative signal estimation error vs the ratio $(m/s\ln(n))$, for MP(a) and OMP(b) algorithms.

Summary

From the experiment we conducted, we observe that the performance, in practice, of the greedy algorithms we tested, is aligned with theoretical results



(a)



(b)

Figure 3.2: Relative signal estimation error vs the ratio $(m/s\ln(n))$, for CoSaMP(a) and IHT(b) algorithms.

presented in this chapter. Especially, the requirement of $m = \mathcal{O}(s\ln(n))$ measurements is valid for all algorithms in the experiment. However, as we have already pointed out, slightly different constants C are observed for each algorithm. In general, OMP and CoSaMP outperform MP and IHT for the case we examined.

Chapter 4

Non-Linear Problem and algorithms

4.1 Formulation of the problem

In Chapters 2 and 3, we considered the case of reconstructing $\mathbf{x}^*$ from the linear measurements $\mathbf{y}$ (Compressed Sensing). As we have seen already, this approach is very useful in a variety of fields in scientific research, however, it is not applicable in several cases of interest. For instance, in statistics and machine learning, a broader class of functions are massively used to perform classification and prediction. Consequently, new methods should be developed, suitable for these applications.

Several works have been proposed using the sparsity inducing ℓ_1-norm as penalty to problems that consider nonlinear models. For example, in [20] the ℓ_1 penalty is added to the logistic cost function. Moreover, in [21], variable selection by logistic regression, using ℓ_1 and $\ell_1+\ell_2$ (elastic - net penalty in the literature) is investigated. Finally, a work concerning the ℓ_1-regularization on general exponential families of functions is [22].

In [23], the definition of nonlinear Compressed Sensing is considered. In this work, the linear scheme used to obtain the measurement vector is replaced by a nonlinear mapping from one vector space to another. Also, an extensive analysis of IHT algorithm for this setting is provided. In addition, some methods are proposed for problems of generic cost functions with sparsity constrains in [24].

An issue that arises from the requirements that guarantee accuracy of ℓ_1 methods is the characterization of the sparsity level of the solution. As it has been showed in [22], the sparsity level of the solution is not known to be optimal. In this chapter, we will consider a greedy algorithm, which does not exploit the ℓ_1-norm, proposed by Sohail Bahmani [25]. In his work, a generic function with specific properties, that will be mentioned in the sequel, is used. The GraSP algorithm (Gradient Support Pursuit) is presented below.

4.2 Gradient Support Pursuit

The GraSP algorithm provides an approximate solution to the problem

$$\underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x}), \text{ s.t. } \|\mathbf{x}\|_0 \leq s, \tag{4.1}$$

The name of the algorithm implies that the gradient (or subgradient) is used to identify the directions of minimization. At a given point $\widehat{\mathbf{x}}$, we select the $2s$ largest in magnitude elements of the gradient. These indices, as well as the support of the current estimate $\widehat{\mathbf{x}}$, are merged to the set $\mathcal{T}$, with cardinality at most $3s$. Then, the cost function is minimized over the set $\mathcal{T}$. Finally, the next point $\widehat{\mathbf{x}}$ is the best s-term approximation of the minimization. The algorithm is summarised in Algorithm 5.

Algorithm 5: Gradient Support Pursuit

Input: $f(\cdot)$, sparsity level s
initialization $\widehat{\mathbf{x}} = \mathbf{0}$
repeat
  $\mathbf{z} = \nabla f(\widehat{\mathbf{x}})$
  $\mathcal{Z} = \text{support}(\mathbf{z}_{2s})$
  $\mathcal{T} = \text{support}(\widehat{\mathbf{x}}) \cup \mathcal{Z}$
  $\mathbf{b} = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x})$, s.t. $\mathbf{x}_{\overline{\mathcal{T}}} = \mathbf{0}$
  $\widehat{\mathbf{x}} = \mathbf{b}_s$: keeps its s largest absolute values and sets the other ones to zero.
until *until halting condition*
Output: $\widehat{\mathbf{x}}$

In the linear setting with squared error as a cost function, several conditions should be satisfied in order to obtain an accurate solution. Observe that, using the squared error as a cost function, GraSP reduces to CoSaMP. Following the same path, by imposing some properties on the cost function, the analysis of GraSP algorithm becomes tractable.

4.2.1 Conditions on the cost function

The properties that guarantee accurate solution for the GraSP algorithm, namely SRH (Stable Restricted Hessian) and SRL (Stable Restricted Linearization), are analogous to RIP. These properties require that locally the curvature of the cost function over the sparse subspaces is bounded from both sides, with bounds of the same order. The definitions of these two properties follow.

Definition 6. *Suppose that f is a twice continuously differentiable function whose Hessian is denoted by $\nabla^2 f(\cdot)$. Furthermore, let*

$$A_k(\mathbf{x}) = \sup\left\{\mathbf{\Delta}^T \nabla^2 f(\mathbf{x})\mathbf{\Delta} \,\middle|\, |\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{\Delta})| \leq k \ , \ \|\mathbf{\Delta}\|_2 = 1\right\}, \quad (4.2)$$

and

$$B_k(\mathbf{x}) = \inf\left\{\mathbf{\Delta}^T \nabla^2 f(\mathbf{x})\mathbf{\Delta} \,\middle|\, |\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{\Delta})| \leq k \ , \ \|\mathbf{\Delta}\|_2 = 1\right\}, \quad (4.3)$$

for all k-sparse vectors $\mathbf{x}$. Then, $f(\cdot)$ is said to have a Stable Restricted Hessian (SRH) with constant μ_k, or in short μ_k-SRH, if $1 \leq \dfrac{A_k(\mathbf{x})}{B_k(\mathbf{x})} \leq \mu_k$.
Note that the SRH property implies convexity over canonical sparse subspaces, but not everywhere.

To generalize the notion of SRH for the case of nonsmooth functions, first we define the restricted subgradient of a function.

Definition 7. *We say that vector $\nabla_f(\mathbf{x})$ is a restricted subgradient of $f : \mathbb{R}^n \to \mathbb{R}$, at point $\mathbf{x}$ if*

$$f(\mathbf{x} + \mathbf{\Delta}) - f(\mathbf{x}) \geq \langle \nabla_f(\mathbf{x}), \mathbf{\Delta} \rangle, \quad (4.4)$$

holds for all k-sparse vectors $\mathbf{\Delta}$.
Notice that $\mathbf{x}$ is not necessarily sparse. The directions of movement $\mathbf{\Delta}$ must be sparse. The existence of restricted subgradients implies convexity in sparse directions. Also, for convex functions, every subgradient can be a restricted subgradient.[1] Finally, we introduce the Bregman Divergence. This notion may be considered as a measure of curvature for non-smooth functions where the Hessian matrix is not defined. The definition of Bregman Divergence is as follows

$$\mathbf{B}_f(\mathbf{x}' \| \mathbf{x}) = f(\mathbf{x}') - f(\mathbf{x}) - \langle \nabla_f(\mathbf{x}), \mathbf{x}' - \mathbf{x} \rangle. \quad (4.5)$$

Now, we proceed to the main definitions for non-smooth functions

Definition 8. *Let $\mathbf{x}$ be a k-sparse vector in $\mathbb{R}^n$. For function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define the functions*

[1] Note that every function (convex or non-convex) has a set of subgradients (the subdifferential) at each point, and if the function is differentiable and convex then it has only one subgradient, the gradient.

$$\alpha_k(\mathbf{x}) = \sup\left\{\frac{1}{\|\boldsymbol{\Delta}\|_2^2}\mathbf{B}_f(\mathbf{x}+\boldsymbol{\Delta}\|\mathbf{x})\middle| |\mathrm{supp}(\mathbf{x})\cup\mathrm{supp}(\boldsymbol{\Delta})| \leq k \ , \ \boldsymbol{\Delta} \neq \mathbf{0}\right\}, \quad (4.6)$$

and

$$\beta_k(\mathbf{x}) = \inf\left\{\frac{1}{\|\boldsymbol{\Delta}\|_2^2}\mathbf{B}_f(\mathbf{x}+\boldsymbol{\Delta}\|\mathbf{x})\middle| |\mathrm{supp}(\mathbf{x})\cup\mathrm{supp}(\boldsymbol{\Delta})| \leq k \ , \ \boldsymbol{\Delta} \neq \mathbf{0}\right\}, \quad (4.7)$$

respectively. Then, $f(\cdot)$ is said to have a Stable Restricted Linearization with constant constant μ_k, or in short μ_k-SRL, if $1 \leq \frac{\alpha_k(\mathbf{x})}{\beta_k(\mathbf{x})} \leq \mu_k$ for all k-sparse vectors.

Definition 8 implies that for any k - sparse vector $\mathbf{x}$, $\alpha_k(\mathbf{x})$ and $\beta_k(\mathbf{x})$ are in order the smallest and largest values that

$$\beta_k(\mathbf{x})\|\boldsymbol{\Delta}\|_2^2 \leq \mathbf{B}_f(\mathbf{x}+\boldsymbol{\Delta}\|\mathbf{x}) \leq \alpha_k(\mathbf{x})\|\boldsymbol{\Delta}\|_2^2, \quad (4.8)$$

holds for all vectors $\boldsymbol{\Delta} \in \mathbb{R}^n$, with card(supp($\mathbf{x}$)$\cup$ supp($\boldsymbol{\Delta}$)) $\leq k$.

4.2.2 Main theorems of convergence

Having defined all the essential quantities, we can proceed with the main theorems for both cases. At first, suppose that:

$$\mathbf{x}^* = \underset{\mathbf{x}}{\mathrm{argmin}} f(\mathbf{x}), \quad \text{s.t.} \quad \|\mathbf{x}\|_0 \leq s. \quad (4.9)$$

Theorem 9. *Suppose that f is a twice continuously differentiable function that has μ_{4s} - SRH with $\mu_{4s} \leq \frac{1+\sqrt{3}}{2}$. Furthermore, suppose that for some $\epsilon > 0$ we have $\|\nabla f(\mathbf{x}^*)_{\mathcal{I}}\|_2 \leq \epsilon B_{4s}(\mathbf{x})$ for all $4s$-sparse $\mathbf{x}$, where $\mathcal{I}$ is the position of the $3s$ largest entries of $\nabla f(\mathbf{x}^*)$ in magnitude. Then, $\widehat{\mathbf{x}}^i$, the estimate at the i-th iteration, satisfies*

$$\|\widehat{\mathbf{x}}^i - \mathbf{x}^*\|_2 \leq 2^{-i}\|\mathbf{x}^*\|_2 + (6+2\sqrt{3})\epsilon. \quad (4.10)$$

Observe that the gradient $\nabla f(\mathbf{x}^*)$ indicates the accuracy of the result. Suppose that the sparse minimum $\mathbf{x}^*$ is sufficiently close to the unconstrained minimum of the cost function, then the estimation error floor will be small because the gradient $\nabla f(\mathbf{x}^*)$ has small magnitude. Also, in cases where $\mathbf{x}^*$ is arbitrary rather than the minimum of the cost function (in statistical estimation $\mathbf{x}^*$ is the target parameter that explains the data), large values of ϵ are implied. Thus we have to expect large estimation error. In this kind of problems, quantity $\|\nabla f(\mathbf{x}^*)\|_2$ can provide a certain interpretation in order to tune the accuracy of the result.

Theorem 10. *Suppose that f is a function that is not necessarily smooth, but it satisfies μ_{4s} - SRL with $\mu_{4s} \leq \frac{3+\sqrt{3}}{4}$. Furthermore, suppose that for $\beta_{4s}(\cdot)$ there exists some $\epsilon > 0$ such that $\|\nabla_f(\mathbf{x}^*)_{\mathcal{I}}\|_2 \leq \epsilon\beta_{4s}(\mathbf{x})$ holds for all $4s$-sparse vectors $\mathbf{x}$ where $\mathcal{I}$ is the position of the $3s$ largest entries of $\nabla_f(\mathbf{x}^*)$ in magnitude. Then, $\widehat{\mathbf{x}}^i$, the estimate at the i-th iteration, satisfies*

$$\|\widehat{\mathbf{x}}^i - \mathbf{x}^*\|_2 \leq 2^{-i}\|\mathbf{x}^*\|_2 + (6 + 2\sqrt{3})\epsilon. \tag{4.11}$$

4.3 Simulations

4.3.1 The Logistic Model

We investigate the performance of GraSP in the logistic model, which is broadly used in machine learning and statistics. In this model, a vector $\mathbf{a} \in \mathbb{R}^n$ of explanatory variables is associated with its label $y \in \{0, 1\}$ by the formula of conditional probability,

$$P(y|\mathbf{a};\mathbf{x}) = \frac{\exp(y\langle\mathbf{a},\mathbf{x}\rangle)}{1 + \exp(\langle\mathbf{a},\mathbf{x}\rangle)}, \tag{4.12}$$

where $\mathbf{x}$ denotes a target vector. For m independent data samples $\{(\mathbf{a}_i, y_i)\}_{i=1}^m$, we can write the joint likelihood as a function of $\mathbf{x}$ (the joint likelihood is the product of the probability (4.12), since the samples are independent). It is easier to maximize the log-likelihood or minimize the negative of the log-likelihood. Thus, the logistic loss,

$$f(\mathbf{x}) = -\frac{1}{m}\log\left(\prod_{i=1}^m P(y_i|\mathbf{a}_i)\right) = -\frac{1}{m}\sum_{i=1}^m \log P(y_i|\mathbf{a}_i). \tag{4.13}$$

Finally

$$f(\mathbf{x}) = \frac{1}{m}\sum_{i=1}^m \log(1 + \exp(\langle\mathbf{a}_i,\mathbf{x}\rangle)) - y_i\langle\mathbf{a}_i,\mathbf{x}\rangle. \tag{4.14}$$

The function (4.14) for the case $m < n$ is merely convex, consequently, they may exist multiple minima. To counter this problem, a regularization penalty is often used. The ℓ_2-norm is the most common regularization term. The cost function (4.14),

$$g(\mathbf{x}) = f(\mathbf{x}) + \frac{\eta}{2}\|\mathbf{x}\|_2^2, \tag{4.15}$$

for a constant η becomes strongly convex, thus, it has a unique minimum. For the ℓ_2-regularized logistic loss cost function, the SRH property can be verified [25].

4.3.2 Testing GraSP

In our simulation, the sparse target vector $\mathbf{x}^*$ has dimension $n = 1000$, and sparsity level $s = 10$, with its entries drawn independently from the standard Gaussian distribution. The measurement matrix is a random draw of a $m \times n$ Gaussian matrix. Each row $\mathbf{a}_i$ of the measurement matrix is associated with a label y_i via the relation (4.12).

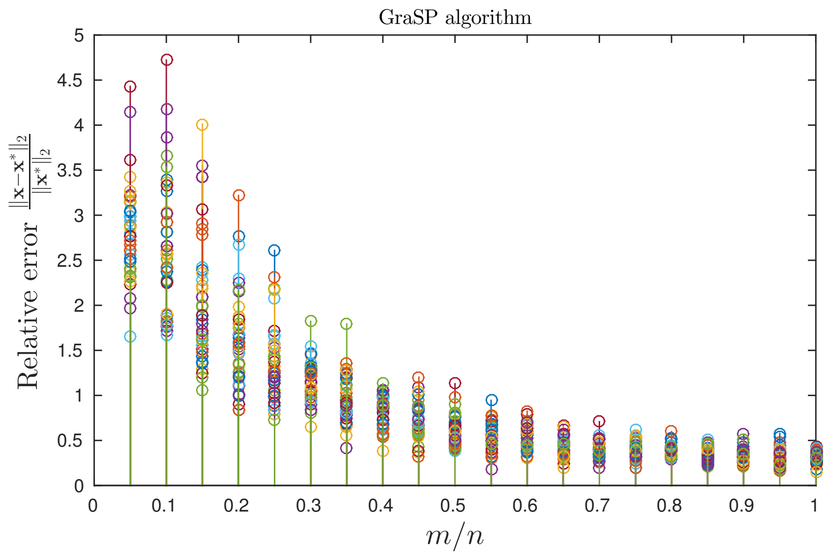
We tested the performance of GraSP algorithm in terms of relative signal estimation error and identification of the support of the target vector, via both the logistic loss cost function (4.14) and ℓ_2-regularized logistic loss cost function (4.15). Also, for the sake of comparison, we ran the LASSO, implemented in the GLMnet package [26] available for MATLAB. The regularization parameter η for the ℓ_2-regularized logistic loss is set to $\eta = (1 - \omega)\sqrt{\frac{\log(n)}{m}}$, with $\omega = 0.8$. For different values of m, namely $m = [50 : 50 : 1000]$, we create the data and run the algorithms for 40 Monte-Carlo trials.

The described process is similar to the one used in [25]. However, we chose to show the accuracy of identifying the target support, as it seems more intuitive in this context. We note that for the minimization step of GraSP in Algorithm 5, we used the gradient descent algorithm. Finally, as stopping criterion, we use a fixed number of iterations ($max\,iters = 100$) in conjunction with the norm of the gradient at each iteration.

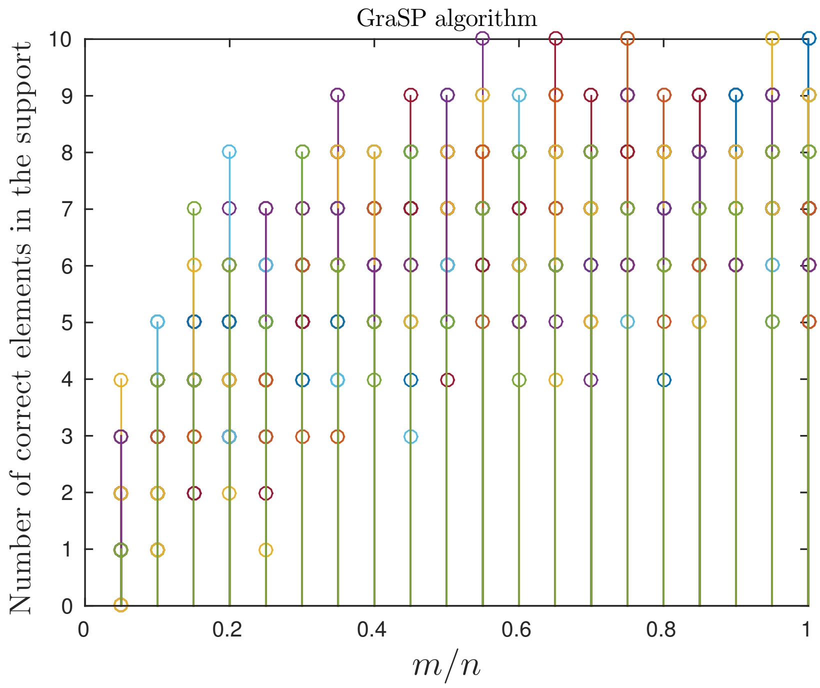
In Figures 4.1, 4.2 and 4.3, we show the performance of the aforementioned methods considering the relative signal estimation error and the accuracy of the estimated support. First, we observe that GraSP for (4.15) attains lower relative signal estimation error than GraSP for (4.14). Both algorithms attain the same accuracy in the estimation of the support. Second, GraSP for (4.15) and the GLMnet attain almost the same performance in terms of relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$. However, the GLMnet estimates the support better than GraSP for (4.15).

Summary

In this chapter, we examined the extension of Compressed Sensing for non-linear measurements. These days, huge amounts of data are collected, usually in a non-linear manner. Numerous scientific fields handle with this type of data and the development of methods that apply to this framework has become crucial. The GraSP algorithm provides an approximate solution to the problem (4.1). Both the properties that the cost function must satisfy



(a)



(b)

Figure 4.1: Relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$ and number of correct entries in the support versus the ratio (m/n) for the (4.14), using GraSP.

and the main theorems for the algorithm are introduced. Simulations showed that GraSP for (4.15) has comparable performance with the GLMnet, which is based on optimization methods. The results for the logistic model, are a strong motivation to examine the performance of GraSP for other Generalised Linear Models (GLMs).

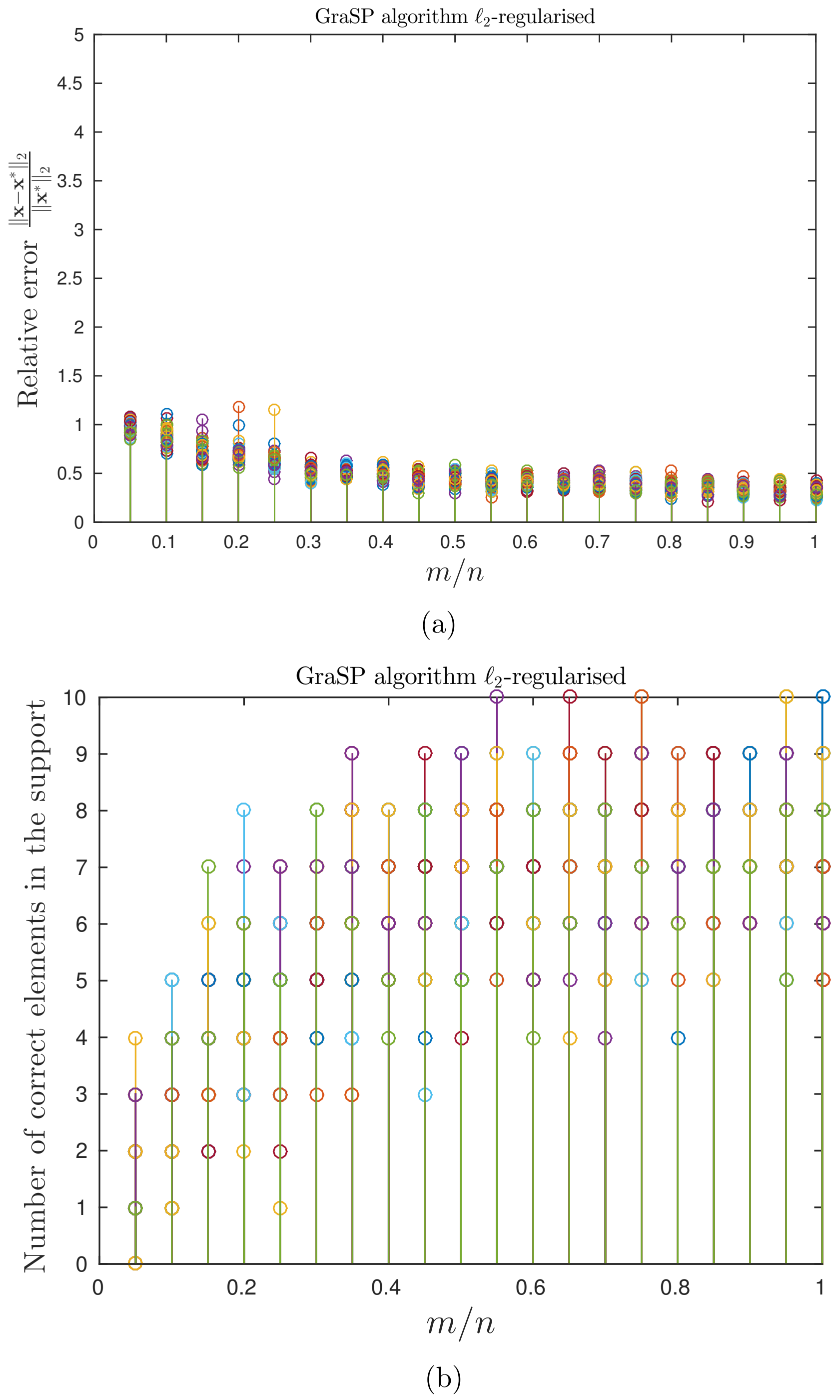


Figure 4.2: Relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$ and number of correct entries in the support versus the ratio (m/n) for the (4.15), using GraSP.

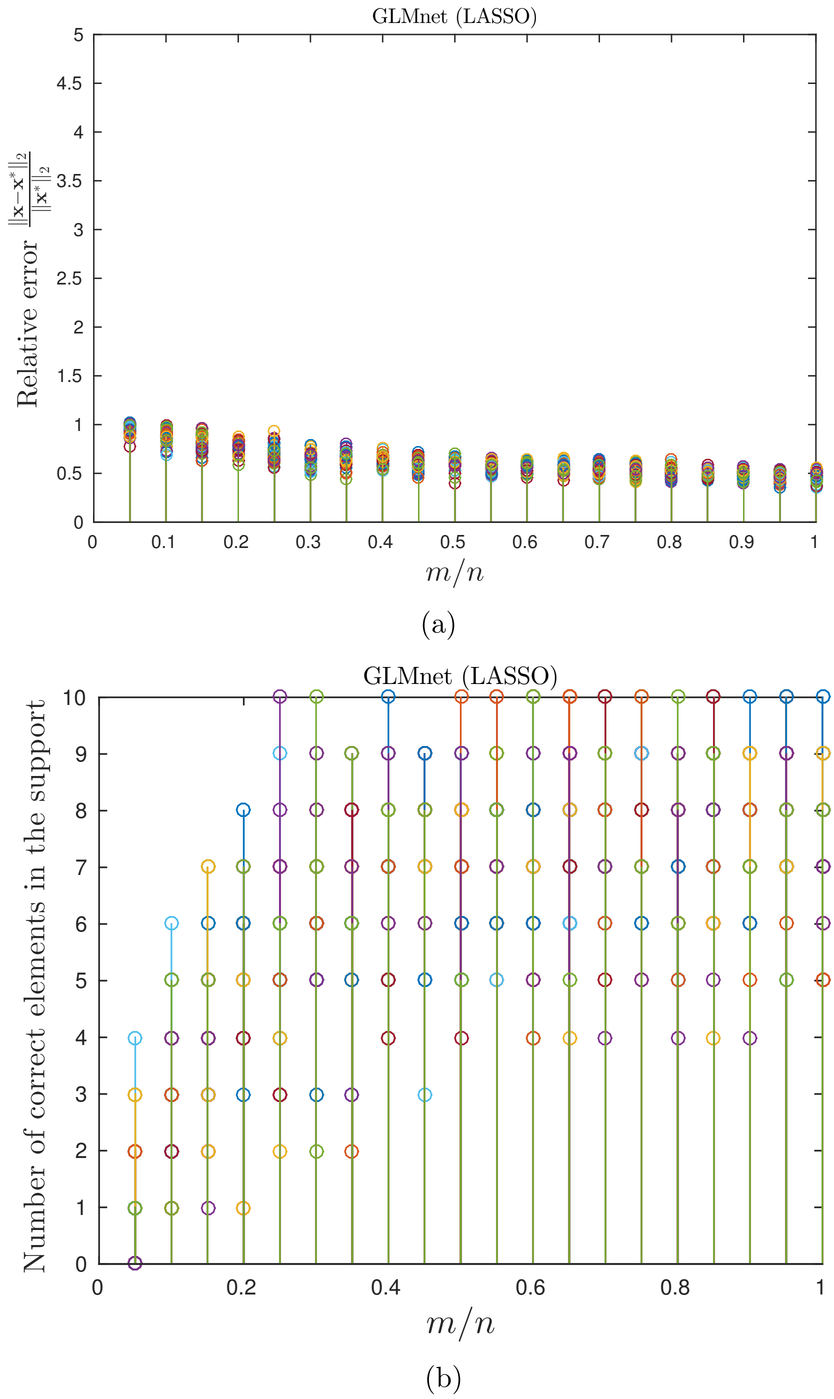


Figure 4.3: Relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$ and number of correct entries in the support versus the ratio (m/n) for GLMnet.

Chapter 5

General Method for sparse recovery from non-linear measurements

5.1 Motivation

In Chapter 4, we pointed out that GraSP is a generalization of CoSaMP for the non-linear framework. Via the experiments we conducted, we observed that not only this generalization is feasible, but also achieves performance similar to the state-of-the-art GLMnet, for some cases. Motivated by these results, we attempt to generalize another familiar greedy algorithm, the OMP algorithm, for the non-linear scenario, namely for the problem (4.1).

In Algorithm 6, we present OMP once again, with an interesting highlight though. Specifically, we notice that $\mathbf{g}^i = \mathbf{A}^T\mathbf{r}^{i-1}$ is actually the gradient at point $\mathbf{x}$, of the cost function $f(\mathbf{x}) = \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$ (line 3 of Algorithm 6). Hence, likewise GraSP, we will use the gradient, $\nabla f(\mathbf{x})$, to choose the directions of minimization for this variation of OMP. Since OMP chooses one element at each iteration, we select the index of the maximum (in magnitude) element of $\nabla f(\mathbf{x})$.

Unfortunately, the main property of OMP, that is, the orthogonalization, is not preserved in the non-linear case. To prevent multiple selections of the same element, we perform an extra check at point (!). More formally, in case one element is already in the support, the second largest in magnitude element is selected. Then, the selected index is merged with the support of the current estimate. Finally, for the minimization step, (line 6), we perform restricted gradient descent. The general method is described in Algorithm 7.

Algorithm 6: OMP Algorithm

Input: measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^m$, function $f(\mathbf{x}) = \|\mathbf{y} - \mathbf{Ax}\|_2^2$

initialization $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \mathbf{0}$

for *iteration i = 1 ; i = i + 1 till the halting criterion is met* **do**

$\quad \mathbf{g}^i = \mathbf{A}^T \mathbf{r}^{i-1} = \nabla f(\mathbf{x^{i-1}})$

$\quad j^i = \underset{j}{\operatorname{argmax}} |\mathbf{g}^i_j| / \|\mathbf{a}_j\|_2$

$\quad \mathcal{T}^i = \mathcal{T}^{i-1} \cup j^i$

$\quad \mathbf{x}^i_{\mathcal{T}^i} = \mathbf{A}^{\dagger}{}_{\mathcal{T}^i} \mathbf{y}, \quad \mathbf{x}^i_{\overline{\mathcal{T}^i}} = \mathbf{0}$

$\quad \mathbf{r}^i = \mathbf{y} - \mathbf{Ax}^i$

Output: $\mathbf{r}^i$, $\mathbf{x}^i$

Algorithm 7: General method for generic cost function

Input: function $f(\cdot)$

initialization $\mathbf{x}^0 = \mathbf{0}$

for *iteration i = 1 ; i = i + 1 till the halting criterion is met* **do**

$\quad \mathbf{g}^i = \nabla f(\mathbf{x^{i-1}})$

$\quad j^i \overset{!}{=} \underset{j}{\operatorname{argmax}} |\mathbf{g}^i_j|$

$\quad \mathcal{T}^i = \mathcal{T}^{i-1} \cup j^i$

$\quad \mathbf{b}^i = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x}), \quad \text{s.t.} \quad \mathbf{x}_{\overline{\mathcal{T}^i}} = \mathbf{0}$

$\quad \mathbf{x}^i = \mathbf{b}^i$

Output: $\mathbf{x}^i$

In Algorithm 7, at point (!) the extra check is performed, to avoid the selection of the same element in the support.

5.2 Simulations

We test the general method described above for the logistic loss cost function (4.14) and (4.15), described in Section 4.3.1. The data are created in the same manner, as described in Section 4.3.2.

At first, we test the performance of the algorithm for a single instance of the created data, namely, for a fixed m, several times. Quantity $\|\nabla f(\mathbf{x})\|_2$ is used as a stopping criterion of the method. We observe that the minimum signal estimation error was attained at s iterations of the algorithm, for all

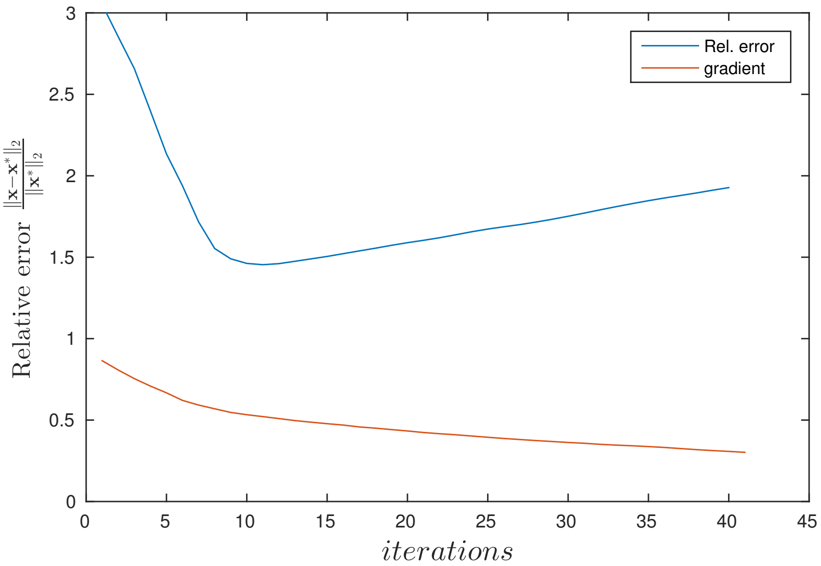


Figure 5.1: Gradient and relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$ vs iterations.

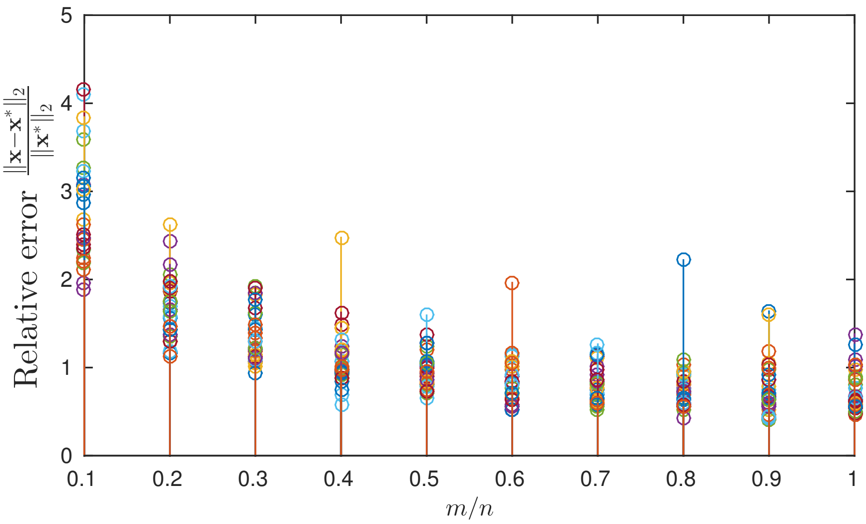
trials. We illustrate one of this experiments, specifically for $m = 400$, in Figure 5.1.

Hence, using this observation, we set $iterations = s$ as the stopping criterion of Algorithm 7. In Figure 5.1, we also observe that since the target $\mathbf{x}^*$ is not the minimizer of the (4.14), the gradient at the estimate $\mathbf{x}$ with the smallest relative signal estimation error is not zero.

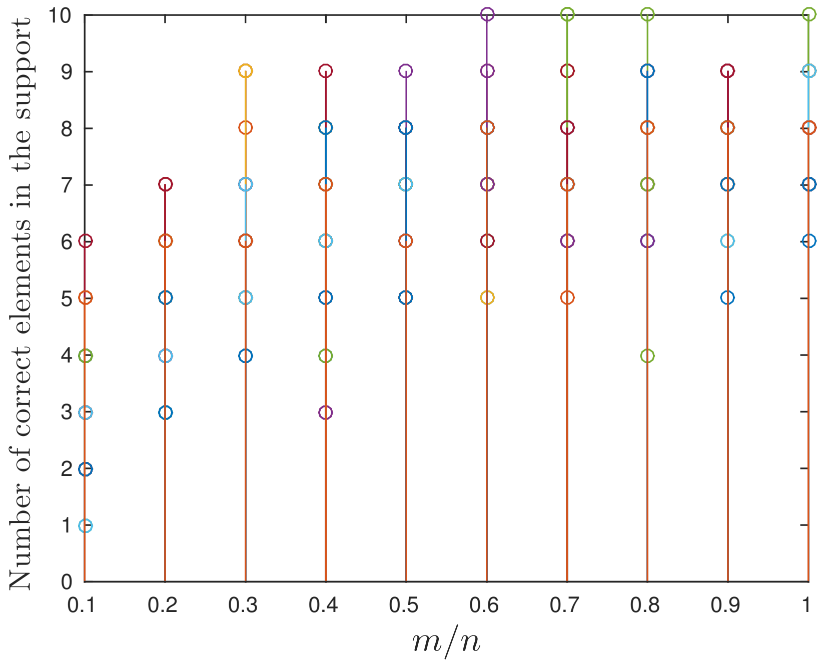
In the sequel, we proceed with testing the performance of Algorithm 7 for 30 Monte-Carlo trials, for different dimensions m. The results are collected and plotted below.

In Figures 5.2 and 5.3, we illustrate the performance of Algorithm 7 for the cost functions (4.14) and (4.15), respectively. At first glance, the proposed iterative method for the different cost functions attain similar performance in terms of relative signal estimation error. In Figures 5.2(b) and 5.3(b), we notice that the method is able to identify most elements of the target support even for small ratios (m/n), for some trials. For larger ratios the number of correct entries in the support also increases.

The performance illustrated in Figures 5.2(a) and 5.3(a) is similar to the performance of GraSP for (4.14), Figure 4.1(a). The proposed method for (4.14) and (4.15) is outperformed by GraSP + ℓ_2 penalty and GLMnet.



(a)

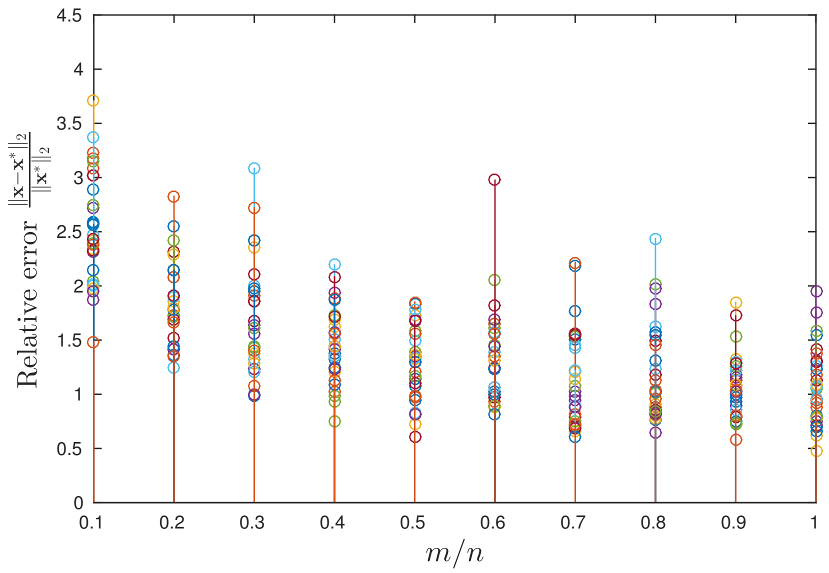


(b)

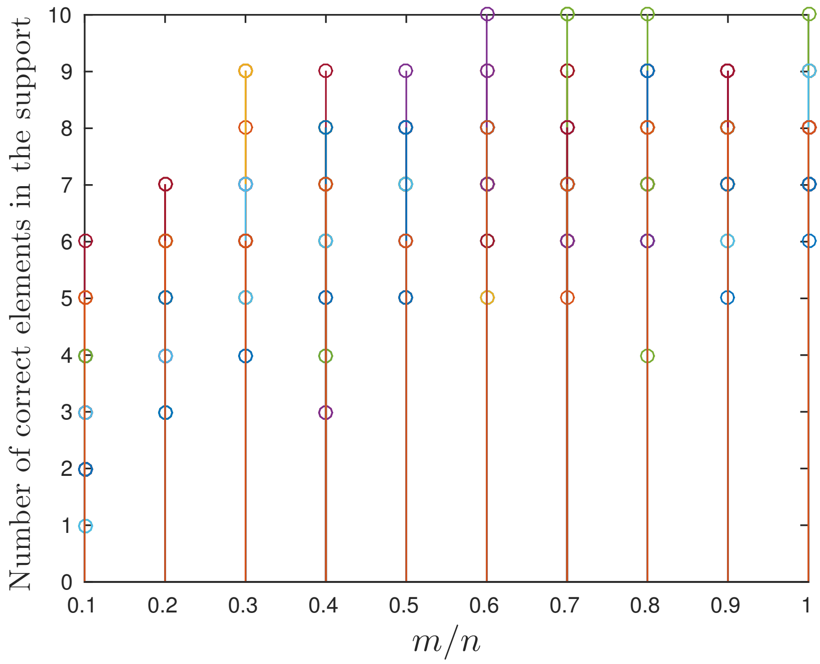
Figure 5.2: Relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$ and number of correct entries in the support versus the ratio (m/n), for Algorithm 7.

Summary

In this chapter, we have investigated the possibility of using a modification of OMP for non-linear Compressed Sensing, introduced in Chapter 4. The Figures 5.2 and 5.3 illustrate that this kind of extension is possible, however, the method is outperformed by the existing algorithms presented in the previous chapter. It would be interesting, though, to work on theoretical guarantees for successful sparse recovery, similar to those for GraSP in Section 4.2.1, for this iterative method. Finally, similar experiments using a class of optimization methods would be an intriguing field of research.



(a)



(b)

Figure 5.3: Relative error $\frac{\|\mathbf{x}-\mathbf{x}^*\|_2}{\|\mathbf{x}^*\|_2}$ and number of correct entries in the support versus the ratio (m/n), for Algorithm 7 + ℓ_2-penalty.

Chapter 6

Conclusion and Future work

In this thesis, we considered the problem of sparse signal reconstruction for both the linear (Compressed Sensing) and the non-linear settings, via greedy algorithms. For the linear model, we observed that the performance of the greedy algorithms we tested is aligned with the theoretical results. Specifically, we observed that, OMP and CoSaMP algorithms outperform MP and IHT. For the non-linear case, we focus on GraSP, a greedy algorithm inspired by CoSaMP. We concluded, through numerical experiments, that GraSP + ℓ_2 regularization attains similar performance with the state-of-the-art GLM-net. Finally, motivated by the results of GraSP, we modified OMP for the non-linear setting. The proposed method is outperformed by the existing algorithms on this framework.

Non-linear models are broadly used in the high-dimensional regime, however, they have attracted less attention than the familiar linear model. The results for the logistic model, though, are a strong motivation to examine the performance of GraSP for other Generalised Linear Models (GLMs). Also, it would be interesting to work on theoretical guarantees, similar to those for GraSP, for successful sparse recovery via the method we proposed in Chapter 5. Finally, similar experiments using a class of optimization methods, modified for this framework, would be an intriguing field of research.

Appendix A

Proofs of theorems

Here we present some proofs that were mentioned in the previous chapters.

Proofs chapter 3

Proof of theorem 3[15, p. 64]

Proof. First, we assume that the sparsest solution of the familiar to us linear system, has all its nonzero values at the beginning of the vector, sorted in *decreasing order* of values $|x_j|$. Consequently, we can write

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \sum_{k=1}^{s} x_k \mathbf{a}_k. \tag{A.1}$$

It is obvious that the measurement vector $\mathbf{y}$ is a linear combination of the first s columns of the matrix $\mathbf{A}$.

At the first iteration of the algorithm the residual is set $\mathbf{r}^0 = \mathbf{y}$. To require that OMP will choose one of the nonzero entries of the vector, we must have that for all $i > s$ (outside the support)

$$|\mathbf{a}_1^T \mathbf{y}| > |\mathbf{a}_i^T \mathbf{y}|, \tag{A.2}$$

holds. We use the definition of the linear system (A.1) in the relation (A.2) to obtain

$$\left| \sum_{k=1}^{s} x_k \mathbf{a}_1^T \mathbf{a}_k \right| > \left| \sum_{k=1}^{s} x_k \mathbf{a}_i^T \mathbf{a}_k \right|. \tag{A.3}$$

To proceed, we work in the same manner as in the two-ortho case in Chapter 3. We estimate a lower bound for the left-hand side and an upper bound for the right-hand side, respectively. First, we consider the left-hand side of (A.3)

$$\left| \sum_{k=1}^{s} x_k \mathbf{a}_1^T \mathbf{a}_k \right| \geq |x_1 \mathbf{a}_1^T \mathbf{a}_1| - \sum_{k=2}^{s} |x_k||\mathbf{a}_1^T \mathbf{a}_k|. \tag{A.4}$$

In the above inequality, we exploit the triangle inequality. Using the fact that the columns of matrix $\mathbf{A}$ are normalized, the definition (2.5) of mutual coherence and the descending order of the entries of $\mathbf{x}$, we obtain

$$|x_1\mathbf{a}_1^T\mathbf{a}_1| - \sum_{k=2}^{s} |x_k||\mathbf{a}_1^T\mathbf{a}_k| \geq |x_1| - \sum_{k=2}^{s} |x_k|\mu(\mathbf{A}) \geq |x_1|\left(1 - \mu(\mathbf{A})\left(s-1\right)\right). \tag{A.5}$$

Similarly, we treat the right-hand side of the inequality (A.3). We use the same tricks as before

$$\begin{aligned} \left|\sum_{k=1}^{s} x_k\mathbf{a}_i^T\mathbf{a}_k\right| &\leq \sum_{k=1}^{s} |x_k||\mathbf{a}_i^T\mathbf{a}_k| \\ &\leq \sum_{k=1}^{s} |x_k|\mu(\mathbf{A}) \\ &\leq |x_1|\mu(\mathbf{A})s. \end{aligned} \tag{A.6}$$

Combining the deduced bounds into (A.3), we have

$$|x_1|\left(1 - \mu(\mathbf{A})\left(s-1\right)\right) > |x_1|\mu(\mathbf{A})s. \tag{A.7}$$

The above implies

$$1 + \mu(\mathbf{A}) > 2\mu(\mathbf{A})s \Rightarrow s < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{A})}\right). \tag{A.8}$$

Satisfying condition (A.8), OMP succeeds in the first step of the algorithm. Next, the residual after its update is still a linear combination of the same s columns of $\mathbf{A}$. Obeying this condition in the next iterations, in conjunction with the orthogonality in the minimization step of Algorithm 2, OMP keeps selecting correct elements in the support. Finally it terminates when the residual is zero at s iterations.

□

Proof of Theorem 7[1, p. 165]

Proof. The inequality (3.36) states that the sequence $(\mathbf{x}^i)$ is bounded. Thus, using the Bolzano-Weierstrass Theorem, the existence of accumulation points is guaranteed.
In order to prove Theorem 7, we have to focus on (3.36). At first, some notation must be introduced, to facilitate the structure of the proof. The steps 4,5,6 of the algorithm in Algorithm 3 are merged to $(CoSaMP_1)$, step 7 is from now on $(CoSaMP_2)$ and finally $(CoSaMP_3)$ refers to step 8.

To prove the inequality (3.36), we work as follows. At first, we note that it is sufficient to prove the inequality for $i \geq 0$,

$$\|\mathbf{x}^{i+1} - \mathbf{x}_{\mathcal{S}}\|_2 \leq \rho\|\mathbf{x}^i - \mathbf{x}_{\mathcal{S}}\|_2 + (1-\rho)\tau\|\mathbf{A}\mathbf{x}_{\overline{\mathcal{S}}} + \mathbf{e}\|_2, \tag{A.9}$$

with $0 < \rho < 1$ and $\tau > 0$. The relation we have to prove is implied by induction. Using the definition of each step ($CoSaMP_1$), ($CoSaMP_2$), ($CoSaMP_3$) at a time, we obtain estimates that are eventually combined to produce the inequality (A.9). At first, we begin with the step ($CoSaMP_3$). We notice that, $\mathbf{x}^{i+1}$ is a better s-sparse approximation (or equally good at least) to $\mathbf{b}^{i+1}$ than $\mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}}$. We make the definition $\mathcal{T}^{i+1} = \mathrm{supp}(\mathbf{x}^{i+1})$, so it follows that $\mathcal{T}^{i+1} \subset \mathcal{S}^{i+1}$.

Thus, overall we have

$$\begin{aligned}
\|(\mathbf{x}_{\mathcal{S}} - \mathbf{x}^{i+1})_{\mathcal{S}^{i+1}}\|_2 &= \|\mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}} - \mathbf{x}^{i+1}\|_2 \\
&= \|\mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}} - \mathbf{b}^{i+1} - \mathbf{x}^{i+1} + \mathbf{b}^{i+1}\|_2 \\
&= \|\mathbf{b}^{i+1} - \mathbf{x}^{i+1} - (\mathbf{b}^{i+1} - \mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}})\|_2 \\
&\leq \|\mathbf{b}^{i+1} - \mathbf{x}^{i+1}\|_2 + |-1|\|\mathbf{b}^{i+1} - \mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}}\|_2 \\
&= \|\mathbf{b}^{i+1} - \mathbf{x}^{i+1}\|_2 + \|\mathbf{b}^{i+1} - \mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}}\|_2 \\
&\leq 2\|\mathbf{b}^{i+1} - \mathbf{x}_{\mathcal{S}\bigcap\mathcal{S}^{i+1}}\|_2 = 2\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2.
\end{aligned} \tag{A.10}$$

Using the fact that $(\mathbf{x}^{i+1}_{\overline{\mathcal{S}^{i+1}}}) = 0$ and $(\mathbf{b}^{i+1}_{\overline{\mathcal{S}^{i+1}}}) = 0$, because they are restricted on the indices with zero values (outside the support), we obtain

$$\begin{aligned}
\|\mathbf{x}_{\mathcal{S}} - \mathbf{x}^{i+1}\|_2^2 &= \|(\mathbf{x}_{\mathcal{S}} - \mathbf{x}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 + \|(\mathbf{x}_{\mathcal{S}} - \mathbf{x}^{i+1})_{\mathcal{S}^{i+1}}\|_2^2 \\
&\leq \|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 + 4\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2^2,
\end{aligned} \tag{A.11}$$

where in the final expression of (A.11), we used the estimate (A.10).

In the sequel, from ($CoSaMP_2$), we have for the vector $\mathbf{Ab}^{i+1}$

$$\langle \mathbf{y} - \mathbf{Ab}^{i+1}, \mathbf{Az} \rangle = 0 \quad \textit{whenever} \quad \mathrm{supp}(\mathbf{z}) \subseteq \mathcal{S}^{i+1}, \tag{A.12}$$

because the residual $\mathbf{y} - \mathbf{Ab}^{i+1}$ is orthogonal to vectors restricted on sets $\subseteq \mathcal{S}^{i+1}$. If we elaborate more on the previous relation, we obtain

$$\begin{aligned}
\langle \mathbf{y} - \mathbf{Ab}^{i+1}, \mathbf{Az} \rangle &= \mathbf{z}^H \mathbf{A}^H (\mathbf{y} - \mathbf{Ab}^{i+1}) \\
&= \langle \mathbf{A}^H (\mathbf{y} - \mathbf{Ab}^{i+1}), \mathbf{z} \rangle = 0.
\end{aligned} \tag{A.13}$$

We conclude by (A.13) that $\langle \mathbf{A}^H(\mathbf{y} - \mathbf{Ab}^{i+1}), \mathbf{z}\rangle = 0$, whenever $\mathrm{supp}(\mathbf{z}) \subseteq \mathcal{S}^{i+1}$, or $(\mathbf{A}^H(\mathbf{y}-\mathbf{Ab}^{i+1}))_{\mathcal{S}^{i+1}} = \mathbf{0}$. Having $\mathbf{y} = \mathbf{Ax}_{\mathcal{S}}+\mathbf{e}'$, where $\mathbf{e}' \coloneqq \mathbf{Ax}_{\overline{\mathcal{S}}}+\mathbf{e}$, we obtain

$$\begin{aligned}
(\mathbf{A}^H(\mathbf{Ax}_{\mathcal{S}} + \mathbf{e}' - \mathbf{Ab}^{i+1}))_{\mathcal{S}^{i+1}} = 0 &\Leftrightarrow (\mathbf{A}^H\mathbf{Ax}_{\mathcal{S}} + \mathbf{A}^H\mathbf{e}' - \mathbf{A}^H\mathbf{Ab}^{i+1})_{\mathcal{S}^{i+1}} = 0 \\
&\Leftrightarrow (\mathbf{A}^H\mathbf{Ax}_{\mathcal{S}} - \mathbf{A}^H\mathbf{Ab}^{i+1})_{\mathcal{S}^{i+1}} = -(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}} \\
&\Leftrightarrow (\mathbf{A}^H\mathbf{A}(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1}))_{\mathcal{S}^{i+1}} = -(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}.
\end{aligned} \tag{A.14}$$

Our aim is to provide an estimate for $\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2$ in terms of $\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2$. By adding the term $(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}$ to both sides of (A.14) and rearranging, we obtain

$$\begin{aligned}(\mathbf{A}^H\mathbf{A}(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1}))_{\mathcal{S}^{i+1}} + (\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}} &= -(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}} + (\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\\ (\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}} &= -(\mathbf{A}^H\mathbf{A}(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1}))_{\mathcal{S}^{i+1}} - (\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}} + (\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}.\end{aligned} \tag{A.15}$$

Next, by extracting the common term in the right-hand side of the deduced equality (A.15), and finally, taking norms, it follows that

$$\begin{aligned}\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 &= \|(\mathbf{Id} - \mathbf{A}^H\mathbf{A})((\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}) - (\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\\ &\leq \|(\mathbf{Id} - \mathbf{A}^H\mathbf{A})((\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}})\|_2 + \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\\ &\overset{(a)}{\leq} \delta_{4s}\|\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1}\|_2 + \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2,\end{aligned} \tag{A.16}$$

where at point (a), we used Lemma 8. We make the assumption, $\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 > \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2/(1-\delta_{4s})$. [1] Using this fact, more precisely that $\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 > \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2$ we elaborate on (A.16), as follows

$$\begin{aligned}\left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)^2 &\leq \left(\delta_{4s}\|\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1}\|_2\right)^2\\ &\leq \delta_{4s}^2\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2^2 + \delta_{4s}^2\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2.\end{aligned} \tag{A.17}$$

Handling the above using the common identity $a^2 - b^2 = (a+b)(a-b)$, it follows

$$\begin{aligned}\delta_{4s}^2\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 &\geq \left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)^2 - \delta_{4s}^2\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2^2\\ &\geq \left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2 + \delta_{4s}\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2\right)\\ &\times \left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2 - \delta_{4s}\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2\right)\\ &\geq (1+\delta_{4s})\left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \frac{1}{(1+\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)\\ &\times (1-\delta_{4s})\left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \frac{1}{(1-\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)\\ &\geq (1^2-\delta_{4s}^2)\left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \frac{1}{(1+\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)\\ &\times \left(\|(\mathbf{x}_{\mathcal{S}} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \frac{1}{(1-\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right).\end{aligned} \tag{A.18}$$

[1] In case the assumption is not true, the inequality (A.19), that is, the estimation we wish to deduce, would be trivial.

We observe that the middle term of the final expression of inequality (A.18), is larger than the last term. Using this observation, as well as dividing with the first term, we deduce

$$\frac{\delta_{4s}^2}{(1-\delta_{4s}^2)}\|(\mathbf{x}_\mathcal{S}-\mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 \geq \left(\|(\mathbf{x}_\mathcal{S}-\mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \frac{1}{(1-\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)^2.$$

In the sequel, by taking the square root of both sides, we obtain

$$\frac{\delta_{4s}}{\sqrt{(1-\delta_{4s}^2)}}\|(\mathbf{x}_\mathcal{S}-\mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2 \geq \left(\|(\mathbf{x}_\mathcal{S}-\mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 - \frac{1}{(1-\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right).$$

Finally, we have

$$\|(\mathbf{x}_\mathcal{S}-\mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2 \leq \frac{\delta_{4s}}{\sqrt{(1-\delta_{4s}^2)}}\|(\mathbf{x}_\mathcal{S}-\mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2 + \frac{1}{(1-\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2. \tag{A.19}$$

For the ($CoSaMP_1$) step, we work as follows. As $\mathcal{L}^{i+1}$, we will define the set of indices of the $2s$ largest absolute entries of $\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i)$. We have already defined that $\mathcal{T}^i$ is the support of the vector $\mathbf{x}^i$. Thus, the next inequality holds, since the restriction to $\mathcal{L}^{i+1}$ of $\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i)$, contains the $2s$ largest absolute values of the vector, while, the restriction to $\mathcal{T}^i \cup S$ contains at least the s largest (see step 7 of Algorithm 3).

Indeed,

$$\|(\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{T}^i\cup\mathcal{S}}\|_2^2 \leq \|(\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{L}^{i+1}}\|_2^2. \tag{A.20}$$

Next, we notice that if we subtract the intersection of the sets $\mathcal{T}^i$ and $\mathcal{L}^{i+1}$, the previous inequality still holds. As a matter of fact

$$\|(\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{T}^i\cup\mathcal{S}\setminus\mathcal{L}^{i+1}}\|_2 \leq \|(\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{L}^{i+1}\setminus\mathcal{T}^i\cup\mathcal{S}}\|_2. \tag{A.21}$$

For the left hand side, we derive by adding and subtracting the terms $\mathbf{x}^i, \mathbf{x}_\mathcal{S}$

$$\|(\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{(\mathcal{T}^i\cup\mathcal{S})\setminus\mathcal{L}^{i+1}}\|_2 \geq \|(\mathbf{x}^i-\mathbf{x}_\mathcal{S})_{\overline{\mathcal{L}^{i+1}}}\|_2 - \|(\mathbf{x}^i-\mathbf{x}_\mathcal{S}+\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{(\mathcal{T}^i\cup\mathcal{S})\setminus\mathcal{L}^{i+1}}\|_2. \tag{A.22}$$

The right hand of the equation (A.21) satisfies,

$$\|(\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{L}^{i+1}\setminus(\mathcal{T}^i\cup\mathcal{S})}\|_2 = \|(\mathbf{x}^i-\mathbf{x}_\mathcal{S}+\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{L}^{i+1}\setminus(\mathcal{T}^i\cup\mathcal{S})}\|_2, \tag{A.23}$$

because $(\mathbf{x}^i-\mathbf{x}_\mathcal{S})_{\mathcal{L}^{i+1}\setminus\mathcal{T}^i\cup\mathcal{S}} = \mathbf{0}$. Now, rearranging the inequality (A.22) and using (A.21) and (A.23), we deduce

$$\begin{aligned}
\|(\mathbf{x}_\mathcal{S}-\mathbf{x}^i)_{\overline{\mathcal{L}^{i+1}}}\|_2 &\leq \|(\mathbf{x}^i-\mathbf{x}_\mathcal{S}+\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{(\mathcal{T}^i\cup\mathcal{S})\setminus\mathcal{L}^{i+1}}\|_2 \\
&+ \|(\mathbf{x}^i-\mathbf{x}_\mathcal{S}+\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{\mathcal{L}^{i+1}\setminus(\mathcal{T}^i\cup\mathcal{S})}\|_2 \\
&\overset{!}{\leq} \sqrt{2}\|(\mathbf{x}^i-\mathbf{x}_\mathcal{S}+\mathbf{A}^H(\mathbf{y}-\mathbf{A}\mathbf{x}^i))_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}}\|_2 \\
&\leq \sqrt{2}\left\|\left((\mathbf{Id}-\mathbf{A}^H\mathbf{A})(\mathbf{x}^i-\mathbf{x}_\mathcal{S})\right)_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}}\right\|_2 \\
&+ \sqrt{2}\|(\mathbf{A}^H\mathbf{e}')_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}}\|_2.
\end{aligned} \tag{A.24}$$

The set $(\mathcal{T}^i \cup \mathcal{S})\triangle\mathcal{L}^{i+1}$ represents the symmetric difference of the two sets, $(\mathcal{L}^{i+1} \setminus (\mathcal{T}^i \cup \mathcal{S}))\cup((\mathcal{T}^i \cup \mathcal{S}) \setminus \mathcal{L}^{i+1}))$. Observe that $(\mathcal{L}^{i+1} \setminus (\mathcal{T}^i \cup \mathcal{S}))\cap((\mathcal{T}^i \cup \mathcal{S}) \setminus \mathcal{L}^{i+1})) = \emptyset$. The sets where the vectors $\left(\mathbf{x}^i - \mathbf{x}_\mathcal{S} + \mathbf{A}^H(\mathbf{y} - \mathbf{A}\mathbf{x}^i)\right)_{(\mathcal{T}^i\cup\mathcal{S})\setminus\mathcal{L}^{i+1}}$ and $\left(\mathbf{x}^i - \mathbf{x}_\mathcal{S} + \mathbf{A}^H(\mathbf{y} - \mathbf{A}\mathbf{x}^i)\right)_{\mathcal{L}^{i+1}\setminus(\mathcal{T}^i\cup\mathcal{S})}$ are supported, are consequently disjoint. Thus, using a version of the Pythagorean theorem at point (!) of (A.24), we conclude in the final expression (one may see the analogy in the familiar Cartesian coordinate system). In the sequel, the equality $\mathbf{y} = \mathbf{A}\mathbf{x}_\mathcal{S} + \mathbf{e}'$ is used. Finally, having $\mathcal{L}^{i+1} \subset \mathcal{S}^{i+1}$ and $\mathcal{T}^{i+1} \subseteq \mathcal{S}^{i+1}$ we derive a lower bound for left hand-side of (A.24)

$$\|(\mathbf{x}_\mathcal{S} - \mathbf{x}^i)_{\overline{\mathcal{L}^{i+1}}}\|_2 \geq \|(\mathbf{x}_\mathcal{S} - \mathbf{x}^i)_{\overline{\mathcal{S}^{i+1}}}\|_2 = \|(\mathbf{x}_\mathcal{S})_{\overline{\mathcal{S}^{i+1}}}\|_2 = \|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2. \tag{A.25}$$

Thus, combining (A.24) and (A.25), we obtain

$$\begin{aligned}
\|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2 &\leq \sqrt{2}\| \left((\mathbf{Id} - \mathbf{A}^H\mathbf{A})(\mathbf{x}^i - \mathbf{x}^\mathcal{S})\right)_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}} \|_2 \\
&\quad + \sqrt{2}\|(\mathbf{A}^H\mathbf{e}')_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}}\|_2 \\
&\overset{(b)}{\leq} \sqrt{2}\delta_{4s}\|\mathbf{x}^i - \mathbf{x}_\mathcal{S}\|_2 + \sqrt{2}\|(\mathbf{A}^H\mathbf{e}')_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}}\|_2,
\end{aligned} \tag{A.26}$$

where at point (b), Lemma 8 is used.

Finally, having derived relations for $(CoSaMP_1)$, $(CoSaMP_2)$, $(CoSaMP_3)$ the thing that remains is to combine them. At first, by applying the relation (A.19) to (A.11) and then the identity $a^2 + (b^2 + c^2) \leq (\sqrt{a^2 + b^2} + c)^2$ we obtain

$$\begin{aligned}
\|\mathbf{x}_\mathcal{S} - \mathbf{x}^{i+1}\|_2^2 &\leq \|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 + 4\|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\mathcal{S}^{i+1}}\|_2^2 \\
&\leq \|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 \\
&\quad + 4\left(\frac{\delta_{4s}}{\sqrt{(1-\delta_{4s}^2)}}\|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2 + \frac{1}{(1-\delta_{4s})}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)^2 \\
&\leq \left(\sqrt{\|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2 + \frac{4\delta_{4s}^2}{1-\delta_{4s}^2}\|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2^2} + \frac{2}{1-\delta_{4s}}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)^2 \\
&\leq \left(\sqrt{\frac{1+3\delta_{4s}^2}{1-\delta_{4s}^2}}\|(\mathbf{x}_\mathcal{S} - \mathbf{b}^{i+1})_{\overline{\mathcal{S}^{i+1}}}\|_2 + \frac{2}{1-\delta_{4s}}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2\right)^2.
\end{aligned} \tag{A.27}$$

Taking the square root in the above inequality and applying (A.26), we deduce

$$\begin{aligned}\|\mathbf{x}_{\mathcal{S}} - \mathbf{x}^{i+1}\|_2 \leq &\sqrt{\frac{2\delta_{4s}^2(1+3\delta_{4s}^2)}{1-\delta_{4s}^2}}\|\mathbf{x}^i - \mathbf{x}_{\mathcal{S}}\|_2 + \sqrt{\frac{2(1+3\delta_{4s}^2)}{1-\delta_{4s}^2}}\|(\mathbf{A}^H\mathbf{e}')_{(\mathcal{T}^i\cup\mathcal{S})\triangle\mathcal{L}^{i+1}})\|_2 \\ &+ \frac{2}{1-\delta_{4s}}\|(\mathbf{A}^H\mathbf{e}')_{\mathcal{S}^{i+1}}\|_2 \\ &\overset{(c)}{\leq} \sqrt{\frac{2\delta_{4s}^2(1+3\delta_{4s}^2)}{1-\delta_{4s}^2}}\|\mathbf{x}^i - \mathbf{x}_{\mathcal{S}}\|_2 + \sqrt{\frac{2(1+3\delta_{4s}^2)}{1-\delta_{4s}^2}}\sqrt{1+\delta_{4s}}\|\mathbf{e}\|_2 \\ &+ \frac{2}{1-\delta_{4s}}\sqrt{1+\delta_{4s}}\|\mathbf{e}\|_2,\end{aligned} \tag{A.28}$$

where at point (c), we used Lemma 9. If we set

$$\begin{aligned}\rho &= \sqrt{\frac{2\delta_{4s}^2(1+3\delta_{4s}^2)}{1-\delta_{4s}^2}} \\ (1-\rho)\tau &= \sqrt{\frac{2(1+3\delta_{4s}^2)}{1-\delta_{4s}^2}}\sqrt{1+\delta_{4s}} + \frac{2}{1-\delta_{4s}}\sqrt{1+\delta_{4s}},\end{aligned} \tag{A.29}$$

the inequality (A.9) holds. The constant ρ must be smaller than one. By solving the inequality $6\delta^2 + 3\delta - 1 < 0$, we observe that it holds when $\frac{\left(-1-\sqrt{\frac{11}{3}}\right)}{4} < \delta < \frac{\left(\sqrt{\frac{11}{3}}-1\right)}{4}$ (between the solutions). Thus, we conclude that requiring $0 < \delta_{4s}^2 < \frac{\left(\sqrt{\frac{11}{3}}-1\right)}{4}$, the inequality holds, leading to the initial statement. □

Proofs chapter 4

Proof of Theorem 9[25, p. 85]

In the beginning, four intermediate results are presented with their proofs, which are then used to prove the first of the main results of Chapter 4.

Proposition 3. *Let* $\mathbf{M}(t)$ *be a matrix-valued function such that for all* $t \in [0,1]$, $\mathbf{M}(t)$ *is symmetric and its eigenvalues lie in interval* $[B(t), A(t)]$ *with* $B(t) > 0$. *Then for any vector* $\mathbf{v}$ *we have*

$$\left(\int_0^1 B(t)dt\right)\|\mathbf{v}\|_2 \leq \left\|\left(\int_0^1 \mathbf{M}(t)dt\right)\mathbf{v}\right\|_2 \leq \left(\int_0^1 A(t)dt\right)\|\mathbf{v}\|_2. \tag{A.30}$$

Proof. At first, denote as λ_{max} and λ_{min} the largest and smallest eigenvalue functions for a set of symmetric positive-semidefinite matrices. At first, the function λ_{max} is convex in virtue of Weyl's inequalities.[2] Also, since $t \in [0, 1]$, the Jensen inequality (integral form) can be exploited. Thus

$$\lambda_{max}\left(\int_0^1 \mathbf{M}(t)dt\right) \leq \int_0^1 \lambda_{max}(\mathbf{M}(t))dt \leq \int_0^1 A(t)dt, \tag{A.31}$$

where $A(t)$ is the possible maximum eigenvalue of $\mathbf{M}(t)$. Moreover we have

$$\left\|\left(\int_0^1 \mathbf{M}(t)dt\right)\mathbf{v}\right\|_2 \leq \lambda_{max}\left(\int_0^1 \mathbf{M}(t)dt\right)\|\mathbf{v}\|_2. \tag{A.32}$$

Combining the three above inequalities the right hand-side of the target inequality 3 is implied. The left-hand side is proved in a similar manner using though the fact that λ_{min} is concave. $\square$

Proposition 4. *Let* $\mathbf{M}(t)$ *be a matrix-valued function such that for all* $t \in [0, 1]$, $\mathbf{M}(t)$ *is symmetric and its eigenvalues lie in interval* $[B(t), A(t)]$ *with* $B(t) > 0$. *If* Λ *is a subset of a row/column indices of* $\mathbf{M}(\cdot)$, *then for any vector* $\mathbf{v}$ *we have*

$$\left\|\left(\int_0^1 \mathbf{P}_\Lambda^T \mathbf{M}(t)\mathbf{P}_{\overline{\Lambda}}dt\right)\mathbf{v}\right\|_2 \leq \int_0^1 \frac{A(t) - B(t)}{2}dt\|\mathbf{v}\|_2. \tag{A.33}$$

Proof. The matrix $\mathbf{M}(t)$ is symmetric, hence it is also diagonalizable. So, for every vector $\mathbf{v}$ holds

$$B(t)\|\mathbf{v}\|_2^2 \leq \mathbf{v}^T\mathbf{M}(t)\mathbf{v} \leq A(t)\|\mathbf{v}\|_2^2. \tag{A.34}$$

Now, for the diagonalizable matrix $\left(\mathbf{M}(t) - \dfrac{A(t)+B(t)}{2}\mathbf{Id}\right)$, derives from

[2] For a more formal explanation, one can visit `https://terrytao.wordpress.com/2010/01/12/254a-notes-3a-eigenvalues-and-sums-of-hermitian-matrices/`

the previous inequality

$$B(t) \leq \frac{\mathbf{v}^T\mathbf{M}(t)\mathbf{v}}{\|\mathbf{v}\|_2^2} \leq A(t)$$

$$B(t) - \frac{A(t)+B(t)}{2} \leq \frac{\mathbf{v}^T\left(\mathbf{M}(t) - \frac{A(t)+B(t)}{2}\mathbf{Id}\right)\mathbf{v}}{\|\mathbf{v}\|_2^2} \leq A(t) - \frac{A(t)+B(t)}{2}$$

$$-\frac{A(t)-B(t)}{2} \leq \frac{\mathbf{v}^T\left(\mathbf{M}(t) - \frac{A(t)+B(t)}{2}\mathbf{Id}\right)\mathbf{v}}{\|\mathbf{v}\|_2^2} \leq \frac{A(t)-B(t)}{2}.$$

Next, let $\widehat{\mathbf{M}}(t) = \left(\mathbf{P}_\Lambda^T\left(\mathbf{M}(t) - \frac{A(t)+B(t)}{2}\mathbf{Id}\right)\mathbf{P}_{\overline{\Lambda}}\right)$. Since $\widehat{\mathbf{M}}(t)$ is a sub-matrix of $\left(\mathbf{M}(t) - \frac{A(t)+B(t)}{2}\mathbf{Id}\right)$, we obtain

$$\|\widehat{\mathbf{M}}(t)\| \leq \left\|\left(\mathbf{M}(t) - \frac{A(t)+B(t)}{2}\mathbf{Id}\right)\right\| \leq \frac{A(t)-B(t)}{2} \tag{A.35}$$

Finally, using the convexity of the norm, Jensen's inequality gives

$$\left\|\left(\int_0^1 \widehat{\mathbf{M}}(t)dt\right)\right\| \leq \int_0^1 \|\widehat{\mathbf{M}}(t)\|dt \leq \int_0^1 \frac{A(t)-B(t)}{2}dt \tag{A.36}$$

□

To proceed this notation is used for simplicity:

$$\begin{aligned}
\alpha_k(\mathbf{p},\mathbf{q}) &= \int_0^1 A_k(t\mathbf{q} + (1-t)\mathbf{p})dt, \\
\beta_k(\mathbf{p},\mathbf{q}) &= \int_0^1 B_k(t\mathbf{q} + (1-t)\mathbf{p})dt, \\
\gamma_k(\mathbf{p},\mathbf{q}) &= \alpha_k(\mathbf{p},\mathbf{q}) - \beta_k(\mathbf{p},\mathbf{q}),
\end{aligned}$$

where $A(t)$,$B(t)$ are defined in Chapter 4. This notation will be used in the following proofs.

Lemma 2. *Let* $\mathcal{R}$ *denote the set* $\operatorname{supp}(\widehat{\mathbf{x}} - \mathbf{x}^*)$. *The current estimate* $\widehat{\mathbf{x}}$ *then satisfies*

$$
\begin{aligned}
\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\overline{\mathcal{Z}}}\|_2 \leq & \frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)+\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}\|\widehat{\mathbf{x}}-\mathbf{x}^*\|_2 \\
& + \frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}+\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}.
\end{aligned}
$$

Proof. From the Algorithm 5, $\mathcal{Z}$ =supp($\mathbf{z}_{2s}$) and card($\mathcal{R}$) $\leq 2s$. Thus, we obtain

$$\|\mathbf{z}_{\mathcal{R}}\|_2 \leq \|\mathbf{z}_{\mathcal{Z}}\|_2.$$

Since $\mathcal{Z}$ is the set of the indices with the largest in magnitude elements from the vector $\mathbf{z}$, it is implied

$$\|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \leq \|\mathbf{z}_{\mathcal{Z}\setminus\mathcal{R}}\|_2. \tag{A.37}$$

From the step 3 of the algorithm, $z = \nabla f(\widehat{\mathbf{x}})$, the left-hand side of the inequality (A.37) becomes

$$
\begin{aligned}
\|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 &= \|\nabla f(\widehat{\mathbf{x}})_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
&= \|\nabla f(\widehat{\mathbf{x}})_{\mathcal{R}\setminus\mathcal{Z}} + \nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}} - \nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
&\geq \|\nabla f(\widehat{\mathbf{x}})_{\mathcal{R}\setminus\mathcal{Z}} - \nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 - \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2.
\end{aligned}
$$

The first term of the previous expression, is a difference of gradients at points $\mathbf{x}^*$,$\widehat{\mathbf{x}}$. By exploiting the the fundamental theorem of calculus

$$\|\nabla f(\widehat{\mathbf{x}})_{\mathcal{R}\setminus\mathcal{Z}} - \nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 = \left\| \int_0^1 \mathbf{P}^T_{\mathcal{R}\setminus\mathcal{Z}} \nabla^2 f(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)(\widehat{\mathbf{x}}-\mathbf{x}^*)dt \right\|_2, \tag{A.38}$$

thus, we obtain

$$
\begin{aligned}
\|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \geq & \left\| \int_0^1 \mathbf{P}^T_{\mathcal{R}\setminus\mathcal{Z}} \nabla^2 f(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)(\widehat{\mathbf{x}}-\mathbf{x}^*)dt \right\|_2 - \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
\geq & \left\| \int_0^1 \mathbf{P}^T_{\mathcal{R}\setminus\mathcal{Z}} \nabla^2 f(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)\mathbf{P}_{\mathcal{R}\setminus\mathcal{Z}}(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}dt \right\|_2 - \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
- & \left\| \int_0^1 \mathbf{P}^T_{\mathcal{R}\setminus\mathcal{Z}} \nabla^2 f(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)\mathbf{P}_{(\mathcal{R}\cap\mathcal{Z})}(\widehat{\mathbf{x}}-\mathbf{x}^*)_{(\mathcal{R}\cap\mathcal{Z})}dt \right\|_2,
\end{aligned}
$$

where we split the set $\mathcal{R}$ to the sets $\mathcal{R}\setminus\mathcal{Z}$ and $\mathcal{R}\cap\mathcal{Z}$. Then, we used the triangle inequality to derive the final inequality. Finally, using the the Proposition 4 and the notation given above, we derive

$$\begin{aligned}
\|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \geq & \left(\int_0^1 B_{2s}(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)dt\right)\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
& - \int_0^1 \frac{A_{2s}(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*) - B_{2s}(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)}{2}dt\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}\cap\mathcal{Z}}\|_2 - \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
& = \beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 - \frac{\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}\cap\mathcal{Z}}\|_2 - \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
& \geq \beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 - \frac{\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 - \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2.
\end{aligned}$$

For the right-hand side of the inequality (A.37), we have

$$\begin{aligned}
\|\mathbf{z}_{\mathcal{Z}\setminus\mathcal{R}}\|_2 &= \|\nabla f(\widehat{\mathbf{x}})_{\mathcal{Z}\setminus\mathcal{R}}\|_2 \\
&= \|\nabla f(\widehat{\mathbf{x}})_{\mathcal{Z}\setminus\mathcal{R}} + \nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}} - \nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 \\
&\leq \|\nabla f(\widehat{\mathbf{x}})_{\mathcal{Z}\setminus\mathcal{R}} + \nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 \\
&= \left\|\int_0^1 \mathbf{P}_{\mathcal{Z}\setminus\mathcal{R}}^T \nabla^2 f(t\widehat{\mathbf{x}}+(1-t)\mathbf{x}^*)\mathbf{P}_{\mathcal{R}}^T(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}}dt\right\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 \\
&\leq \frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\mathcal{R}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 \\
&= \frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2.
\end{aligned}$$

Finally, by combining the inequalities for each side, we obtain

$$\begin{aligned}
\frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 &\geq \|\mathbf{z}_{\mathcal{Z}\setminus\mathcal{R}}\|_2 \\
&\geq \|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \\
&\geq \beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{(\mathcal{R}\setminus\mathcal{Z})}\|_2 - \frac{\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 \\
&- \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2.
\end{aligned} \tag{A.39}$$

With some reordering of (A.39), we deduce

$$\begin{aligned}
\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{(\mathcal{R}\setminus\mathcal{Z})}\|_2 \leq & \frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2 + \frac{\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 \\
& + \|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2.
\end{aligned}$$

Observe that $\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{(\mathcal{R}\setminus\mathcal{Z})}\|_2 = \|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\overline{\mathcal{Z}}}\|_2$, since $\mathcal{R} = \mathrm{supp}(\widehat{\mathbf{x}}-\mathbf{x}^*)$. Thus, we derive

$$\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\overline{\mathcal{Z}}}\|_2 \leq \frac{\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)+\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)\|_2 + \frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}. \tag{A.40}$$

The desired result is deduced. □

Lemma 3. *The vector* $\mathbf{b}$ *given by*

$$\mathbf{b} = \text{argmin} f(\mathbf{x}), \tag{A.41}$$
$$\text{s.t.} \quad \mathbf{x}_{\overline{\mathcal{T}}} = \mathbf{0}, \tag{A.42}$$

satisfies

$$\|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2 \leq \frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{T}}\|_2}{\beta_{4s}(\mathbf{b}, \mathbf{x}^*)} + \frac{\gamma_{4s}(\mathbf{b}, \mathbf{x}^*)}{2\beta_{4s}(\mathbf{b}, \mathbf{x}^*)}\|\mathbf{x}^*_{\overline{\mathcal{T}}}\|_2. \tag{A.43}$$

Proof. At first, the fundamental theorem of calculus is used, namely,

$$\nabla f(\mathbf{x}^*) - \nabla f(\mathbf{b}) = \int_0^1 \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})(\mathbf{x}^* - \mathbf{b}) dt. \tag{A.44}$$

Since $\mathbf{b}$ is the solution of (A.41) optimization problem, it is necessary that $\nabla f(\mathbf{b})_{\mathcal{T}} = \mathbf{0}$. So, back to (A.44),

$$\begin{aligned}
\nabla f(\mathbf{x}^*)_{\mathcal{T}} &= \left(\int_0^1 \mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b}) dt\right)(\mathbf{x}^* - \mathbf{b}) \\
&= \left(\int_0^1 \mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}} dt\right)(\mathbf{x}^* - \mathbf{b})_{\mathcal{T}} \\
&+ \left(\int_0^1 \mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\overline{\mathcal{T}}} dt\right)(\mathbf{x}^* - \mathbf{b})_{\overline{\mathcal{T}}}.
\end{aligned} \tag{A.45}$$

From the assumptions of the main theorem, f has $\mu_{4s} - SRH$ and card($\mathcal{T} \cup$ supp$(t\mathbf{x}^* + (1-t)\mathbf{b})) \leq 4s$ for all $t \in [0,1]$. Thus, using (4.2), (4.3), we can write

$$B_{4s}(t\mathbf{x}^* + (1-t)\mathbf{b}) \leq \lambda_{min}(\mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}})$$
$$\beta_{4s}(\mathbf{b}, \mathbf{x}^*) \leq \int_0^1 \lambda_{min}(\mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}}) \leq \lambda_{min}\left(\int_0^1 \mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}} dt\right)$$

where in the last expression, the use Proposition 3. Similarly

$$\alpha_{4s}(\mathbf{b}, \mathbf{x}^*) \geq \int_0^1 \lambda_{max}(\mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}}) \geq \lambda_{max}\left(\int_0^1 \mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}} dt\right). \tag{A.46}$$

Since 0 is not an eigenvalue of the matrix $\mathbf{W} = \left(\int_0^1 \mathbf{P}^T_{\mathcal{T}} \nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}} dt\right)$, the matrix is invertible, and in fact positive-definite. Thus, for the inverse matrix holds

$$\frac{1}{\alpha_{4s}(\mathbf{b}, \mathbf{x}^*)} \leq \lambda_{min}(\mathbf{W}^{-1}) \leq \lambda_{max}(\mathbf{W}^{-1}) \leq \frac{1}{\beta_{4s}(\mathbf{b}, \mathbf{x}^*)}. \tag{A.47}$$

By multiplying both sides of (A.45) with the inverse, we obtain

$$\begin{aligned}\mathbf{W}^{-1}\nabla f(\mathbf{x}^*)_{\mathcal{T}} &= \mathbf{W}^{-1}\left(\int_0^1 \mathbf{P}_{\mathcal{T}}^T\nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\mathcal{T}}dt\right)(\mathbf{x}^* - \mathbf{b})_{\mathcal{T}} \\ &\quad + \mathbf{W}^{-1}\left(\int_0^1 \mathbf{P}_{\mathcal{T}}^T\nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\overline{\mathcal{T}}}dt\right)\mathbf{x}^*_{\overline{\mathcal{T}}} \\ &= (\mathbf{x}^* - \mathbf{b})_{\mathcal{T}} + \mathbf{W}^{-1}\left(\int_0^1 \mathbf{P}_{\mathcal{T}}^T\nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\overline{\mathcal{T}}}dt\right)\mathbf{x}^*_{\overline{\mathcal{T}}},\end{aligned}$$

where the vector $\mathbf{b}_{\overline{\mathcal{T}}} = \mathbf{0}$. Denote $\mathcal{S}^* = \mathrm{supp}(\mathbf{x}^*)$. We use the triangle inequality, the relation (A.47) and finally Proposition 4 to obtain

$$\begin{aligned}\|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2 &= \|\mathbf{W}^{-1}\nabla f(\mathbf{x}^*)_{\mathcal{T}}\|_2 - \mathbf{W}^{-1}\left(\int_0^1 \mathbf{P}_{\mathcal{T}}^T\nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\overline{\mathcal{T}}}dt\right)\mathbf{x}^*_{\overline{\mathcal{T}}} \\ &\le \|\mathbf{W}^{-1}\nabla f(\mathbf{x}^*)_{\mathcal{T}}\|_2 + \mathbf{W}^{-1}\left(\int_0^1 \mathbf{P}_{\mathcal{T}}^T\nabla^2 f(t\mathbf{x}^* + (1-t)\mathbf{b})\mathbf{P}_{\overline{\mathcal{T}}\cap\mathcal{S}^*}dt\right)\mathbf{x}^*_{\overline{\mathcal{T}}\cap\mathcal{S}^*} \\ &\le \frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{T}}\|_2}{\beta_{4s}(\mathbf{b},\mathbf{x}^*)} + \frac{\gamma_{4s}(\mathbf{b},\mathbf{x}^*)}{2\beta_{4s}(\mathbf{b},\mathbf{x}^*)}\|\mathbf{x}^*_{\overline{\mathcal{T}}}\|_2.\end{aligned}$$

□

Lemma 4. *(Iteration Invariant). The estimation error in the current iteration, $\|\widehat{\mathbf{x}} - \mathbf{x}^*\|_2$ and that in the next iteration, $\|\mathbf{b}_{\mathcal{S}} - \mathbf{x}^*\|_2$, , are related by the inequality*

$$\begin{aligned}\|\mathbf{b}_{\mathcal{S}} - \mathbf{x}^*\|_2 &\le \frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*) + \gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}\left(1 + \frac{\gamma_{4s}(\mathbf{b},\mathbf{x}^*)}{\beta_{4s}(\mathbf{b},\mathbf{x}^*)}\right)\|\widehat{\mathbf{x}} - \mathbf{x}^*\|_2 \\ &\quad + \left(1 + \frac{\gamma_{4s}(\mathbf{b},\mathbf{x}^*)}{\beta_{4s}(\mathbf{b},\mathbf{x}^*)}\right)\frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)} \\ &\quad + 2\frac{\|\nabla f(\mathbf{x}^*)_{T}\|_2}{\beta_{4s}(\mathbf{b},\mathbf{x}^*)}.\end{aligned}$$

Proof. At first, we have $\widehat{\mathbf{x}}_{\overline{\mathcal{T}}} = \mathbf{0}$ by definition. Also, since $\mathcal{Z} \subseteq \mathcal{T}$, $\overline{\mathcal{T}} \subseteq \overline{\mathcal{Z}}$ holds. So

$$\|(\widehat{\mathbf{x}} - \mathbf{x}^*)_{\overline{\mathcal{T}}}\|_2 \le \|(\widehat{\mathbf{x}} - \mathbf{x}^*)_{\overline{\mathcal{T}}}\|_2. \tag{A.48}$$

Thus, from Proposition 2,

$$\begin{aligned}\|\mathbf{x}^*_{\overline{\mathcal{T}}}\|_2 = \|(\widehat{\mathbf{x}} - \mathbf{x}^*)_{\overline{\mathcal{T}}}\|_2 &\le \frac{\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*) + \gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{2\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}\|(\widehat{\mathbf{x}} - \mathbf{x}^*)\|_2 \\ &\quad + \frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}.\end{aligned} \tag{A.49}$$

Moreover, from Algorithm 5, $\mathbf{b}_{\mathcal{S}} = \widehat{\mathbf{x}}$ and $\mathbf{b}_{\mathcal{S}\cap\overline{\mathcal{T}}} = \mathbf{0}$. Then, by triangle inequality (splitting the support to two complementary sets)

$$\begin{aligned}\|\mathbf{b}_{\mathcal{S}} - \mathbf{x}^*\|_2 &\leq \|(\mathbf{b}_{\mathcal{S}} - \mathbf{x}^*_{\mathcal{T}}\|_2 + \|\mathbf{x}^*)_{\overline{\mathcal{T}}}\|_2 \\ &\leq \|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2 + \|\mathbf{b}_{\mathcal{S}} - \mathbf{b}\|_2 + \|\mathbf{x}^*_{\overline{\mathcal{T}}}\|_2 \\ &\leq 2\|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2 + \|\mathbf{x}^*_{\overline{\mathcal{T}}}\|_2,\end{aligned} \tag{A.50}$$

where we added and subtracted the vector $\mathbf{b}$, and we used the triangle inequality in the deduced expression. Furthermore, we have $\|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2 \geq \|\mathbf{b}_{\mathcal{S}} - \mathbf{b}\|_2$, because $\mathbf{b}_{\mathcal{S}}$ is the best s-approximation of $\mathbf{b}$. Finally, we use Lemma 3 to bound $2\|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2$ term of (A.50) and (A.49) to deduce the desired result. □

Now, having these intermediate results, the proof of theorem for smooth functions follows. It is obvious by definition that for $k \leq k'$, and any vector $\mathbf{z}$, the bounds become less tight. So $A_k(\mathbf{z}) \leq A_{k'}(\mathbf{z})$. Consequently, $\mu_k \leq \mu_{k'}$. Also, since $\dfrac{A_k(\mathbf{x})}{B_k(\mathbf{x})} \leq \mu_k$, $\dfrac{\alpha_k(\mathbf{p},\mathbf{q})}{\beta_k(\mathbf{p},\mathbf{q})} \leq \mu_k$ is easily implied, and thereby $\dfrac{\gamma_k(\mathbf{p},\mathbf{q})}{\beta_k(\mathbf{p},\mathbf{q})} \leq \mu_k - 1$. If we apply these results to Lemma 4, we have

$$\begin{aligned}\|\widehat{\mathbf{x}}^i - \mathbf{x}^*\|_2 &\leq (\mu_{4s} - 1)\mu_{4s}\|\widehat{\mathbf{x}}^{i-1} - \mathbf{x}^*\|_2 + 2\frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{T}}\|_2}{\beta_{4s}(\mathbf{b},\mathbf{x}^*)} \\ &+ \mu_{4s}\frac{\|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}}^{i-1},\mathbf{x}^*)} \\ &\leq (\mu_{4s}^2 - \mu_{4s})\|\widehat{\mathbf{x}}^{i-1} - \mathbf{x}^*\|_2 + 2\varepsilon + 2\mu_{4s}\varepsilon.\end{aligned}$$

Having, $\dfrac{\|\nabla f(\mathbf{x}^*)_{\mathcal{T}}\|_2}{\beta_{4s}(\mathbf{b},\mathbf{x}^*)} \geq \dfrac{\|\nabla f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 + \|\nabla f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}}^{i-1},\mathbf{x}^*)}$, and the assumption $\mu_{4s} \leq \dfrac{1+\sqrt{3}}{2}$, we derive

$$\|\widehat{\mathbf{x}}^i - \mathbf{x}^*\|_2 \leq \frac{1}{2}\|\widehat{\mathbf{x}}^{i-1} - \mathbf{x}^*\|_2 + (3+\sqrt{3})\varepsilon. \tag{A.51}$$

Similarly with proof of Theorem 7 the proof is obtained recursively.

Proof of theorem 9[25, p. 92]

Similarly, we proceed with the second result for non-smooth cost functions. Using the definition of Bregman divergence of Chapter 4 and interchanging $\mathbf{x}$ and $\mathbf{x}+\mathbf{\Delta}$ we deduce for the sum

$$\begin{aligned}\mathbf{B}_f(\mathbf{x}+\mathbf{\Delta}\|\mathbf{x})+\mathbf{B}_f(\mathbf{x}\|\mathbf{x}+\mathbf{\Delta}) &= f(\mathbf{x}+\mathbf{\Delta})-f(\mathbf{x})-\langle\nabla_f(\mathbf{x}),\mathbf{\Delta})\rangle\\ &\quad+f(\mathbf{x})-f(\mathbf{x}+\mathbf{\Delta})-\langle\nabla_f(\mathbf{x}+\mathbf{\Delta}),-\mathbf{\Delta}\rangle\\ &=-(\langle\nabla_f(\mathbf{x}),\mathbf{\Delta})\rangle+\langle\nabla_f(\mathbf{x}+\mathbf{\Delta}),-\mathbf{\Delta}\rangle)\\ &=\langle\nabla_f(\mathbf{x}+\mathbf{\Delta})-\nabla_f(\mathbf{x}),\mathbf{\Delta}\rangle.\end{aligned} \tag{A.52}$$

Also, from relation (4.8) it is obvious

$$[\beta_k(\mathbf{x}+\mathbf{\Delta})+\beta_k(\mathbf{x})]\|\mathbf{\Delta}\|_2^2\le\langle\nabla_f(\mathbf{x}+\mathbf{\Delta})-\nabla_f(\mathbf{x}),\mathbf{\Delta}\rangle\le[\alpha_k(\mathbf{x}+\mathbf{\Delta})+\alpha_k(\mathbf{x})]\|\mathbf{\Delta}\|_2^2. \tag{A.53}$$

We provide some notation to make the following results easier to the reader. At first, we focus on two vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$ and we denote $\mathrm{card}(\mathrm{supp}(\mathbf{x}_1)\cup\mathrm{supp}(\mathbf{x}_2))\le r$. Also, $\mathbf{\Delta}=\mathbf{x}_1-\mathbf{x}_2$ and $\mathrm{supp}(\mathbf{\Delta})=\mathcal{R}$. Similarly $\mathbf{\Delta}'=\nabla_f(\mathbf{x}_1)-\nabla_f(\mathbf{x}_2)$. Moreover, we denote $\alpha_k(\mathbf{x}_1)+\alpha_k(\mathbf{x}_2)$ as well as $\beta_k(\mathbf{x}_1)+\beta_k(\mathbf{x}_2)$ by $\overline{\alpha}_k(\mathbf{x}_1,\mathbf{x}_2)$ or $\overline{\alpha}_k$ and $\overline{\beta}_k(\mathbf{x}_1,\mathbf{x}_2)$ or $\overline{\beta}_k$, respectively. Finally, we have $\overline{\gamma}_k=\overline{\gamma}_k(\mathbf{x}_1,\mathbf{x}_2)=\overline{\alpha}_k(\mathbf{x}_1,\mathbf{x}_2)-\overline{\beta}_k(\mathbf{x}_1,\mathbf{x}_2)$.
In the sequel, some intermediate results are presented concerning the Bregman divergence. The way this proof is structured is similar to Theorem's 8.

Proposition 5. *The following inequalities hold for $\mathcal{R}'\subseteq\mathcal{R}$*

$$\left|\|\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2-\overline{\alpha}_r\langle\mathbf{\Delta}',\mathbf{\Delta}_{\mathcal{R}'}\rangle\right|\le\overline{\gamma}_r\|\mathbf{\Delta}_{\mathcal{R}'}\|_2\|\mathbf{\Delta}\|_2,$$
$$\left|\|\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2-\overline{\beta}_r\langle\mathbf{\Delta}',\mathbf{\Delta}_{\mathcal{R}'}\rangle\right|\le\overline{\gamma}_r\|\mathbf{\Delta}_{\mathcal{R}'}\|_2\|\mathbf{\Delta}\|_2.$$

Proof. From (4.8), for any $t\in\mathbb{R}$ we have

$$\beta_r(\mathbf{x}_1)\|\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2t^2\le\mathbf{B}_f(\mathbf{x}_1-t\Delta'_{\mathcal{R}'}\|\mathbf{x}_1)\le\alpha_r(\mathbf{x}_1)\|\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2t^2, \tag{A.54}$$
$$\beta_r(\mathbf{x}_2)\|\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2t^2\le\mathbf{B}_f(\mathbf{x}_2+t\Delta'_{\mathcal{R}'}\|\mathbf{x}_2)\le\alpha_r(\mathbf{x}_2)\|\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2t^2. \tag{A.55}$$

Observe that

$$\mathbf{B}_f(\mathbf{x}_1-(\mathbf{\Delta}-t\mathbf{\Delta}'_{\mathcal{R}'})\|\mathbf{x}_1)=\mathbf{B}_f(\mathbf{x}_2+t\mathbf{\Delta}'_{\mathcal{R}'}\|\mathbf{x}_1). \tag{A.56}$$

Thus, we can write

$$\beta_r(\mathbf{x}_1)\|\mathbf{\Delta}-t\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2\le\mathbf{B}_f(\mathbf{x}_2+t\mathbf{\Delta}'_{\mathcal{R}'}\|\mathbf{x}_1)\le\alpha_r(\mathbf{x}_1)\|\mathbf{\Delta}-t\mathbf{\Delta}'_{\mathcal{R}'}\|_2^2. \tag{A.57}$$

Similarly, for $\mathbf{x}_2$

$$\beta_r(\mathbf{x}_2)\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \leq \mathbf{B}_f(\mathbf{x}_1 - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|\mathbf{x}_2) \leq \alpha_r(\mathbf{x}_2)\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2. \tag{A.58}$$

By taking the sum for (A.54), (A.55) and using (A.52), we obtain

$$[\beta_r(\mathbf{x}_1) + \beta_r(\mathbf{x}_2)]\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 \leq \mathbf{B}_f(\mathbf{x}_1 - t\Delta'_{\mathcal{R}'}\|\mathbf{x}_1) + \mathbf{B}_f(\mathbf{x}_2 + t\Delta'_{\mathcal{R}'}\|\mathbf{x}_2) \leq [\alpha_r(\mathbf{x}_1) + \alpha_r(\mathbf{x}_2)]\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 \tag{A.59}$$

$$\overline{\beta}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 \leq \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle t - f(\mathbf{x}_1) - f(\mathbf{x}_2) + f(\mathbf{x}_1 - t\boldsymbol{\Delta}'_{\mathcal{R}'}) + f(\mathbf{x}_2 + t\boldsymbol{\Delta}'_{\mathcal{R}'}) \leq \overline{\alpha}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2. \tag{A.60}$$

Similarly, for (A.57) and(A.58)

$$\overline{\beta}_r\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \leq \langle \boldsymbol{\Delta}', \boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - f(\mathbf{x}_1) - f(\mathbf{x}_2) + f(\mathbf{x}_1 - t\boldsymbol{\Delta}'_{\mathcal{R}'}) + f(\mathbf{x}_2 + t\boldsymbol{\Delta}'_{\mathcal{R}'}) \leq \overline{\alpha}_r\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2. \tag{A.61}$$

Now, by subtracting (A.61) from (A.60), we derive

$$\overline{\beta}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 - \overline{\alpha}_r\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \leq \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle t - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\rangle \leq \overline{\alpha}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 - \overline{\beta}_r\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \tag{A.62}$$

$$\overline{\beta}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 - \overline{\alpha}_r\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \leq 2\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle t - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}\rangle \leq \overline{\alpha}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 - \overline{\beta}_r\|\boldsymbol{\Delta} - t\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2. \tag{A.63}$$

Next, we expand the familiar identity $a^2 - b^2$ in (A.63). Then, we separate the deduced inequality to the two sides, that is

$$\overline{\gamma}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 + \overline{\alpha}_r\|\boldsymbol{\Delta}\|_2^2 + 2\Bigg(\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - \overline{\alpha}\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle\Bigg)t - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}\rangle \geq 0, \tag{A.64}$$

$$\overline{\gamma}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 - \overline{\beta}_r\|\boldsymbol{\Delta}\|_2^2 - 2\Bigg(\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - \overline{\beta}\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle\Bigg)t + \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}\rangle \geq 0. \tag{A.65}$$

The above inequalities are quadratics in terms of t. The constants of the polynomials, in both (A.64) and (A.65), are bounded from above by $\overline{\gamma}\|\boldsymbol{\Delta}\|_2^2$ (an implication of (A.53)). Thus, we have

$$\overline{\gamma}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 + 2\Bigg(\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - \overline{\alpha}\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle\Bigg)t + \overline{\gamma}_r\|\boldsymbol{\Delta}\|_2^2 \geq 0, \tag{A.66}$$

$$\overline{\gamma}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 t^2 - 2\Bigg(\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - \overline{\beta}\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle\Bigg)t + \overline{\gamma}_r\|\boldsymbol{\Delta}\|_2^2 \geq 0. \tag{A.67}$$

Since the above quadratics are non-negative for all $t \in \mathbb{R}$, their discriminant is negative. So

$$\Bigg(\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - \overline{\alpha}\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle\Bigg)^2 - \overline{\gamma}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \overline{\gamma}_r\|\boldsymbol{\Delta}\|_2^2 \leq 0, \tag{A.68}$$

$$\Bigg(\langle \boldsymbol{\Delta}', \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle - \overline{\beta}\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}'_{\mathcal{R}'}\rangle\Bigg)^2 - \overline{\gamma}_r\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 \overline{\gamma}_r\|\boldsymbol{\Delta}\|_2^2 \leq 0, \tag{A.69}$$

which implies the desired result. □

Proposition 6. *Suppose $\mathcal{R}' \subseteq \mathcal{R}$. The following inequalities hold*

$$\begin{aligned}\left|\overline{\alpha}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2^2 - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}_{\mathcal{R}'}\rangle\right| &\leq \overline{\gamma}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2 \|\boldsymbol{\Delta}\|_2,\\ \left|\overline{\beta}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2^2 - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}_{\mathcal{R}'}\rangle\right| &\leq \overline{\gamma}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2 \|\boldsymbol{\Delta}\|_2.\end{aligned}$$

The proof of Proposition 6 is analogous to the proof of Preposition 5. Thus it is ommited.

Corollary 1. *The inequality*

$$\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2 \geq \overline{\beta}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2 - \overline{\gamma} \|\boldsymbol{\Delta}_{\mathcal{R}\backslash\mathcal{R}'}\|_2, \tag{A.70}$$

holds for $\mathcal{R}' \subseteq \mathcal{R}$.

Proof. We have

$$\overline{\alpha}_r^2 \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2^2 - \|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 = -\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 + \overline{\alpha}_r \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}_{\mathcal{R}'}\rangle + \overline{\alpha}_r \left[\overline{\alpha}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2^2 - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}_{\mathcal{R}'}\rangle\right]. \tag{A.71}$$

Now, from Propositions 5 and 6, holds

$$-\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2^2 + \overline{\alpha}_r \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}_{\mathcal{R}'}\rangle + \overline{\alpha}_r \left[\overline{\alpha}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2^2 - \langle \boldsymbol{\Delta}', \boldsymbol{\Delta}_{\mathcal{R}'}\rangle\right] \leq \overline{\gamma}_r \|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2 \|\boldsymbol{\Delta}\|_2 + \overline{\alpha}_r \overline{\gamma}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2 \|\boldsymbol{\Delta}\|_2. \tag{A.72}$$

Hence

$$\begin{aligned}\|\boldsymbol{\Delta}'_{\mathcal{R}'}\|_2 &\geq \overline{\alpha}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2 - \overline{\gamma}_r \|\boldsymbol{\Delta}\|_2\\ &\geq \overline{\beta}_r \|\boldsymbol{\Delta}_{\mathcal{R}'}\|_2 - \overline{\gamma}_r \|\boldsymbol{\Delta}_{\mathcal{R}\backslash\mathcal{R}'}\|_2,\end{aligned}$$

where we split the set $\mathcal{R}$ to the sets $\mathcal{R}'$ and $\mathcal{R}\backslash\mathcal{R}'$. □

Proposition 7. *Suppose that $\mathcal{K}$ is a subset of $\overline{\mathcal{R}}$ with at most k elements. Then we have*

$$\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2 \leq \overline{\gamma}_{k+r} \|\boldsymbol{\Delta}\|_2. \tag{A.73}$$

Proof. For any $t \in \mathbb{R}$,

$$\beta_{k+r}(\mathbf{x}_1) \|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 t^2 \leq \mathbf{B}_f(\mathbf{x}_1 + t\Delta'_{\mathcal{K}} \| \mathbf{x}_1) \leq \alpha_{k+r}(\mathbf{x}_1) \|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 t^2, \tag{A.74}$$

$$\beta_{k+r}(\mathbf{x}_2) \|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 t^2 \leq \mathbf{B}_f(\mathbf{x}_2 - t\Delta'_{\mathcal{K}} \| \mathbf{x}_2) \leq \alpha_{k+r}(\mathbf{x}_2) \|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 t^2. \tag{A.75}$$

So

$$\beta_{k+r}(\mathbf{x}_1)\|\boldsymbol{\Delta}+t\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 \le \mathbf{B}_f(\mathbf{x}_2 - t\boldsymbol{\Delta}'_{\mathcal{K}}\|\mathbf{x}_1) \le \alpha_{k+r}(\mathbf{x}_1)\|\boldsymbol{\Delta}+t\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2, \tag{A.76}$$

$$\beta_{k+r}(\mathbf{x}_2)\|\boldsymbol{\Delta}+t\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 \le \mathbf{B}_f(\mathbf{x}_1 + t\boldsymbol{\Delta}'_{\mathcal{K}}\|\mathbf{x}_2) \le \alpha_{k+r}(\mathbf{x}_2)\|\boldsymbol{\Delta}+t\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2. \tag{A.77}$$

By subtracting the sum of (A.76) and (A.77) from the sum of (A.74) and (A.75), we obtain

$$\overline{\beta}_{k+r}\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2t^2 - \overline{\alpha}_{k+r}\|\boldsymbol{\Delta}+t\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 \le -2t\langle\boldsymbol{\Delta}',\boldsymbol{\Delta}'_{\mathcal{K}}\rangle - \langle\boldsymbol{\Delta}',\boldsymbol{\Delta}\rangle \le \overline{\alpha}_{k+r}\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2t^2 - \overline{\beta}_{k+r}\|\boldsymbol{\Delta}+t\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2. \tag{A.78}$$

Observe that $\langle\boldsymbol{\Delta}',\boldsymbol{\Delta}'_{\mathcal{K}}\rangle = \|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2$ and $\langle\boldsymbol{\Delta},\boldsymbol{\Delta}'_{\mathcal{K}}\rangle = 0$, since $\mathcal{K}$ is a subset of $\overline{\mathcal{R}}$. Thus, using (A.78) and (A.53), we deduce

$$\overline{\gamma}_{k+r}\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2t^2 \pm 2\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2t + \overline{\gamma}_{k+r}\|\boldsymbol{\Delta}\|_2^2 \ge 0, \tag{A.79}$$

hold for all t. Therefore, the above quadratics of t must not have positive discriminants. Thus we must have

$$\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^4 - \overline{\gamma}_{k+r}^2\|\boldsymbol{\Delta}\|_2^2\|\boldsymbol{\Delta}'_{\mathcal{K}}\|_2^2 \le 0, \tag{A.80}$$

which yields the result. □

In the sequel, we present the proof of a lemma similar to Lemma 2.

Lemma 5. *Let $\mathcal{R}$ denote the set* supp $(\widehat{\mathbf{x}}-\mathbf{x}^*)$. *The current estimate $\widehat{\mathbf{x}}$ then satisfies*

$$\begin{aligned}\|(\widehat{\mathbf{x}}-\mathbf{x}^*)_{\overline{\mathcal{Z}}}\|_2 \le{}& \frac{\gamma_{4s}(\widehat{\mathbf{x}},\mathbf{x}^*)+\gamma_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}{\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}\|\widehat{\mathbf{x}}-\mathbf{x}^*\|_2\\ &+\frac{\|\nabla_f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}+\nabla_f(\mathbf{x}^*)_{\mathcal{Z}\setminus\mathcal{R}}\|_2}{\beta_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)}.\end{aligned}$$

Proof. In a similar manner, $\mathcal{Z}$ =supp$(\mathbf{z}_{2s})$ and card$(\mathcal{R}) \le 2s$. Thus, we obtain

$$\|\mathbf{z}_{\mathcal{R}}\|_2 \le \|\mathbf{z}_{\mathcal{Z}}\|_2.$$

Then,

$$\|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 \le \|\mathbf{z}_{\mathcal{Z}\setminus\mathcal{R}}\|_2. \tag{A.81}$$

We are going to bound each side of the inequality (A.81).
For the left-hand side, we obtain

$$\begin{aligned}\|\mathbf{z}_{\mathcal{R}\setminus\mathcal{Z}}\|_2 &= \|\nabla_f(\widehat{\mathbf{x}})_{\mathcal{R}\setminus\mathcal{Z}}\|_2\\ &\ge \|\big(\nabla_f(\widehat{\mathbf{x}})-\nabla_f(\mathbf{x}^*)\big)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 - \|\nabla_f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2\\ &\ge \overline{\beta}_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}},\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 - \overline{\gamma}_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}},\mathbf{x}^*)_{\mathcal{R}\cap\mathcal{Z}}\|_2 - \|\nabla_f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2\\ &\overset{(d)}{\ge} \overline{\beta}_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}},\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2 - \overline{\gamma}_{2s}(\widehat{\mathbf{x}},\mathbf{x}^*)\|(\widehat{\mathbf{x}},\mathbf{x}^*)\|_2 - \|\nabla_f(\mathbf{x}^*)_{\mathcal{R}\setminus\mathcal{Z}}\|_2,\end{aligned}$$

where at point (d), the Corollary 1 is used ($\mathcal{R}$ is split to $\mathcal{R}\backslash\mathcal{Z}$ and $\mathcal{R}\cap\mathcal{Z}$). The last expression is straightforward from the assumption $\mathcal{R} = \mathrm{supp}(\widehat{\mathbf{x}}-\mathbf{x}^*)$.

Also,

$$\begin{aligned}
\|\mathbf{z}_{\mathcal{Z}\backslash\mathcal{R}}\|_2 &= \|\nabla_f(\widehat{\mathbf{x}})_{\mathcal{Z}\backslash\mathcal{R}}\|_2 \\
&\leq \|\big(\nabla_f(\widehat{\mathbf{x}}) - \nabla_f(\mathbf{x}^*)\big)_{\mathcal{Z}\backslash\mathcal{R}}\|_2 + \|\nabla_f(\mathbf{x}^*)_{\mathcal{Z}\backslash\mathcal{R}}\|_2 \\
&\leq \overline{\gamma}_{4s}(\widehat{\mathbf{x}}, \mathbf{x}^*)\|(\widehat{\mathbf{x}}, \mathbf{x}^*)\|_2 + \|\nabla_f(\mathbf{x}^*)_{\mathcal{Z}\backslash\mathcal{R}}\|_2
\end{aligned}$$

Combining the bounds and relation (A.81), we deduce

$$\overline{\gamma}_{4s}(\widehat{\mathbf{x}}, \mathbf{x}^*)\|(\widehat{\mathbf{x}}, \mathbf{x}^*)\|_2 + \|\nabla_f(\mathbf{x}^*)_{\mathcal{Z}\backslash\mathcal{R}}\|_2 \geq \overline{\beta}_{2s}(\widehat{\mathbf{x}}, \mathbf{x}^*)\|(\widehat{\mathbf{x}}, \mathbf{x}^*)_{\mathcal{R}\backslash\mathcal{Z}}\|_2 - \overline{\gamma}_{2s}(\widehat{\mathbf{x}}, \mathbf{x}^*)\|(\widehat{\mathbf{x}}, \mathbf{x}^*)\|_2 - \|\nabla_f(\mathbf{x}^*)_{\mathcal{R}\backslash\mathcal{Z}}\|_2. \tag{A.82}$$

Reordering (A.82), followed by the observation $(\widehat{\mathbf{x}}, \mathbf{x}^*)_{\mathcal{R}\backslash\mathcal{Z}} = (\widehat{\mathbf{x}}, \mathbf{x}^*)_{\overline{\mathcal{Z}}}$, leads to the desired result. □

Lemma 6. *The vector* $\mathbf{b}$ *where*

$$\mathbf{b} = \mathrm{argmin} f(\mathbf{x}), \tag{A.83}$$

$$\text{s.t. } \mathbf{x}_{\overline{\mathcal{T}}} = \mathbf{0}, \tag{A.84}$$

satisfies

$$\|\mathbf{x}^*_{\mathcal{T}} - \mathbf{b}\|_2 \leq \frac{\|\nabla_f(\mathbf{x}^*)_{\mathcal{T}}\|_2}{\overline{\beta}_{4s}(\mathbf{x}^*, \mathbf{b})} + \left(1 + \frac{\overline{\gamma}_{4s}(\mathbf{x}^*, \mathbf{b})}{2\overline{\beta}_{4s}(\mathbf{x}^*, \mathbf{b})}\right)\|\mathbf{x}^*_{\overline{\mathcal{T}}}\|_2. \tag{A.85}$$

The proof is deduced by the Corollary 1.

Lemma 7. *The estimation error of the current iterate* $\|\widehat{\mathbf{x}} - \mathbf{x}^*\|_2$ *and that of the next iterate* $\|\mathbf{b}_S - \mathbf{x}^*\|_2$ *are related by the inequality*

$$\begin{aligned}
\|\mathbf{b}_S - \mathbf{x}^*\|_2 \leq &\left(1 + \frac{2\overline{\gamma}_{4s}(\mathbf{x}^*, \mathbf{b})}{\overline{\beta}_{4s}(\mathbf{x}^*, \mathbf{b})}\right)\frac{\overline{\gamma}_{4s}(\widehat{\mathbf{x}}, \mathbf{x}^*) + \overline{\gamma}_{2s}(\widehat{\mathbf{x}}, \mathbf{x}^*)}{\overline{\beta}_{2s}(\widehat{\mathbf{x}}, \mathbf{x}^*)}\|\widehat{\mathbf{x}} - \mathbf{x}^*\|_2 \\
&+ \left(1 + \frac{2\overline{\gamma}_{4s}(\mathbf{x}^*, \mathbf{b})}{\overline{\beta}_{4s}(\mathbf{x}^*, \mathbf{b})}\right)\frac{\|\nabla_f(\mathbf{x}^*)_{\mathcal{R}\backslash\mathcal{Z}}\|_2 + \|\nabla_f(\mathbf{x}^*)_{\mathcal{Z}\backslash\mathcal{R}}\|_2}{\overline{\beta}_{2s}(\widehat{\mathbf{x}}, \mathbf{x}^*)} \\
&+ 2\frac{\|\nabla_f(\mathbf{x}^*)_{\mathcal{T}}\|_2}{\overline{\beta}_{4s}(\mathbf{b}, \mathbf{x}^*)}.
\end{aligned}$$

Finally, using all the above the proof of Theorem 10 follows.

Proof. Theorem 10. Let the vectors involved in the j - th iteration of the algorithm be denoted by superscript (j). Given that $\mu_{4s} \leq \dfrac{3+\sqrt{3}}{4}$ we have

$$\frac{\overline{\gamma}_{4s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*)}{\overline{\beta}_{4s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*)} \leq \frac{\sqrt{3}-1}{4}, \tag{A.86}$$

$$\left(1 + \frac{2\overline{\gamma}_{4s}(\mathbf{x}^*, \mathbf{b})}{\overline{\beta}_{4s}(\mathbf{x}^*, \mathbf{b})}\right) \leq \frac{1+\sqrt{3}}{2}. \tag{A.87}$$

Thereby,

$$\begin{aligned}\left(1 + \frac{2\overline{\gamma}_{4s}(\mathbf{x}^*, \mathbf{b})}{\overline{\beta}_{4s}(\mathbf{x}^*, \mathbf{b})}\right)\frac{\overline{\gamma}_{4s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*) + \overline{\gamma}_{2s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*)}{\overline{\beta}_{2s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*)} &\leq \left(\frac{1+\sqrt{3}}{2}\right) \times \left(\frac{2\overline{\gamma}_{4s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*)}{\overline{\beta}_{4s}(\widehat{\mathbf{x}}^j, \mathbf{x}^*)}\right) \\ &\leq \left(\frac{1+\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}-1}{4}\right) = \frac{1}{2}.\end{aligned}$$

Now, from Lemma 7 we can write

$$\|\widehat{\mathbf{x}}^{(j+1)} - \mathbf{x}^*\|_2 \leq \frac{1}{2}\|\widehat{\mathbf{x}}^j, \mathbf{x}^*\|_2 + (3+\sqrt{3})\varepsilon. \tag{A.88}$$

Recursively the above inequality for $j = 0, 1, ..., i-1$ yields

$$\|\widehat{\mathbf{x}} - \mathbf{x}^*\|_2 \leq \left(\frac{1}{2}\right)^i \|\mathbf{x}^*\|_2 + (6+2\sqrt{3})\varepsilon. \tag{A.89}$$

□

Appendix B

Basic results

Vector and Matrix Norms

We are familiar with the notion of euclidean norm from basic linear algebra. Here a more general definition is presented.

Definition 9. *The p-norm (or ℓ_p norm) is on $\mathbb{R}^n$ or $\mathbb{C}^n$ for $1 \leq p < \infty$ is*

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \tag{B.1}$$

and for $p = \infty$

$$\|\mathbf{x}\|_\infty = \max_{j \in [n]} |x_j|. \tag{B.2}$$

The p-norm satisfies the familiar properties of homogeneity and triangle inequality. Now, we proceed with definitions for the matrix norms.

Definition 10. *Let $\mathbf{A} : X \longrightarrow Y$ be a linear map between two normed spaces (vector spaces endowed with a norm). The operator norm of $\mathbf{A}$ is*

$$\|\mathbf{A}\| = \sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|. \tag{B.3}$$

In particular, we have for a matrix $\mathbf{A}$ and $1 \leq p,q \leq \infty$ between the spaces l_p and ℓ_q, the matrix (operator) norm as

$$\|\mathbf{A}\|_{p\to q} = \sup_{\|\mathbf{x}_p\|=1} \|\mathbf{A}\mathbf{x}\|_q. \tag{B.4}$$

The above definition is the induced matrix norm. An intuition of this definition is that $\|\mathbf{A}\|$ is the maximum extend of stretching of unit ball vector by matrix $\mathbf{A}$. Some special cases,

$$\|\mathbf{A}\|_{1\to 1} = \max_{k \in [n]} \sum_{j=1}^{m} |a_{jk}|, \qquad \|\mathbf{A}\|_{\infty\to\infty} = \max_{j \in [m]} \sum_{k=1}^{n} |a_{jk}|.$$

The first equation is maximum absolute column sum of the matrix and the other one is the same for the rows of the matrix. Also a very common norm is, for $\mathbf{A} \in \mathbb{R}^{m\times n}$

$$\|\mathbf{A}\|_{2\to 2} = \sqrt{\lambda_{\max}\left(\mathbf{A}^T\mathbf{A}\right)} = \sigma_{\max}\left(\mathbf{A}\right). \tag{B.5}$$

If the $\mathbf{A}$ is a symmetric (hermitian) matrix, then $\|\mathbf{A}\|_{2\to 2}$ is the maximum eigenvalue of $\mathbf{A}$. Next, a very important theorem is stated, used thoroughly to relate the properties of the measurement matrix. Gershgorin circle theorem gives information about the position of eigenvalues of a square matrix.

Theorem 11. *The eigenvalues of a $n \times n$ matrix $\mathbf{A}$, with entries $a_{i,j}$, lie in the union of n disks $d_i = d_i(c_i, r_i)$ centered at $c_i = a_{i,i}$ and with radius*

$$r_i = \sum_{j \neq i} |a_{i,j}|. \tag{B.6}$$

Note that theorem (11) implies that, if λ is an eigenvalue of a square matrix, then

$$|\lambda - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}|. \tag{B.7}$$

Tools useful to Appendix A

A theorem used during this work is the Bolzano-Weierstrass Theorem that states

Theorem 12. *Every bounded infinite subset of $\mathbb{R}^p$ has an accumulation point.*

Also a form of the previous theorem applicable to sequences is

Theorem 13. *A bounded sequence (a_k) has a convergent subsequence.*

The proofs are ommited.

Lemma 8. *Given vectors $\mathbf{u}, \mathbf{v}, \in \mathbb{C}^n$ and an index set $\mathcal{S} \subset [n]$, the following holds*

$$|\langle \mathbf{u}, (\mathbf{Id} - \mathbf{A}^H \mathbf{A})\mathbf{v} \rangle| \leq \delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \tag{B.8}$$

if $\text{card}(\text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})) \leq t$, and

$$\|((\mathbf{Id} - \mathbf{A}^H \mathbf{A})\mathbf{v})_{\mathcal{S}}\|_2 \leq \delta_t \|\mathbf{v}\|_2, \tag{B.9}$$

if $\text{card}(\text{supp}(\mathcal{S}) \cup \text{supp}(\mathbf{v})) \leq t$.

Lemma 9. *Given $\mathbf{e} \in \mathbb{C}^n$ and an index set $\mathcal{S} \subset [n]$ with card($\mathcal{S}$) $= s$,*

$$\|(\mathbf{A}^H \mathbf{e})_{\mathcal{S}}\|_2 \leq \sqrt{1 + \delta_s} \|\mathbf{e}\|_2, \tag{B.10}$$

holds.

Fin.

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