MIMO capacity estimation using MIMObit

Andreas Sgourakis

School of Electrical and Computer Engineering

Technical University of Crete

Diploma Thesis Committee

Associate Professor George Karystinos, Thesis Supervisor

Professor Aggelos Bletsas

Professor Nick Buris

March 2019

Acknowledgements

First of all, I would like to thank my thesis supervisor, Associate Professor George Karystinos, for entrusting me with this project and for all his guidance and support throughout this work.

I would like to express my great appreciation to Professor Nick Buris for giving us the opportunity to exploit MIMObit's capabilities since it was an integral part of this work.

My sincere thanks to Professor Aggelos Bletsas for being part of my thesis committee.

I would also like to thank my friends for the great time we spent together during this journey and the memorable moments we had. Finally, I would like to thank my family for their constant support throughout my studies.

Abstract

In this work, we study the performance of multiple-input multiple-output (MIMO) wireless systems. Specifically, we examine their capacity for different system scenaria and with different propagation models. The MIMO systems analyzed include the single-user case where multiple antennas are used and the multiuser case with two users present and a single antenna at each of them and the base station. We examine the capacity properties in the case of unknown and known propagation channel. In the case where the latter is infeasible, we present ways to estimate the optimal power allocation and approach the system capacity. To simulate communication systems under various propagation environments, we use Neben's MIMObit which offers an integrated platform that fully takes into account the antennas used at the transmitter and the receiver. We evaluate the accuracy of simulation results provided by MIMObit using dipole antennas created in its graphical user interface and in CST Microwave Studio. Finally, we make some capacity estimation comparisons using the channels produced by MIMObit and Matlab.

Contents

List of Figures

Chapter 1

Information Theory Background

Information theory is a fundamental part of communication theory as it studies the transmission, processing, extraction, and utilization of information. We give a brief description of some important quantities of information that will be used later in this thesis.

1.1 Entropy

Entropy is defined for a random variable and can be interpreted as a measure of the amount of uncertainty associated with the aforementioned random variable.

Entropy of a discrete random variable

Let X be a discrete random variable. The values that it takes on are depicted as x_i and the probability mass function as p_x. The entropy of X is defined as

$$H(X) := -\sum_i p_x(x_i) \log_2 p_x(x_i) \quad \text{bits/symbol.} \tag{1.1}$$

The entropy of X depends only on the distribution of the random variable X and not on the values that it takes on. It can also be seen that $H(X) \geq 0$, since $0 \leq p_x \leq 1$.

Binary Entropy

The entropy of a Bernoulli valued random variable X which takes on its values with probabilities p and $1-p$ is called binary entropy and given by

$$\mathcal{H}(p) := -p \log_2 p - (1-p) \log_2 (1-p) \quad \text{bits/symbol.} \tag{1.2}$$

It can be shown that the binary entropy is maximized when X is uniformly distributed. In that way, the amount of uncertainty reaches a maximum. This is demonstrated in Figure 1.1.

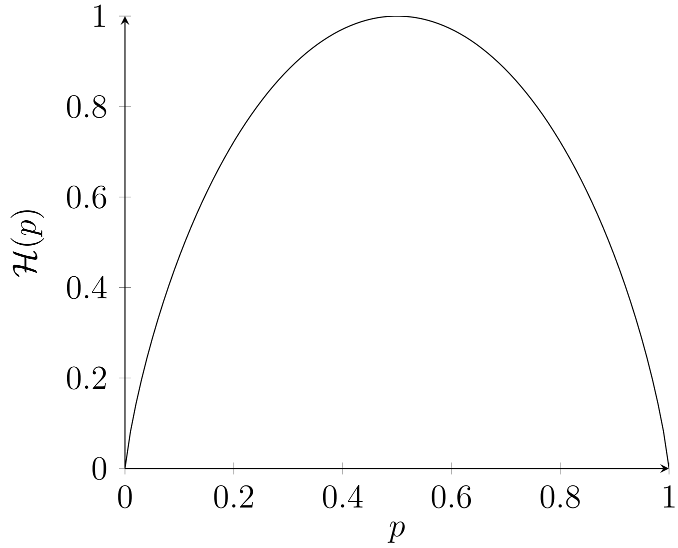


Figure 1.1: Binary entropy.

Entropy of a continuous random variable

Let X be a continuous random variable with probability density function f_x. The differential entropy of X is defined as:

$$h(X) := -\int_{-\infty}^{+\infty} f_X(x) \log_2 f_X(x)\; dx \quad \text{bits/symbol.} \tag{1.3}$$

Note that, as in the discrete random variable case, the differential entropy depends only on the probability density function of the random variable. One important difference to consider is that the differential entropy can be negative.

Entropy of a Gaussian random variable

In this thesis, we mostly deal with continuous random variables that follow the normal distribution. That is,

$$X \sim \mathcal{N}(\mu, \sigma^2), \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{1.4}$$

The differential entropy of X is

$$h(X) = \frac{1}{2}\log_2(2\pi e \sigma^2) \quad \text{bits/symbol.} \tag{1.5}$$

The proof is given in Chapter 6.

Entropy of a Gaussian random vector

A random vector $\boldsymbol{X} = [X_1, X_2, ..., X_n]^T$ is Gaussian if the random variables $X_1, X_2, ..., X_n$ are jointly Gaussian. The mean vector is defined as $\boldsymbol{\mu} = \big[E[X_1], E[X_2], ..., E[X_n]\big]^T$

and the covariance matrix as $\boldsymbol{K_x} = E\big[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T\big]$. In general,

$$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu},\, \boldsymbol{K_x}), \qquad f_{X_1,X_2,\ldots,X_n}(x_1, x_2, ..., x_n) = \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}}. \tag{1.6}$$

The differential entropy of $\boldsymbol{X}$ is

$$h(X_1, X_2, ..., X_n) = \frac{1}{2}\log_2\big[(2\pi e)^n \det(\boldsymbol{K_x})\big] \quad \text{bits/symbol.} \tag{1.7}$$

The proof is given in Chapter 6.

Entropy of a complex Gaussian random variable

The random variable $Z = X + jY$, where X and Y are real i.i.d. Gaussian random variables, is considered as a complex Gaussian random variable. In this thesis, we deal with complex Gaussian random variables that are circularly symmetric, which implies that $e^{j\phi}Z$ has the same distribution as Z for any real ϕ. Hence,

$$E[Z] = E[e^{j\phi}Z] = e^{j\phi}E[Z], \tag{1.8}$$
$$E[Z^2] = E[(e^{j\phi}Z)^2] = e^{2j\phi}E[Z^2]. \tag{1.9}$$

Equation (1.8) leads to $E[Z] = 0$, which implies $E[X] = 0$ and $E[Y] = 0$. Equation (1.9) leads to $E[Z^2] = 0$ which implies $E[X^2] = E[Y^2]$. The proof of the latter is given in Chapter 6. Taking into account these results, Z is circularly symmetric if X and Y are i.i.d with zero mean and equal variance, i.e.,

$$X \sim \mathcal{N}(0,\, \sigma^2), \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}, \tag{1.10}$$

$$Y \sim \mathcal{N}(0,\, \sigma^2), \qquad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{y^2}{2\sigma^2}}. \tag{1.11}$$

Since X and Y are independent, their joint PDF is the product of their PDFs, i.e.,

$$f_Z(z) = f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{y^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{1.12}$$
$$= \frac{1}{2\pi\sigma^2}e^{-\frac{|z|^2}{2\sigma^2}} = \frac{1}{\pi\sigma_z^2}e^{-\frac{|z|^2}{\sigma_z^2}}. \tag{1.13}$$

The above result indicates that $Z \sim \mathcal{CN}(0,\, \sigma_z^2)$ where $\sigma_z^2 = \sigma_x^2 + \sigma_y^2 = \sigma^2 + \sigma^2 = 2\sigma^2$. Using the definition of differential entropy, it can be proved that

$$h(Z) = \log_2(\pi e \sigma_z^2) \quad \text{bits/symbol.} \tag{1.14}$$

Entropy of a complex Gaussian random vector

A random vector $\boldsymbol{Z} = [Z_1, Z_2, ..., Z_n]^T$ is complex Gaussian if the random variables $Z_1, Z_2, ..., Z_n$ are jointly complex Gaussian. In this thesis, we deal with complex Gaussian random vectors which are circularly symmetric, which implies that the mean vector is $\boldsymbol{\mu} = \left[E[Z_1], E[Z_2], ..., E[Z_n]\right]^T = 0$, the pseudo-covariance matrix is $E[\boldsymbol{Z}\boldsymbol{Z}^T] = 0$, and the covariance matrix is $\boldsymbol{K_z} = E[\boldsymbol{Z}\boldsymbol{Z}^H]$, i.e.,

$$\boldsymbol{Z} \sim \mathcal{CN}(0,\ K_z), \qquad f_{Z_1,Z_2,...,Z_n}(z_1, z_2, ..., z_n) = \frac{e^{-\boldsymbol{Z}^H \boldsymbol{K_z}^{-1} \boldsymbol{Z}}}{\pi^n \det(\boldsymbol{K_z})}. \tag{1.15}$$

Using the definition of differential entropy, it can be proved that

$$h(Z_1, Z_2, ..., Z_n) = \log_2[(\pi e)^n \det(\boldsymbol{K_z})] \quad \text{bits/symbol}. \tag{1.16}$$

Maximization of differential entropy

The normal distribution maximizes the differential entropy, under constraints of mean and variance. Let g_X be a Gaussian PDF with mean μ and variance σ^2 and f_X an arbitrary PDF with the same mean and variance. The Kullback-Leibler distance between the two probability density functions is defined as

$$D_{\mathrm{KL}}(f\|g) = \int_{-\infty}^{+\infty} f_X \log_2 \frac{f_X}{g_X}\, dx \geq 0 \tag{1.17}$$

which implies that

$$h(g_X) \geq h(f_X). \tag{1.18}$$

The proof is given in Chapter 6.

1.2 Mutual Information

The mutual information, denoted by $I(X;Y)$, is a measure of dependence between random variables X and Y. Explicitly it measures the amount of information that X contains about Y and vice versa. It is symmetric in X and Y and always nonnegative, i.e.,

$$I(X;Y) = I(Y;X) \geq 0. \tag{1.19}$$

Formally, the mutual information is expressed as the Kullback-Leibler distance of the product of the marginal distributions $p_X(x)p_Y(y)$ of the two random variables X and Y from the random variables' joint distribution $p_{X,Y}(x,y)$, i.e.,

$$I(X;Y) = D_{\mathrm{KL}}(p_{X,Y}\|p_X p_Y). \tag{1.20}$$

Generally, the Kullback-Leibler distance between two probability mass functions p and q is defined as

$$D_{\mathrm{KL}}(p\|q) = \sum_x p_X \log_2 \frac{p_X}{q_X} \geq 0. \tag{1.21}$$

Therefore, the mutual information between X and Y is defined as

$$I(X;Y) = \sum_x \sum_y p_{X,Y} \log_2 \frac{p_{X,Y}}{p_X p_Y} \quad \text{bits/symbol}. \tag{1.22}$$

Figure 1.2: Communication system.

Using equation (1.22), we conclude with the more meaningful result

$$I(X;Y) = H(Y) - H(Y|X) \quad \text{bits/symbol.} \tag{1.23}$$

The proof is given in Chapter 6. This result shows that the mutual information between X and Y is the reduction in uncertainty of Y when we know the random variable X. By symmetry, it also follows that

$$I(X;Y) = H(X) - H(X|Y) \quad \text{bits/symbol.} \tag{1.24}$$

Another relation between mutual information and entropy is

$$I(X;Y) = H(X) + H(Y) - H(X,Y) \tag{1.25}$$

since it is known that

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y). \tag{1.26}$$

1.3 Channel Capacity

Channel capacity defines a tight upper bound on the rate at which information can be reliably transmitted over a communication channel. Let X and Y be random variables depicting the input and the output, respectively, of a channel. The capacity is given by the maximum of the mutual information between the input and output of the channel, where the maximization is with respect to the input distribution, i.e.,

$$C = \max_{f_X} I(X;Y) \quad \text{bits/symbol.} \tag{1.27}$$

Real AWGN channel

We consider the real AWGN channel where the output y is the sum of the input x and zero-mean Gaussian noise w. For the input x, it is required that

$$E\left\{x^2\right\} \leq P. \tag{1.28}$$

The signal model is defined as

$$y = x + w, \qquad\qquad w \sim \mathcal{N}(0,\, N). \tag{1.29}$$

The capacity of the channel is the maximization of the mutual information between the input x and the output y over all possible distributions on the input that satisfy the power constraint, i.e.,

$$C = \max_{f_x : E\left\{x^2\right\} \leq P} I(x;y), \tag{1.30}$$

which implies that the capacity of the real AWGN channel is

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right) = \frac{1}{2}\log_2\Big(1 + \text{SNR}\Big) \quad \text{bits/symbol.} \tag{1.31}$$

The proof is given in Chapter 6.

Complex AWGN channel

Dealing with complex random variables, the input x can take on any complex value with an average power constraint of P per complex input value, i.e.,

$$y = x + w, \qquad w \sim \mathcal{CN}(0,\, N). \tag{1.32}$$

The best way to derive the capacity is to consider this complex channel as two real AWGN channels. Each real component corresponds with SNR $= (P/2)/(N/2) = P/N$ in order to meet the overall SNR. We already know the capacity of the real AWGN channel. Hence, the capacity of the complex AWGN channel is

$$C = \frac{1}{2}\log_2\left(1+\frac{P}{N}\right) + \frac{1}{2}\log_2\left(1+\frac{P}{N}\right) = \log_2\left(1+\frac{P}{N}\right) = \log_2\left(1+\mathrm{SNR}\right). \tag{1.33}$$

Alternatively, we can derive this formula by working directly with complex random variables, i.e.,

$$C = \max_{f_x : E\left\{|x|^2\right\} \le P} I(x;y). \tag{1.34}$$

The proof is given in Chapter 6.

Chapter 2

Single-user Case

In this chapter, we introduce the single user scenario where the communication is performed between one transmitter and one receiver. We calculate the maximum achievable rate of reliable communication based on the fading channel model assuming that the transmitter and the receiver have one antenna each. Later on, we analyze the performance of the system, in terms of capacity, when they both have multiple antennas. We also take into account the scenario where the transmitter has multiple antennas and the receiver a single antenna, and vice versa. To keep things simple, we assume that not only the receiver, but also the transmitter can track the channel.

2.1 Single-input Single-output (SISO) System

SISO stands for a single-input single-output system, which practically means that there is one single antenna at the transmitter and the receiver.

Fading channel

In wireless communications, the electromagnetic waves that propagate may be reflected, refracted, or diffracted. This causes constructive and destructive interference of the multiple signal paths between the transmitter and receiver. In addition, the propagating waves may also attenuate due to natural phenomena. As a result, the transmitted signal experiences fading and the propagation environment is represented by a fading channel coefficient h. This channel is an extention of the complex AWGN channel. The only difference is that every input symbol is subject to a fading channel coefficient. This channel coefficient varies according to a distribution that best approaches the propagation environment. In our study, we consider the Rayleigh fading distribution which is suited to situations where there are large numbers of signal paths and reflections and no dominant propagation along the line of sight between the transmitter and the receiver.

Slow-fading channel

In the slow-fading scenario, the channel state remains constant during the transmission of information, which means that every input symbol is subject to the same fading coefficient. The signal model is defined as

$$y[m] = hx[m] + w[m], \qquad w[m] \sim \mathcal{CN}(0,\, N), \tag{2.1}$$

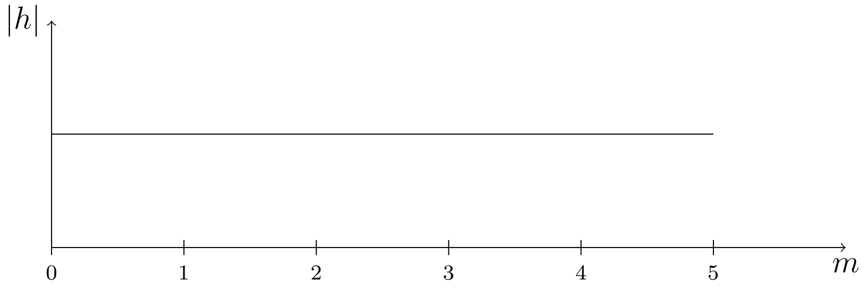


Figure 2.1: Slow fading.

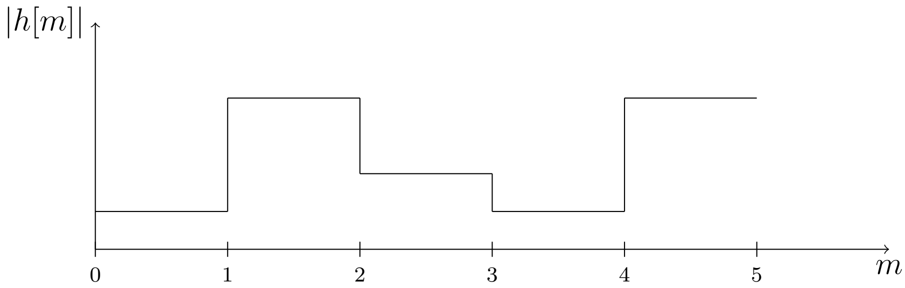


Figure 2.2: Fast fading.

where m denotes time. The capacity of the SISO channel is

$$C = \max_{f_x : E\left\{|x|^2\right\} \leq P} I(x; y) \tag{2.2}$$

which implies that

$$C = \log_2\left(1 + \frac{P|h|^2}{N}\right) \quad \text{bits/symbol.} \tag{2.3}$$

The proof is given in Chapter 6.

Fast-fading channel

In the fast fading scenario, the channel state varies during the transmission of information. For simplicity reasons, we assume that it varies in every transmitted symbol. The signal model is defined as

$$y[m] = h[m]x[m] + w[m], \qquad w[m] \sim \mathcal{CN}(0,\, N). \tag{2.4}$$

Each symbol experiences a different fade, where $h[m]$ are i.i.d. across different time instances m. Assuming L channel coefficients, we could calculate the capacity of each channel state, so that the mean capacity would be

$$C = \frac{1}{L}\sum_{m=1}^{L} \log_2\left(1 + \frac{P|h[m]|^2}{N}\right) \quad \text{bits/symbol.} \tag{2.5}$$

Waterfilling power allocation

If the transmitter had knowledge of the future channel states across time, the capacity formula of the fast fading channel could be optimized. Considering L channel states and a total power of LP, we can allocate more power when the channel gain (i.e. $|h[m]|$) is good and less or even no power when the channel gain is poor. Hence, we have

$$C = \max_{p_1,\dots,p_L} \frac{1}{L} \sum_{m=1}^{L} \log_2 \left(1 + \frac{p_m |h[m]|^2}{N}\right) \tag{2.6}$$

subject to the power constraints

$$\sum_{m=1}^{L} p_m = LP, \qquad p_m \geq 0, \qquad m = 1, ..., L. \tag{2.7}$$

We have to maximize a multivariable function subject to the constraint that another multivariable function equals a constant. Thus we use the Lagrangian function.

$$\mathcal{L}(\lambda, p_1, p_2, ..., p_L) := \sum_{m=1}^{L} \log_2 \left(1 + \frac{p_m |h[m]|^2}{N}\right) - \lambda \sum_{m=1}^{L} p_m \tag{2.8}$$

The new variable λ is called Lagrange multiplier. We need to calculate the power vector which is defined as

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix}. \tag{2.9}$$

We differentiate $\mathcal{L}$ with respect to the real vector $\boldsymbol{p}$. Then we set the gradient of $\mathcal{L}$ equal to the zero vector.

$$\nabla_{\boldsymbol{p}} \mathcal{L} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial p_1} \\ \frac{\partial \mathcal{L}}{\partial p_2} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial p_L} \end{bmatrix} = \mathbf{0}. \tag{2.10}$$

For simplicity, we are going to calculate only the first element of the vector. The other elements can be calculated similarly.

$$\frac{\partial \mathcal{L}}{\partial p_1} = 0 \tag{2.11}$$

$$\frac{\frac{|h[1]|^2}{N}}{1+\frac{p_1|h[1]|^2}{N}} - \lambda = 0 \tag{2.12}$$

$$\frac{\frac{|h[1]|^2}{N}}{1+\frac{p_1|h[1]|^2}{N}} = \lambda \tag{2.13}$$

$$\lambda\left(1+\frac{p_1|h[1]|^2}{N}\right) = \frac{|h[1]|^2}{N} \tag{2.14}$$

$$\left(1+\frac{p_1|h[1]|^2}{N}\right) = \frac{|h[1]|^2}{N}\frac{1}{\lambda} \tag{2.15}$$

$$\frac{p_1|h[1]|^2}{N} = \frac{|h[1]|^2}{N}\frac{1}{\lambda} - 1 \tag{2.16}$$

$$p_1 = \left(\frac{1}{\lambda} - \frac{N}{|h[1]|^2}\right). \tag{2.17}$$

Hence, we get the following result

$$\boldsymbol{p} = \begin{bmatrix} \left(\frac{1}{\lambda} - \frac{N}{|h[1]|^2}\right)^+ \\ \left(\frac{1}{\lambda} - \frac{N}{|h[2]|^2}\right)^+ \\ \vdots \\ \left(\frac{1}{\lambda} - \frac{N}{|h[L]|^2}\right)^+ \end{bmatrix} \tag{2.18}$$

where $x^+ := \max(x, 0)$. To find the parameter $\frac{1}{\lambda}$, we use the power constraint of equation (2.7) to obtain

$$\frac{1}{\lambda} - \frac{N}{|h[1]|^2} + \frac{1}{\lambda} - \frac{N}{|h[2]|^2} + ... + \frac{1}{\lambda} - \frac{N}{|h[L]|^2} = LP \tag{2.19}$$

$$\implies \frac{1}{\lambda} = \frac{LP + \frac{N}{|h[1]|^2} + \frac{N}{|h[2]|^2} + ... + \frac{N}{|h[L]|^2}}{L}. \tag{2.20}$$

We use the solution of (2.20) to find the power vector of (2.18). If any of the power elements turns to be zero, we reallocate the power to the remaining channel states. Practically, this means that we return to equation (2.20) and find the new parameter $\frac{1}{\lambda}$, without taking into account the channel states that correspond to zero power. This is a repetitive process and is called the waterfilling algorithm. Note that all the parameters stated in the procedure below (e.g., $\boldsymbol{n}$, $\boldsymbol{p}$) refer to local parameters that change accordingly only in this local environment. What really matters is the output parameter $\frac{1}{\lambda}$.

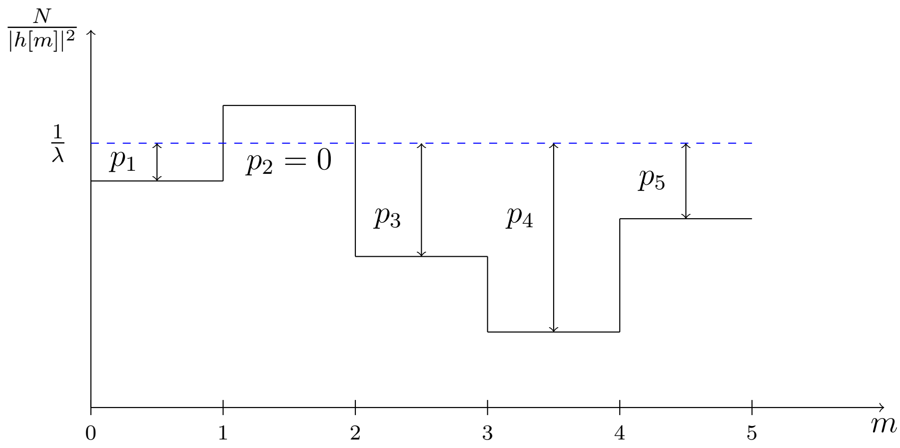


Figure 2.3: Waterfilling power allocation.

Algorithm 1 Waterfilling

1: **procedure** WATERFILLING
2: **Input:** $\boldsymbol{n} \leftarrow \frac{N}{|h[1]|^2}, \frac{N}{|h[2]|^2}, \ldots, \frac{N}{|h[L]|^2},\ P_{total} \leftarrow LP,\ L$
3: $S \leftarrow$ summation of all elements in $\boldsymbol{n}$
4: $\frac{1}{\lambda} \leftarrow \frac{P_{total}+S}{L}$
5: $\boldsymbol{p} \leftarrow \left(\frac{1}{\lambda} - \boldsymbol{n}\right)^+$
6: **Loop:** check for zero elements in $\boldsymbol{p}$
7: **If true**
8: set elements in $\boldsymbol{n}$ corresponding to zero power equal to zero
9: $Z \leftarrow$ number of zero elements in $\boldsymbol{n}$
10: $S \leftarrow$ summation of all elements in $\boldsymbol{n}$
11: $\frac{1}{\lambda} \leftarrow \frac{P_{total}+S}{L-Z}$
12: $\boldsymbol{p} \leftarrow \left(\frac{1}{\lambda} - \boldsymbol{n}\right)^+$
13: **goto** line 6
14: **else**
15: **Output:** $\frac{1}{\lambda}$

After finding the final value of $\frac{1}{\lambda}$, we substitute it in (2.18) to calculate the optimal power vector. Hence, the capacity is proved to be

$$C = \frac{1}{L} \sum_{m=1}^{L} \log_2 \left(1 + \frac{p_m |h[m]|^2}{N}\right) \quad \text{bits/symbol.} \tag{2.21}$$

An alternative approach that was tested and proved to give the same result regarding the parameter $\frac{1}{\lambda}$ is the following.

Algorithm 2 Waterfilling

procedure WATERFILLING
  Input: $\boldsymbol{n} \leftarrow \frac{N}{|h[1]|^2}, \frac{N}{|h[2]|^2}, ..., \frac{N}{|h[L]|^2},\ P_{total} \leftarrow LP,\ L$
  $n_{min} \leftarrow$ minimum element in $\boldsymbol{n}$
  $\frac{1}{\lambda} \leftarrow n_{min} + \frac{P_{total}}{L}$
  $\boldsymbol{p} \leftarrow \left(\frac{1}{\lambda} - \boldsymbol{n}\right)^+$
  $P_{allocated} \leftarrow$ summation of all elements in $\boldsymbol{p}$
  While $|P_{total} - P_{allocated}| > tolerance$ **do** ▷ e.g. $tolerance \leftarrow 10^{-5}$
  $\frac{1}{\lambda} \leftarrow \frac{1}{\lambda} + \frac{P_{total} - P_{allocated}}{L}$
  $\boldsymbol{p} \leftarrow \left(\frac{1}{\lambda} - \boldsymbol{n}\right)^+$
  $P_{allocated} \leftarrow$ summation of all elements in $\boldsymbol{p}$
  end loop
  Output: $\frac{1}{\lambda}$

So far, we have considered the waterfilling power allocation, where the transmitter is assumed to know the future channel states over time. In practice, the transmitter is able to know only the distribution f_h of the fading process.

Consider a virtual channel with significantly large length $L_{\text{virtual}} \gg 1$, which means L_{virtual} i.i.d. channel states according to the known fading process. The transmitter performs the waterfilling algorithm for that virtual channel subject to a power constraint of $L_{\text{virtual}}P$ and stores the parameter $\frac{1}{\lambda(f_h)}$. Then, it uses this parameter to find the optimal power allocation for the actual transmission channel. Therefore, the optimal power depends only on the current channel state of the actual transmission channel and not on its future channel states, hence

$$P(h[m]) = \left(\frac{1}{\lambda(f_h)} - \frac{N}{|h[m]|^2}\right)^+, \tag{2.22}$$

$$C = \frac{1}{L}\sum_{m=1}^{L} \log_2\left(1 + \frac{P(h[m])|h[m]|^2}{N}\right) \quad \text{bits/symbol.} \tag{2.23}$$

Interestingly, we noticed that the total power constraint of LP is not met by the summation of all power elements in equation (2.22). We estimated the distribution of the total power for various numbers of transmissions subject to a power constraint of $LP = 200$. Note that, if we increase the number of transmissions L, then we should reduce the power P per channel state in order to keep the overall power constraint of LP constant. The noise power N remains constant, so the average SNR reduces when we increase the number of transmissions.

In Figure 2.4, we plot the PDF of the total power. The plot shows that, for larger number of transmissions L, it is more likely to get a total power close to the power constraint. In any case, we noticed that the power constraint is on average satisfied, when the number of transmissions (or, equivalently, the channel length) $\gg 1$. As a result, it is meaningful to evaluate the performance of the following algorithms for different number of transmissions and signal-to-noise ratios.

- **Equal Power Allocation:** Equation (2.5).
- **Waterfilling using $\boldsymbol{\lambda(h)}$:** Equation (2.21).

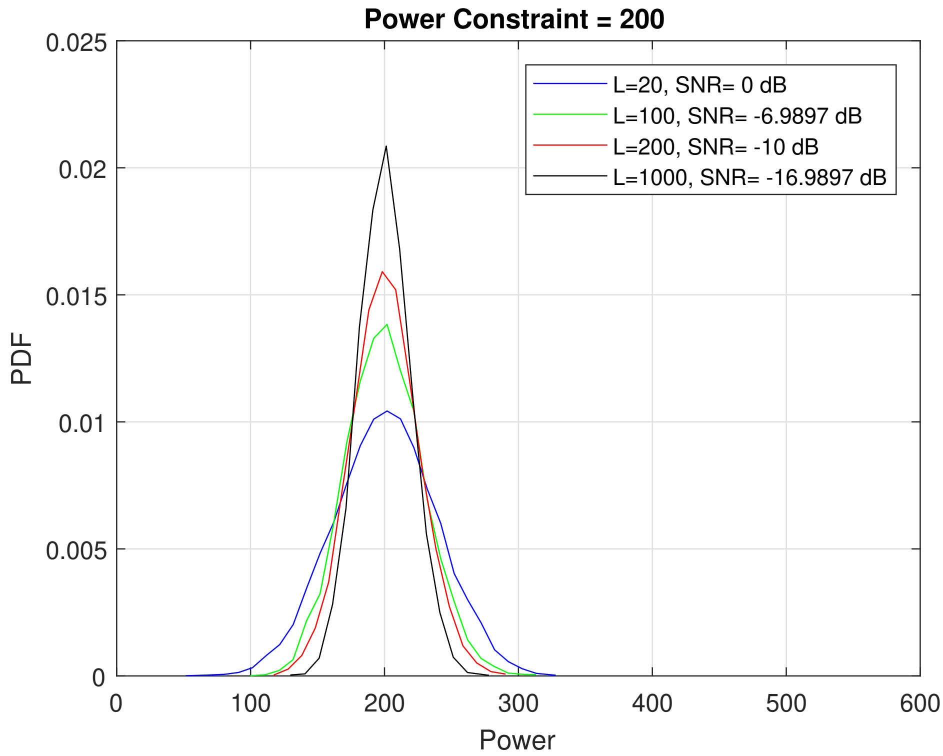


Figure 2.4: PDF of power (SISO channel).

- **Waterfilling using $\lambda(f_h)$-Power Violation:** Equation (2.23).
- **Waterfilling using $\lambda(f_h)$-Power Correction:** Equation (2.23). When the total power is about to exceed the power constraint, we correct the power to the appropriate channel state and allocate zero power to the remaining ones. Otherwise, the remaining power is allocated to the last channel state. Thus, the power constraint is satisfied.

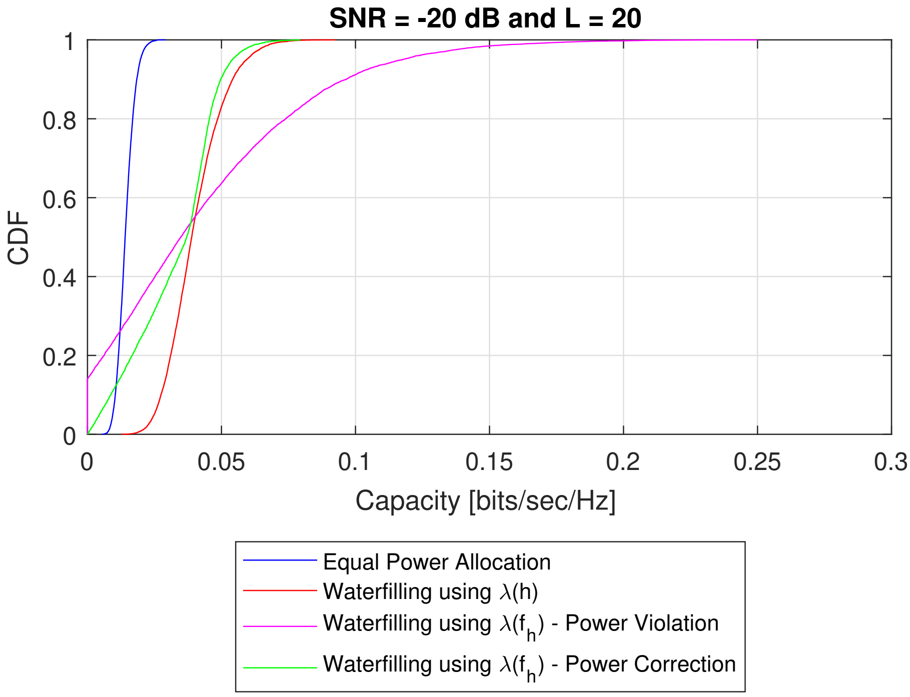
Simulation results

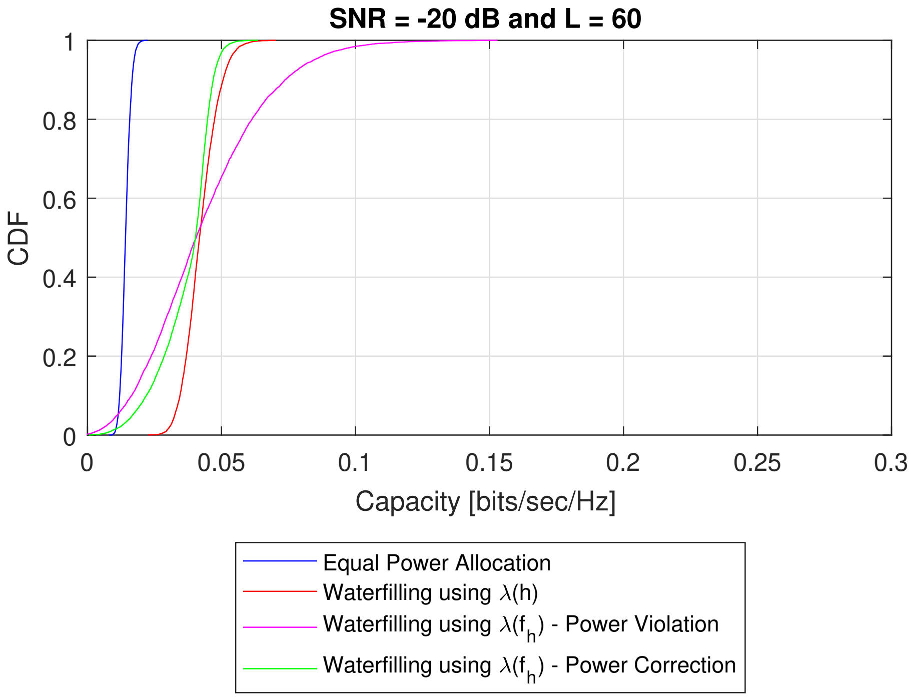
We calculated 10^4 capacity values for each algorithm and plotted the CDF of them for different signal-to-noise ratios and number of transmissions. In addition, we calculated the mean capacity of these values. It is clear that as we increase the SNR, the capacity performance of these algorithms tends to be the same. We define the ergodic capacity of the fast fading channel as follows.

$$C = \mathbb{E}\left[\log_2\left(1 + \frac{P(h)|h|^2}{N}\right)\right] \quad \text{bits/symbol}, \tag{2.24}$$

where

$$P(h) = \left(\frac{1}{\lambda} - \frac{N}{|h|^2}\right)^+, \quad \mathbb{E}[P(h)] = P. \tag{2.25}$$


SNR = -20 dB and L = 20
CDF
Capacity [bits/sec/Hz]
Equal Power Allocation
Waterfilling using λ(h)
Waterfilling using λ(f_h) - Power Violation
Waterfilling using λ(f_h) - Power Correction


SNR = -20 dB and L = 60
CDF
Capacity [bits/sec/Hz]
Equal Power Allocation
Waterfilling using λ(h)
Waterfilling using λ(f_h) - Power Violation
Waterfilling using λ(f_h) - Power Correction

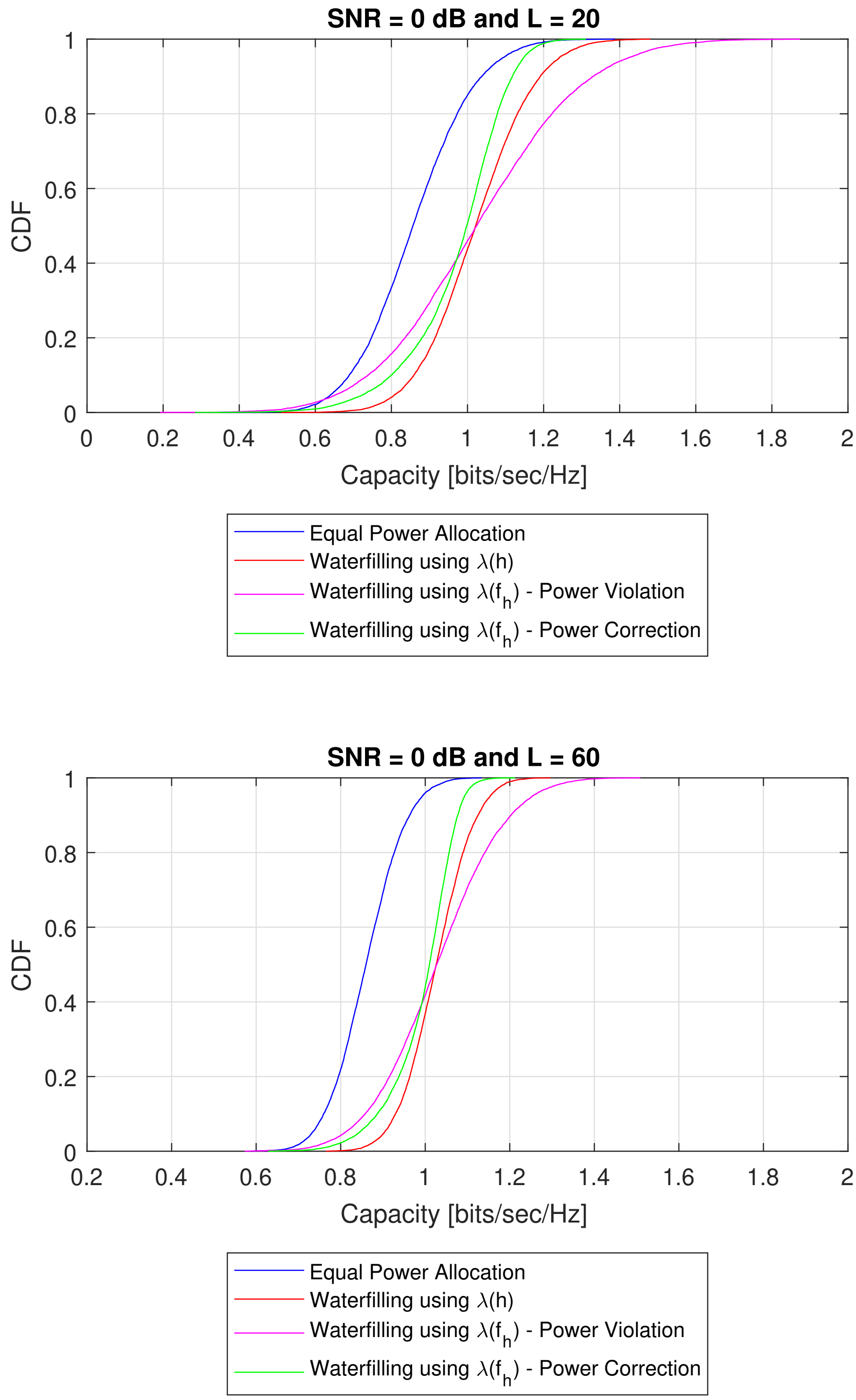
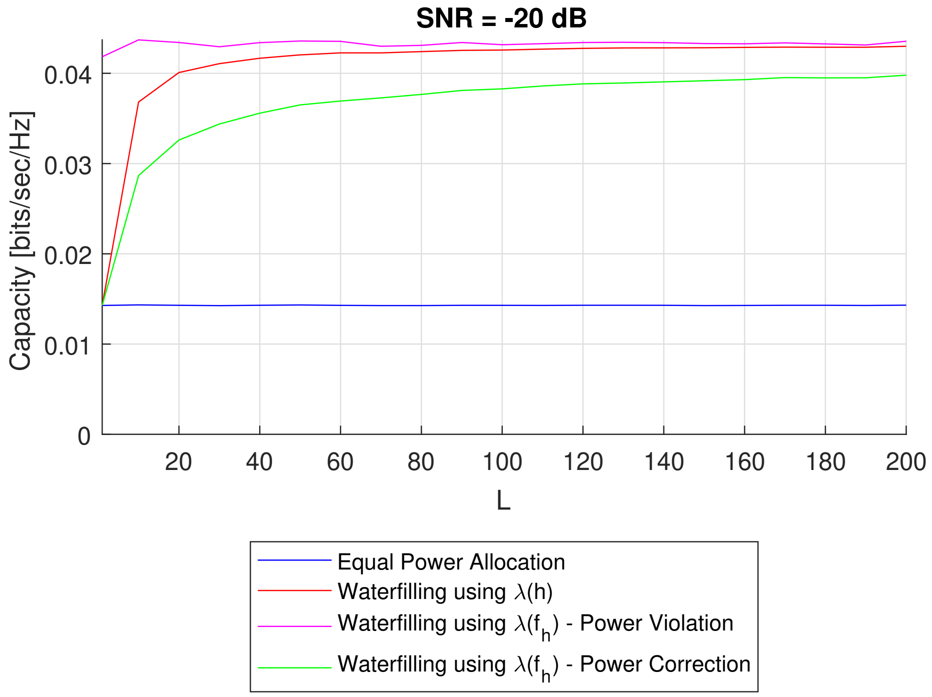
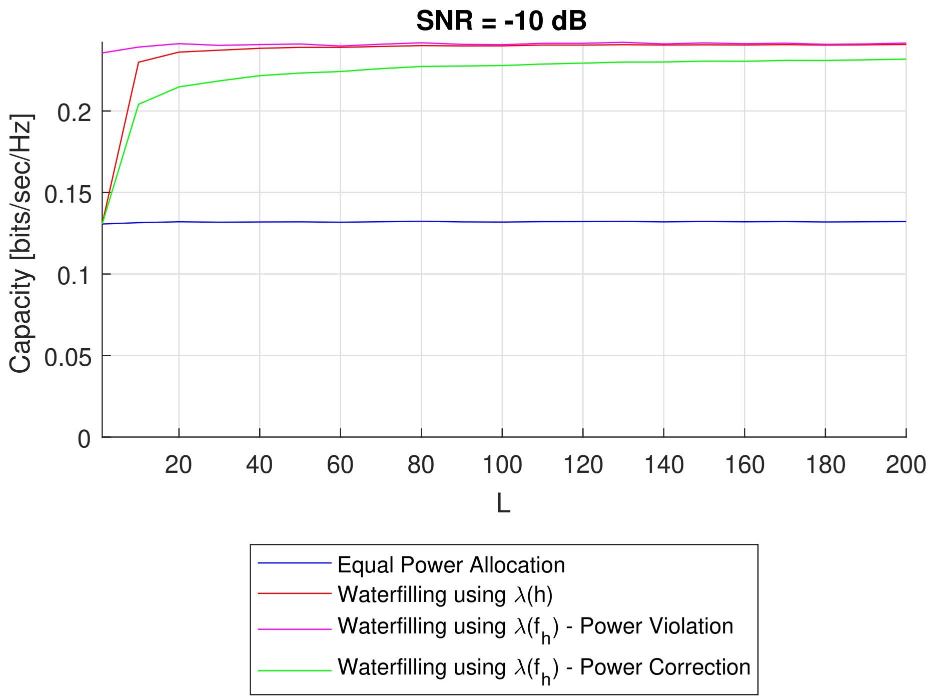


Figure 2.5: CDF of capacity (SISO channel).


SNR = -20 dB
Capacity [bits/sec/Hz]
0
0.01
0.02
0.03
0.04
20
40
60
80
100
120
140
160
180
200
L
Equal Power Allocation
Waterfilling using λ(h)
Waterfilling using λ(f_h) - Power Violation
Waterfilling using λ(f_h) - Power Correction


SNR = -10 dB
Capacity [bits/sec/Hz]
0
0.05
0.1
0.15
0.2
20
40
60
80
100
120
140
160
180
200
L
Equal Power Allocation
Waterfilling using λ(h)
Waterfilling using λ(f_h) - Power Violation
Waterfilling using λ(f_h) - Power Correction

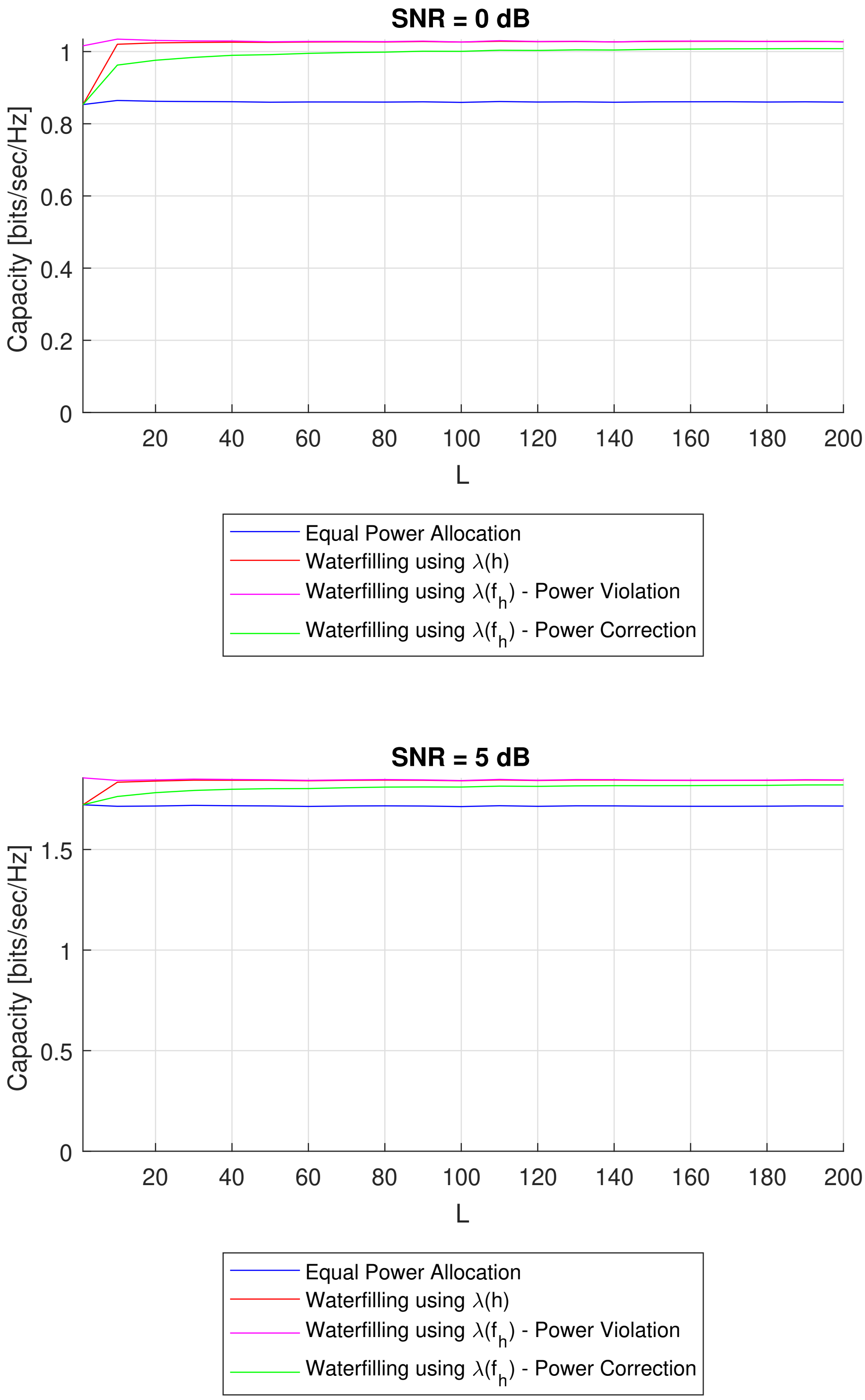


Figure 2.6: Mean capacity vs channel length (SISO channel).

2.2 Multiple-input Multiple-output (MIMO) System

MIMO stands for a multiple-input multiple-output system, which practically means that there are multiple antennas at the transmitter and the receiver.

Slow-fading channel

We calculate the capacity of each channel state, focusing on the slow fading scenario, where the channel remains constant during the transmission of information. Assuming n_t transmit antennas and n_r receive antennas, the signal model is defined as

$$\boldsymbol{y}[m] = \boldsymbol{H}\boldsymbol{x}[m] + \boldsymbol{w}[m], \quad \boldsymbol{w}[m] \sim \mathcal{CN}(0,\, N\boldsymbol{I}_{n_r}), \tag{2.26}$$

where we have

$$\begin{bmatrix} y_1[m] \\ y_2[m] \\ \vdots \\ y_{n_r}[m] \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n_t} \\ h_{21} & h_{22} & & \\ \vdots & & \ddots & \\ h_{n_r1} & & & h_{n_rn_t} \end{bmatrix} \begin{bmatrix} x_1[m] \\ x_2[m] \\ \vdots \\ x_{n_t}[m] \end{bmatrix} + \begin{bmatrix} w_1[m] \\ w_2[m] \\ \vdots \\ w_{n_r}[m] \end{bmatrix}. \tag{2.27}$$

Hence, we get

$$y_1[m] = h_{11}x_1[m] + h_{12}x_2[m] + ... + h_{1n_t}x_{n_t}[m] + w_1[m], \tag{2.28}$$

$$\vdots$$

$$y_{n_r}[m] = h_{n_r1}x_1[m] + h_{n_r2}x_2[m] + ... + h_{n_rn_t}x_{n_t}[m] + w_{n_r}[m]. \tag{2.29}$$

Note that $h_{ij} \sim \mathcal{CN}(0,\, 1)$ where $i = 1, ..., n_r$ and $j = 1, ..., n_t$. We assume that all elements in the channel matrix are i.i.d. according to that fading distribution.

Singular value decomposition (SVD)

The channel matrix $\boldsymbol{H}$ can be expressed in terms of its singular value decomposition.

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\boldsymbol{H}} \tag{2.30}$$

where $\boldsymbol{U}_{n_r\times n_r}, \boldsymbol{V}_{n_t\times n_t}$ are unitary matrices, which means that

$$\boldsymbol{U}\boldsymbol{U}^H = \boldsymbol{U}^H\boldsymbol{U} = \boldsymbol{I}, \tag{2.31}$$

$$\boldsymbol{V}\boldsymbol{V}^H = \boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}. \tag{2.32}$$

$\boldsymbol{\Sigma}_{n_r\times n_t}$ is a diagonal matrix of the form

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}. \tag{2.33}$$

The diagonal elements of $\boldsymbol{\Sigma}$ are nonnegative real numbers and are called singular values of the matrix $\boldsymbol{H}$, where

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r. \tag{2.34}$$

Since the elements of matrix $\boldsymbol{H}$ are i.i.d., it holds that

$$r = \text{rank}(\boldsymbol{H}) = \min(n_t, n_r). \tag{2.35}$$

The transmitter can perform pre-processing of the input symbols while the receiver can perform post-processing of the output symbols. Pre-processing and post-processing are based on the SVD of the channel matrix.

$$\tilde{\boldsymbol{x}}[m] = \boldsymbol{V} \begin{bmatrix} x_1[m] \\ x_2[m] \\ \vdots \\ x_{n_t}[m] \end{bmatrix}, \quad \tilde{\boldsymbol{y}}[m] = \boldsymbol{U}^H \begin{bmatrix} y_1[m] \\ y_2[m] \\ \vdots \\ y_{n_r}[m] \end{bmatrix}. \tag{2.36}$$

As a result, a new noise vector is also defined as

$$\tilde{\boldsymbol{w}}[m] = \boldsymbol{U}^H \begin{bmatrix} w_1[m] \\ w_2[m] \\ \vdots \\ w_{n_r}[m] \end{bmatrix}. \tag{2.37}$$

We use the signal model defined in equation (2.26) and apply pre-processing to obtain

$$\boldsymbol{y}[m] = \boldsymbol{H}\tilde{\boldsymbol{x}}[m] + \boldsymbol{w}[m] \tag{2.38}$$
$$= \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H\tilde{\boldsymbol{x}}[m] + \boldsymbol{w}[m] \tag{2.39}$$
$$= \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H\boldsymbol{V}\boldsymbol{x}[m] + \boldsymbol{w}[m] \tag{2.40}$$
$$= \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{x}[m] + \boldsymbol{w}[m]. \tag{2.41}$$

Then, we apply post-processing to obtain

$$\boldsymbol{U}^H\boldsymbol{y}[m] = \boldsymbol{U}^H\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{x}[m] + \boldsymbol{U}^H\boldsymbol{w}[m], \tag{2.42}$$
$$\Rightarrow \tilde{\boldsymbol{y}}[m] = \boldsymbol{\Sigma}\boldsymbol{x}[m] + \tilde{\boldsymbol{w}}[m]. \tag{2.43}$$

We just proved that a MIMO channel can be decomposed into a set of r parallel channels. Note that since the matrix $\boldsymbol{\Sigma}$ has a maximum of r diagonal elements, we can transmit r symbols at each time.

$$\begin{bmatrix} \tilde{y}_1[m] \\ \vdots \\ \tilde{y}_r[m] \end{bmatrix} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} x_1[m] \\ \vdots \\ x_r[m] \end{bmatrix} + \begin{bmatrix} \tilde{w}_1[m] \\ \vdots \\ \tilde{w}_r[m] \end{bmatrix}. \tag{2.44}$$

The parallel channels are defined as

$$\tilde{y}_1[m] = \sigma_1 x_1[m] + \tilde{w}_1[m], \tag{2.45}$$
$$\vdots$$
$$\tilde{y}_r[m] = \sigma_r x_r[m] + \tilde{w}_r[m]. \tag{2.46}$$

Therefore, the capacity of the MIMO channel is the sum of capacities of each individual channel, i.e.,

$$C = \sum_{i=1}^{r} \log_2\left(1 + \frac{p_i\sigma_i^2}{N}\right) \quad \text{bits/symbol} \tag{2.47}$$

where the optimal power vector is given by the waterfilling algorithm

$$\boldsymbol{p} = \begin{bmatrix} \left(\frac{1}{\lambda} - \frac{N}{\sigma_1^2}\right)^+ \\ \left(\frac{1}{\lambda} - \frac{N}{\sigma_2^2}\right)^+ \\ \vdots \\ \left(\frac{1}{\lambda} - \frac{N}{\sigma_r^2}\right)^+ \end{bmatrix} \tag{2.48}$$

subject to the power constraint

$$\sum_{i=1}^{r} p_i = P, \quad p_i \geq 0, \quad i = 1, ..., r. \tag{2.49}$$

Fast-fading channel

For the fast-fading scenario, the signal model is defined as

$$\boldsymbol{y}[m] = \boldsymbol{H}[m]\boldsymbol{x}[m] + \boldsymbol{w}[m], \quad \boldsymbol{w}[m] \sim \mathcal{CN}(0,\, N\boldsymbol{I}_{n_r}). \tag{2.50}$$

We have proved that each time the MIMO channel can be decomposed into a set of parallel channels, where the power allocation to each of them is given by the waterfilling algorithm. This can be considered as waterfilling over space, as we take advantage of the MIMO system capabilities. Assuming L transmissions, we calculate the capacity of each of them in order to calculate the mean capacity

$$C = \frac{1}{L}\sum_{m=1}^{L}\sum_{i=1}^{r} \log_2\left(1 + \frac{p_{m,i}\sigma_{m,i}^2}{N}\right) \quad \text{bits/symbol} \tag{2.51}$$

subject to the power constraint

$$\sum_{m=1}^{L}\sum_{i=1}^{r} p_{m,i} = LP, \quad p_{m,i} \geq 0, \quad i = 1, ..., r \quad m = 1, ..., L. \tag{2.52}$$

Obviously that involves waterfilling over space and time. Using the same approach discussed in the SISO case, we define the ergodic capacity of the fast-fading channel as

$$C = \sum_{i=1}^{r} \mathbb{E}\left[\log_2\left(1 + \frac{P(\sigma_i)\sigma_i^2}{N}\right)\right] \quad \text{bits/symbol} \tag{2.53}$$

where

$$P(\sigma_i) = \left(\frac{1}{\lambda} - \frac{N}{\sigma_i^2}\right)^+, \quad \sum_{i=1}^{r} \mathbb{E}[P(\sigma_i)] = P. \tag{2.54}$$

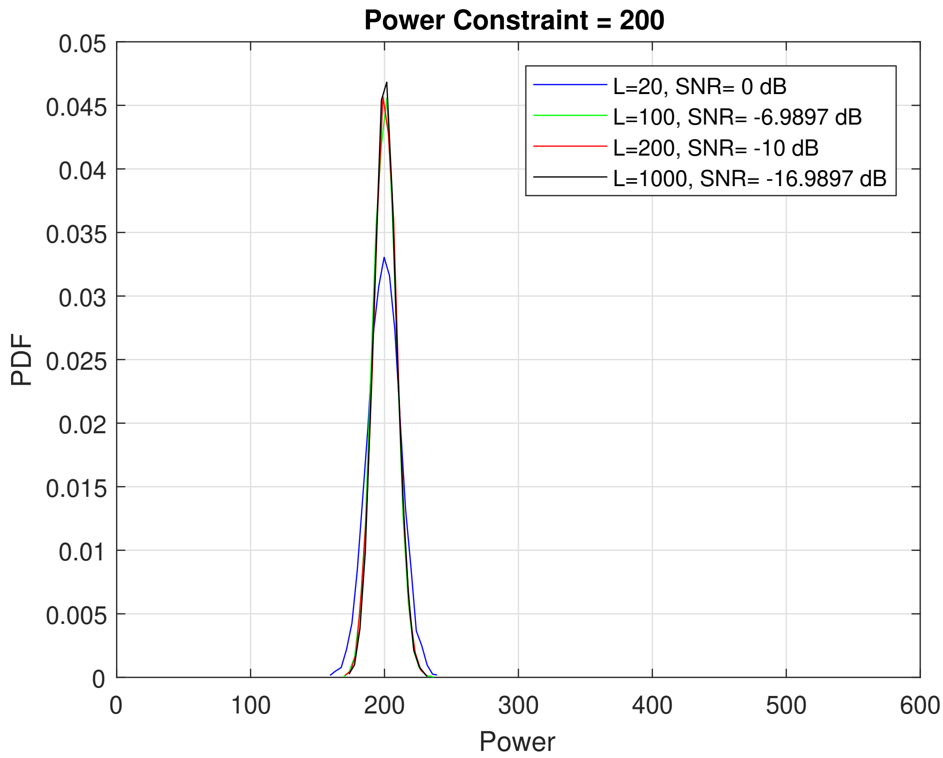


Figure 2.7: PDF of power (2x4 MIMO channel)

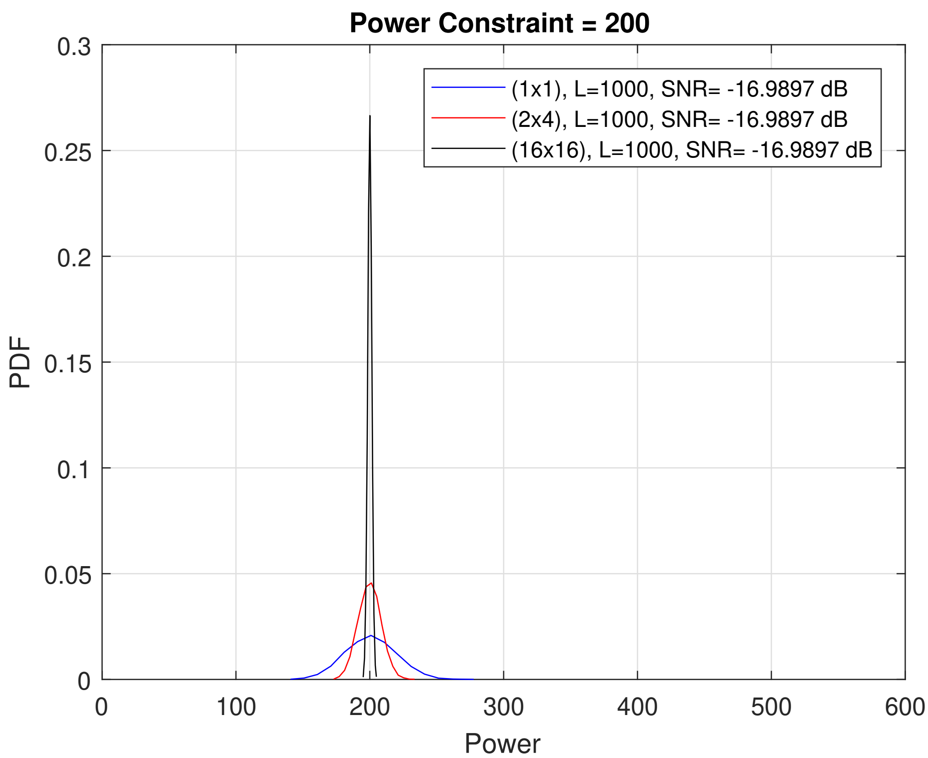
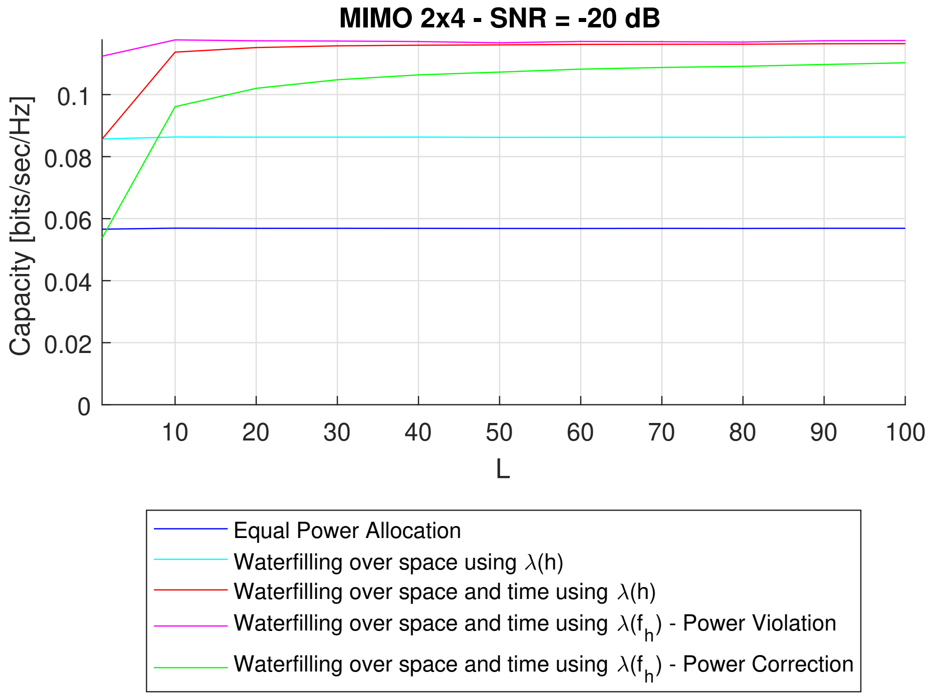
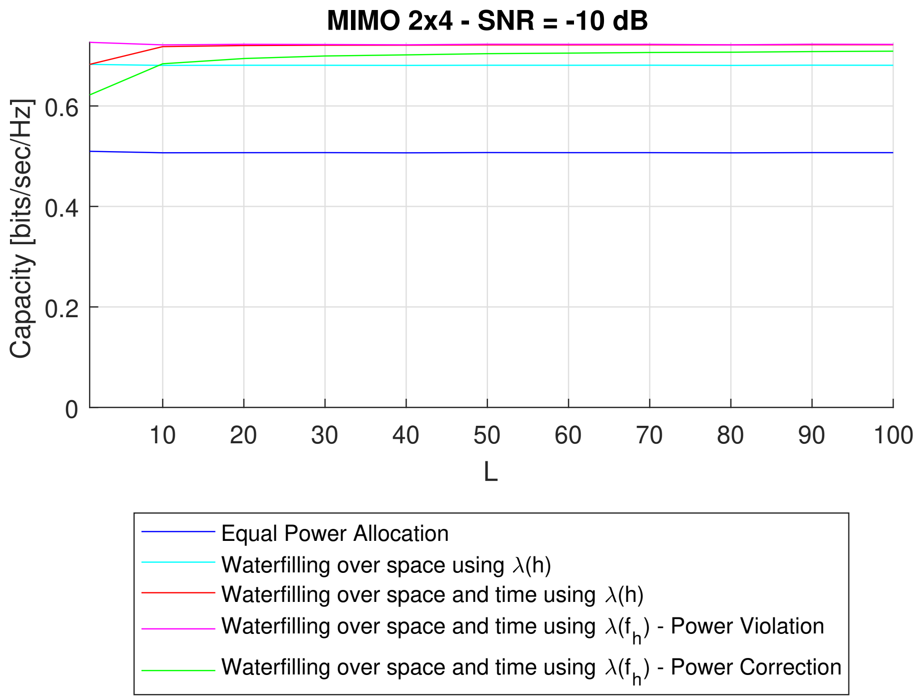


Figure 2.8: PDF of power (1x1, 2x4, 16x16 MIMO channel)


MIMO 2x4 - SNR = -20 dB
Capacity [bits/sec/Hz]
0
0.02
0.04
0.06
0.08
0.1
10
20
30
40
50
60
70
80
90
100
L
Equal Power Allocation
Waterfilling over space using λ(h)
Waterfilling over space and time using λ(h)
Waterfilling over space and time using λ(f_h) - Power Violation
Waterfilling over space and time using λ(f_h) - Power Correction


MIMO 2x4 - SNR = -10 dB
Capacity [bits/sec/Hz]
0
0.2
0.4
0.6
10
20
30
40
50
60
70
80
90
100
L
Equal Power Allocation
Waterfilling over space using λ(h)
Waterfilling over space and time using λ(h)
Waterfilling over space and time using λ(f_h) - Power Violation
Waterfilling over space and time using λ(f_h) - Power Correction

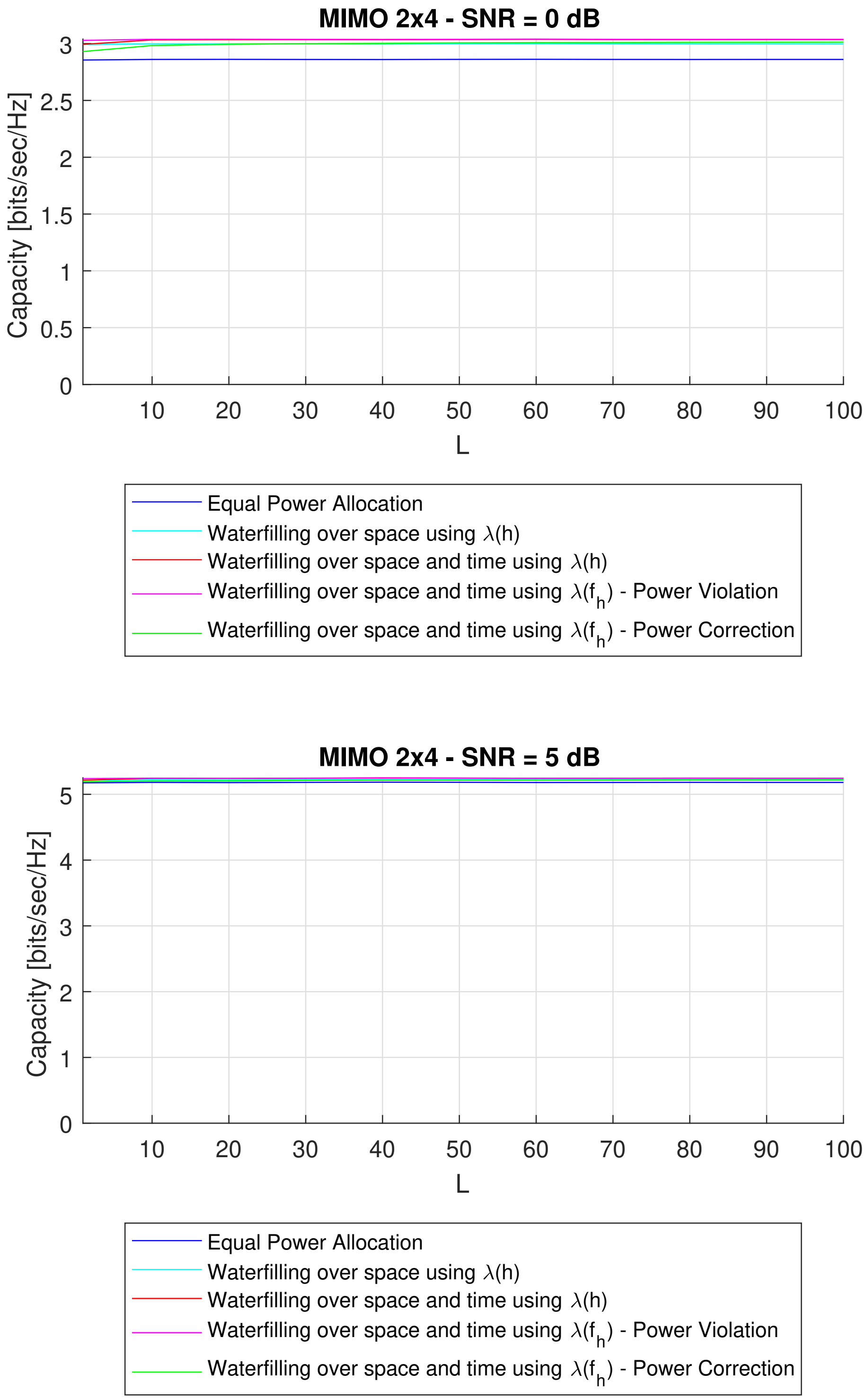


Figure 2.9: Mean capacity vs channel length (2x4 MIMO).

Simulation results

This time the PDF graph is narrower, Figure 2.7, in comparison with the one referring to a SISO system. It is clear that as we increase the number of antennas involved in the system, the range of values of the total power becomes negligibly small around the power constraint. That is confirmed by Figure 2.8. We evaluated the performance of the known four algorithms, including one in which each time only waterfilling over space is performed. Regarding the algorithm which involves power correction, the acceptable power is allocated to the parallel channels of the appropriate transmission according to the waterfilling algorithm. Specifically, when the total power does not exceed the power constraint, we refer to the last transmission. Otherwise, we refer to the transmission at which the power violation occurs. We obtained 10^4 capacity values and calculated the mean capacity of them. It is obvious that a MIMO $M \times N$ system is equivalent with a $N \times M$ regarding capacity, when the transmissison channel is known at the transmitter. In our case we tested a MIMO system with two transmit and four receive antennas. Consider that an important feature of a MIMO system, when the transmitter can track the channel and apply proper pre-processing, is not just the optimal power allocation but the fact that the receiver can decode seperately a set of independent transmitted symbols. That can be seen when we compare equations (2.28) and (2.29) with (2.45) and (2.46).

2.3 Multiple-input Single-output (MISO) System

MISO stands for a multiple-input single-output system. Thus, we consider multiple antennas at the transmitter and one antenna at the receiver.

Slow-fading channel

Assuming n_t antennas at the transmitter and one antenna at the receiver, the signal model is defined as

$$y[m] = \boldsymbol{h}^H \boldsymbol{x}[m] + w[m], \qquad w[m] \sim \mathcal{CN}(0,\, N). \tag{2.55}$$

The transmitter tracks the channel, so it performs pre-processing of the input vector based on the transmission channel. The goal is to maximize the received power in order to maximize capacity as well, since capacity is computed as

$$C = \log_2(1 + \text{SNR}) \quad \text{bits/symbol}. \tag{2.56}$$

We transmit one symbol at each time and we denote the optimal filter as $\boldsymbol{f}$. The system's SNR is defined as

$$\text{SNR} = \frac{E\left\{|\boldsymbol{h}^H \boldsymbol{x}[m]|^2\right\}}{E\left\{|w[m]|^2\right\}} = \frac{E\left\{|\boldsymbol{h}^H \boldsymbol{f} x[m]|^2\right\}}{E\left\{|w[m]|^2\right\}}. \tag{2.57}$$

The input symbol $x[m]$ has a power constraint of P, so we need to maximize the quantity $\boldsymbol{h}^H \boldsymbol{f}$ subject to that power constraint. The dot product of two vectors is defined as

$$\boldsymbol{h}^H \boldsymbol{f} = \|\boldsymbol{h}\| \|\boldsymbol{f}\| \cos(\theta) \tag{2.58}$$

where θ is the angle between the two vectors. The dot product between them is maximized when the two vectors are collinear along the same direction, because in that case we get

$$\theta = 0 \implies \cos(\theta) = 1. \tag{2.59}$$

Therefore,

$$\boldsymbol{f} = c\boldsymbol{h}, \tag{2.60}$$

where $c > 0$. However, in order to have the total transmit power constraint satisfied, we need to define the constant c as follows

$$c = \frac{1}{\|\boldsymbol{h}\|} \implies \boldsymbol{f} = \frac{\boldsymbol{h}}{\|\boldsymbol{h}\|}. \tag{2.61}$$

The input vector after pre-processing at the transmitter is defined as

$$\tilde{\boldsymbol{x}}[m] = \frac{\boldsymbol{h}}{\|\boldsymbol{h}\|}x[m]. \tag{2.62}$$

Using the signal model in equation (2.55) we get

$$\begin{aligned} y[m] &= \boldsymbol{h}^H\tilde{\boldsymbol{x}}[m] + w[m] && (2.63)\\ &= \boldsymbol{h}^H\frac{\boldsymbol{h}}{\|\boldsymbol{h}\|}x[m] + w[m] && (2.64)\\ &= \frac{\|\boldsymbol{h}\|^2}{\|\boldsymbol{h}\|}x[m] + w[m] && (2.65)\\ &= \|\boldsymbol{h}\|x[m] + w[m]. && (2.66) \end{aligned}$$

It was proved that a MISO system corresponds to an equivalent SISO. We could also calculate directly the SNR in order to compute capacity.

$$\mathrm{SNR} = \frac{E\left\{|\boldsymbol{h}^H\frac{\boldsymbol{h}}{\|\boldsymbol{h}\|}x[m]|^2\right\}}{E\left\{|w[m]|^2\right\}} = \frac{E\left\{\left|\|\boldsymbol{h}\|x[m]\right|^2\right\}}{E\left\{|w[m]|^2\right\}} = \frac{E\left\{\left|x[m]\right|^2\right\}\|\boldsymbol{h}\|^2}{E\left\{|w[m]|^2\right\}} = \frac{P\|\boldsymbol{h}\|^2}{N}. \tag{2.67}$$

An alternative approach could have been the singular value decomposition of the channel vector. Pre-processing could be applied in the exact same way as we did in the MIMO case. The only difference is that there is no need to apply any post-processing since the matrix $U = 1$. The transmission channel is a vector so its rank $= 1$. Hence, there is only one singular value

$$\sigma_1 = \|\boldsymbol{h}\|. \tag{2.68}$$

The equivalent representation of the channel model is

$$y[m] = \sigma_1 x[m] + w[m]. \tag{2.69}$$

As a result, the capacity of the MISO channel is

$$C = \log_2\left(1 + \frac{P\|\boldsymbol{h}\|^2}{N}\right) \quad \text{bits/symbol} \tag{2.70}$$

or, alternatively,

$$C = \log_2\left(1 + \frac{P\sigma_1^2}{N}\right) \quad \text{bits/symbol.} \tag{2.71}$$

Fast-fading channel

In this case, the channel does not remain constant during the transmission of information. The signal model is defined as

$$y[m] = \boldsymbol{h}^H[m]\boldsymbol{x}[m] + w[m], \qquad w[m] \sim \mathcal{CN}(0,\, N). \tag{2.72}$$

Assuming L channel states, we calculate the capacity of each channel state so as to calculate the mean capacity

$$C = \frac{1}{L}\sum_{m=1}^{L} \log_2\left(1 + \frac{p_m\|\boldsymbol{h}[m]\|^2}{N}\right) \quad \text{bits/symbol} \tag{2.73}$$

subject to the power constraint

$$\sum_{m=1}^{L} p_m = LP, \quad p_m \geq 0, \quad m = 1, ..., L. \tag{2.74}$$

The optimal power allocation involves waterfilling. We also define the ergodic capacity of the fast-fading channel as

$$C = \mathbb{E}\left[\log_2\left(1 + \frac{P(\boldsymbol{h})\|\boldsymbol{h}\|^2}{N}\right)\right] \quad \text{bits/symbol} \tag{2.75}$$

where

$$P(\boldsymbol{h}) = \left(\frac{1}{\lambda} - \frac{N}{\|\boldsymbol{h}\|^2}\right)^+, \quad \mathbb{E}[P(\boldsymbol{h})] = P. \tag{2.76}$$

2.4 Single-input Multiple-output (SIMO) System

SIMO stands for a single-input multiple-output system. Hence, there is a single antenna at the transmitter and multiple antennas at the receiver.

Slow-fading channel

Assuming one antenna at the transmitter and n_r antennas at the receiver, the signal model is considered as

$$\boldsymbol{y}[m] = \boldsymbol{h}x[m] + \boldsymbol{w}[m], \qquad \boldsymbol{w}[m] \sim \mathcal{CN}(0,\, N\boldsymbol{I}_{n_r}). \tag{2.77}$$

The receiver can always track the channel, so it can perform post-processing of the output vector based on the transmission channel. We need to choose the optimal filter so as to maximize the received power and capacity as well. We denote this filter as $\boldsymbol{f}$. Hence, we get

$$\boldsymbol{f}^H\boldsymbol{y}[m] = \boldsymbol{f}^H(\boldsymbol{h}x[m] + \boldsymbol{w}[m]) = \boldsymbol{f}^H\boldsymbol{h}x[m] + \boldsymbol{f}^H\boldsymbol{w}[m]. \tag{2.78}$$

The system's SNR is defined as

$$\mathrm{SNR} = \frac{E\left\{|\boldsymbol{f}^H \boldsymbol{h} x[m]|^2\right\}}{E\left\{|\boldsymbol{f}^H \boldsymbol{w}[m]|^2\right\}}. \tag{2.79}$$

We need to maximize the quantity $\boldsymbol{f}^H \boldsymbol{h}$, where the dot product between the two vectors is defined as

$$\boldsymbol{f}^H \boldsymbol{h} = \|\boldsymbol{f}\| \|\boldsymbol{h}\| \cos(\theta). \tag{2.80}$$

We have already proved that the optimal filter is

$$\boldsymbol{f} = c\boldsymbol{h} \tag{2.81}$$

where $c > 0$. In this case, there is no power constraint since the filter is applied at the receiver. However, the SNR remains the same regardless the selection of the constant c. Specifically, the signal power is considered as

$$E\left\{\left|c\boldsymbol{h}^H \boldsymbol{h} x[m]\right|^2\right\} = E\left\{\left|c\|\boldsymbol{h}\|^2 x[m]\right|^2\right\} = c^2 \|\boldsymbol{h}\|^4 E\left\{|x[m]|^2\right\} = c^2 \|\boldsymbol{h}\|^4 P \tag{2.82}$$

whereas the noise power is

$$E\left\{\left|c\boldsymbol{h}^H \boldsymbol{w}[m]\right|^2\right\} = E\left\{(c\boldsymbol{h}^H \boldsymbol{w}[m])(c\boldsymbol{h}^H \boldsymbol{w}[m])^H\right\} = c\boldsymbol{h}^H E\left\{\boldsymbol{w}[m]\boldsymbol{w}^H[m]\right\} \boldsymbol{h} c = c\boldsymbol{h}^H N \boldsymbol{I}_{n_r} \boldsymbol{h} c = c^2 \|\boldsymbol{h}\|^2 N. \tag{2.83}$$

Therefore, the SNR of the system is

$$\mathrm{SNR} = \frac{E\left\{|\boldsymbol{f}^H \boldsymbol{h} x[m]|^2\right\}}{E\left\{|\boldsymbol{f}^H \boldsymbol{w}[m]|^2\right\}} = \frac{c^2 \|\boldsymbol{h}\|^4 P}{c^2 \|\boldsymbol{h}\|^2 N} = \frac{P\|\boldsymbol{h}\|^2}{N}. \tag{2.84}$$

The singular-value decomposition of the channel vector leads to the same result as well. Post-processing could be applied in the same way as we did in the MIMO case. This time there is no need to apply pre-processing since the matrix $V = 1$. The transmission channel is a vector so its rank = 1, which means that there is only one singular value

$$\sigma_1 = \|\boldsymbol{h}\|. \tag{2.85}$$

The equivalent representation of the channel model is

$$\tilde{\boldsymbol{y}}[m] = \sigma_1 x[m] + \tilde{\boldsymbol{w}}[m]. \tag{2.86}$$

As a result, the capacity of the SIMO channel is

$$C = \log_2\left(1 + \frac{P\|\boldsymbol{h}\|^2}{N}\right) \quad \text{bits/symbol} \tag{2.87}$$

or, alternatively,

$$C = \log_2\left(1 + \frac{P\sigma_1^2}{N}\right) \quad \text{bits/symbol}. \tag{2.88}$$

Fast-fading channel

The signal model is

$$\boldsymbol{y}[m] = \boldsymbol{h}[m]x[m] + \boldsymbol{w}[m], \qquad \boldsymbol{w}[m] \sim \mathcal{CN}(0,\, N\boldsymbol{I}_{n_r}). \tag{2.89}$$

We have proved that the capacities of a MISO system and a SIMO system are the same, when the transmitter can track the channel. As a result the fast fading scenario of a SIMO system is identical to that of a MISO.

2.5 Capacity Performance of MIMO Systems

In this section, we evaluate the capacity performance of MIMO systems in comparison with SISO and SIMO systems. We do not take into account MISO systems, since their capacity formula is identical to that of SIMO systems when the transmission channel is known. Considering $L = 20$ transmissions, we used the Rayleigh fading model and applied the waterfilling algorithm to the actual transmission channel using $\lambda(h)$. Calculating the mean capacity, we can clearly evaluate the benefits in capacity of a MIMO system.

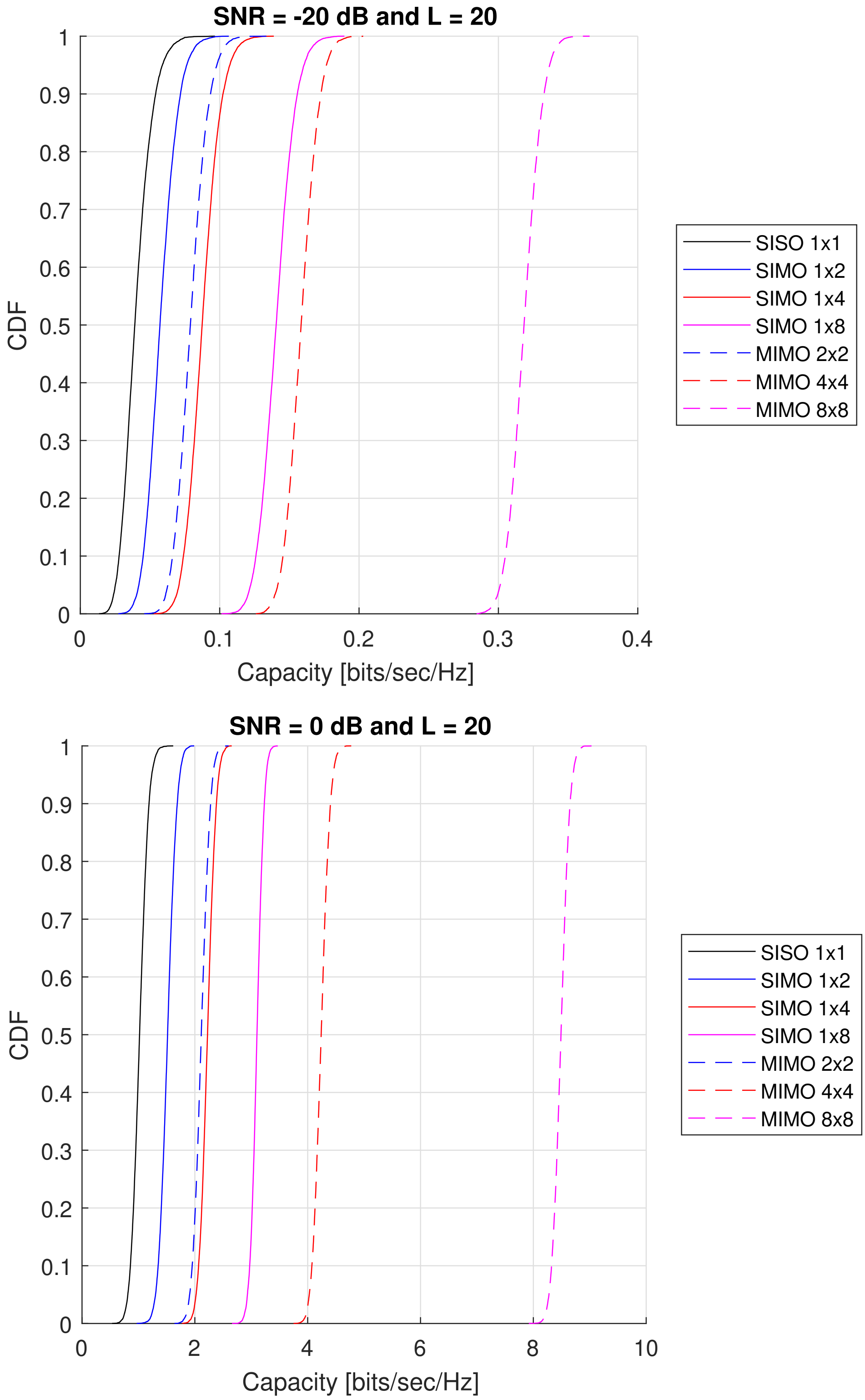


Figure 2.10: CDF of capacity (SISO/SIMO/MIMO).

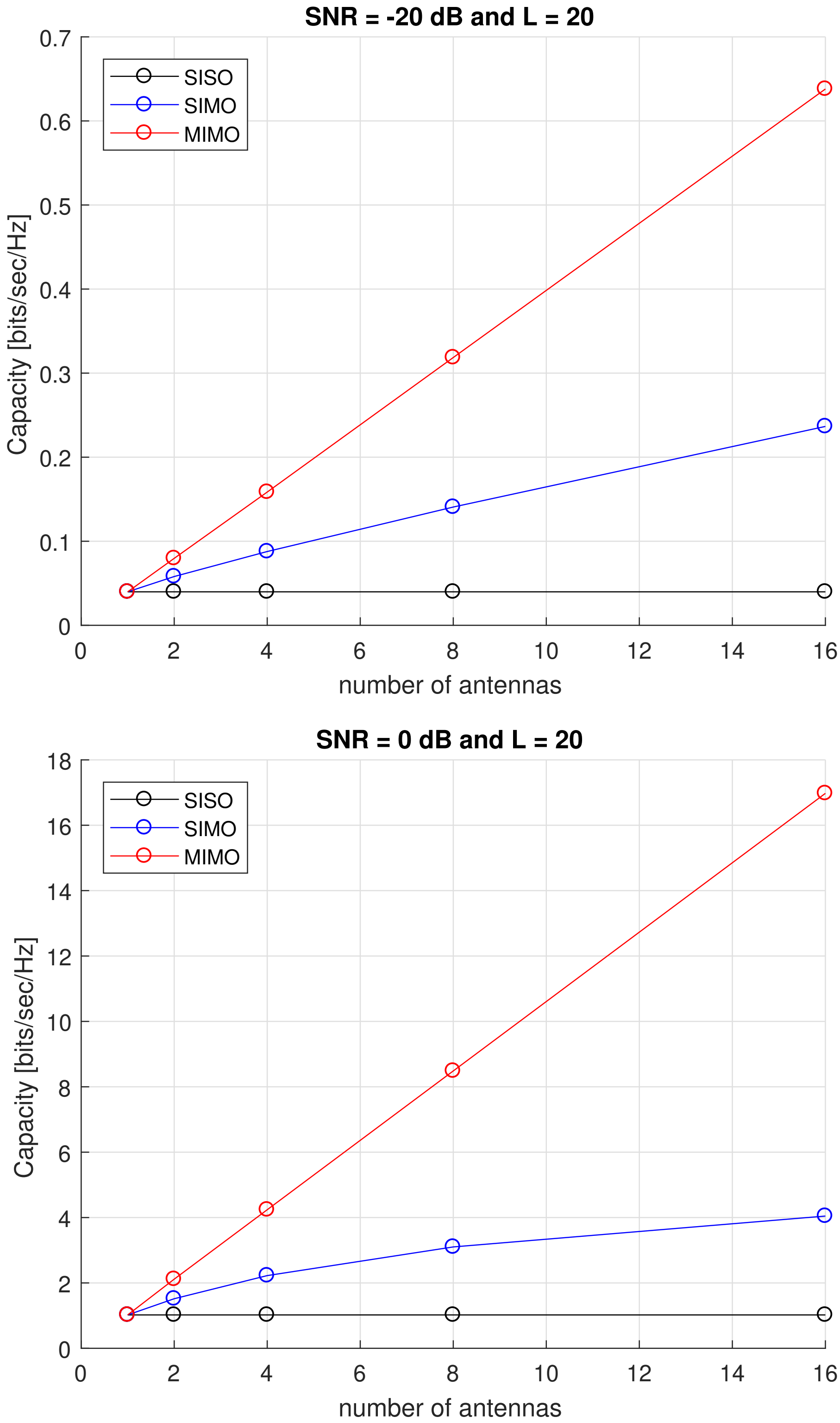


Figure 2.11: Mean capacity vs number of antennas (SISO/SIMO/MIMO).

Chapter 3

Multi-user Case

In this chapter, we study the multiuser scenario. The communication is performed between multiple users and the base station. When the transmission is from the users to the base station we define the uplink channel and when the transmission is from the base station to the users we define the downlink channel. In the multiuser scenario, we are interested in finding the maximum total rate and the maximum common rate at which the users can simultaneously reliably communicate (i.e., uplink channel) or be communicated to (i.e., downlink channel). These quantities are called sum capacity and symmetric capacity of the channel respectively. In the multi-user scenario many combinations of rates can occur and as a result we can define a capacity region. We analyze the case where there is a single antenna at each user and at the base station. To keep things simple, we assume that the channel state information is available at the transmitter, which means that it can perfectly track the channel. We also mention the capacity formula for the case where the transmitter cannot track the channel (i.e., no Tx CSI) in the fast-fading scenario.

3.1 Uplink Channel

In the uplink channel, the users transmit information to the base station. We consider the fading channel model with two users, so that we can depict the capacity region.

Slow-fading channel

The base station receives the sum of the two signals transmitted by the users, so the signal model is

$$y[m] = h_1 x_1[m] + h_2 x_2[m] + w[m], \qquad w[m] \sim \mathcal{CN}(0,\, N). \tag{3.1}$$

Each input symbol $x_k[m]$ has an average power constraint of P_k where $k = 1, 2$ and $P_1 + P_2 = P$. Assuming that the base station can always decode the users' data, there are two available strategies that conclude to the same sum rate. The base station first decodes the signal of user 1 and then the signal of user 2. In order to decode the signal transmitted from user 1, it first treats the signal of user 2 as interference. We calculate the SNR of each user so as to define its rate.

$$\mathrm{SNR}_{\mathrm{user}_1} = \frac{E\left\{|h_1 x_1[m]|^2\right\}}{E\left\{|h_2 x_2[m] + w[m]|^2\right\}} = \frac{P_1|h_1|^2}{P_2|h_2|^2 + N}. \tag{3.2}$$

The proof is given in Chapter 6. When the base station successfully decodes the signal of user 1, it subtracts it from the received signal so there is only the signal transmitted from user 2 and the noise left in the system. This process is called successive interference cancellation (SIC). When SIC is applied, it decodes the signal transmitted from user 2. Hence, we get

$$\mathrm{SNR}_{\mathrm{user}_2} = \frac{E\left\{|h_2 x_2[m]|^2\right\}}{E\left\{|w[m]|^2\right\}} = \frac{P_2|h_2|^2}{N}. \tag{3.3}$$

The achievable rate pairs of the two users are defined as

$$R_1 \le \log_2\left(1 + \frac{P_1|h_1|^2}{P_2|h_2|^2 + N}\right) \quad \text{bits/symbol}, \tag{3.4}$$

$$R_2 \le \log_2\left(1 + \frac{P_2|h_2|^2}{N}\right) \quad \text{bits/symbol}. \tag{3.5}$$

The sum rate achieved is

$$R_1 + R_2 \le \log_2\left(1 + \frac{P_1|h_1|^2}{P_2|h_2|^2 + N}\right) + \log_2\left(1 + \frac{P_2|h_2|^2}{N}\right) \tag{3.6}$$

$$= \log_2\left[\left(1 + \frac{P_1|h_1|^2}{P_2|h_2|^2 + N}\right)\left(1 + \frac{P_2|h_2|^2}{N}\right)\right] \tag{3.7}$$

$$= \log_2\left[\left(\frac{P_1|h_1|^2 + P_2|h_2|^2 + N}{P_2|h_2|^2 + N}\right)\left(\frac{P_2|h_2|^2 + N}{N}\right)\right] \tag{3.8}$$

$$= \log_2\left(\frac{P_1|h_1|^2 + P_2|h_2|^2 + N}{N}\right) \tag{3.9}$$

$$= \log_2\left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2}{N}\right) \quad \text{bits/symbol}. \tag{3.10}$$

Note that the base station can reverse the decoding order of the signals transmitted by the two users. In this case, we have

$$\mathrm{SNR}_{\mathrm{user}_1} = \frac{E\left\{|h_1 x_1[m]|^2\right\}}{E\left\{|w[m]|^2\right\}} = \frac{P_1|h_1|^2}{N}, \tag{3.11}$$

$$\mathrm{SNR}_{\mathrm{user}_2} = \frac{E\left\{|h_2 x_2[m]|^2\right\}}{E\left\{|h_1 x_1[m] + w[m]|^2\right\}} = \frac{P_2|h_2|^2}{P_1|h_1|^2 + N}. \tag{3.12}$$

Therefore, the achievable rates of the two users and the sum rate achieved are

$$R_1 \leq \log_2\left(1+\frac{P_1|h_1|^2}{N}\right) \quad \text{bits/symbol,} \tag{3.13}$$

$$R_2 \leq \log_2\left(1+\frac{P_2|h_2|^2}{P_1|h_1|^2+N}\right) \quad \text{bits/symbol,} \tag{3.14}$$

$$R_1+R_2 \leq \log_2\left(1+\frac{P_1|h_1|^2}{N}\right)+\log_2\left(1+\frac{P_2|h_2|^2}{P_1|h_1|^2+N}\right) \tag{3.15}$$

$$= \log_2\left[\left(1+\frac{P_1|h_1|^2}{N}\right)\left(1+\frac{P_2|h_2|^2}{P_1|h_1|^2+N}\right)\right] \tag{3.16}$$

$$= \log_2\left[\left(\frac{P_1|h_1|^2+N}{N}\right)\left(\frac{P_2|h_2|^2+P_1|h_1|^2+N}{P_1|h_1|^2+N}\right)\right] \tag{3.17}$$

$$= \log_2\left(\frac{P_1|h_1|^2+P_2|h_2|^2+N}{N}\right) \tag{3.18}$$

$$= \log_2\left(1+\frac{P_1|h_1|^2+P_2|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.19}$$

We can now define the capacity region, Figure 3.1; the area that includes all possible rate pairs. The two decoding strategies discussed above refer to the operation points A and B respectively. All the other points on the line segment AB can be obtained by time-sharing between those two points. In terms of sum rate, all operation points on the line segment AB conclude to the same result. Depending on the power splits related to the symbols transmitted by the two users, we can define various forms of the pentagon. We explicitly analyze those forms when we distinguish cases for the channel gains. In these cases, we also analyze the sum capacity and the symmetric capacity as well.

Symmetric case $|h_1| = |h_2|$

We consider the case where the two users experience the same channel gain (i.e., $|h_1| = |h_2| = |h|$). The sum rate achieved is

$$R_1+R_2 \leq \log_2\left(1+\frac{P_1|h|^2+P_2|h|^2}{N}\right) \tag{3.20}$$

$$= \log_2\left(1+\frac{(P_1+P_2)|h|^2}{N}\right) \tag{3.21}$$

$$= \log_2\left(1+\frac{P|h|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.22}$$

In this scenario, all possible power splits between the two users lead to the same sum rate, which it turns to be equivalent to the case where only one user transmits using all the available power. Hence, the sum capacity is

$$C_{\text{sum}} = \log_2\left(1+\frac{P|h|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.23}$$

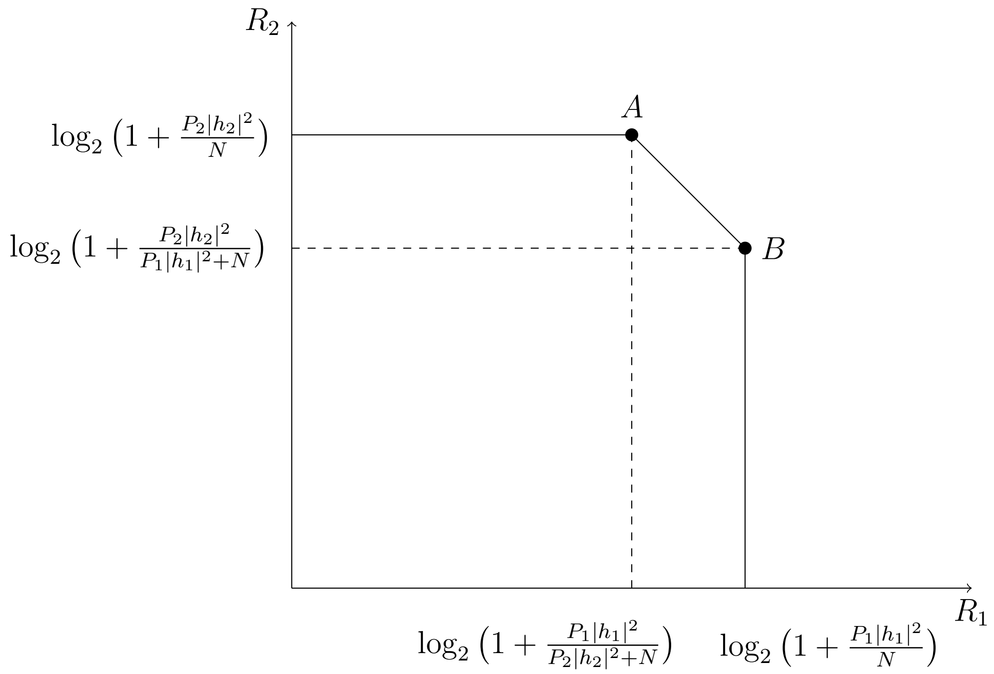


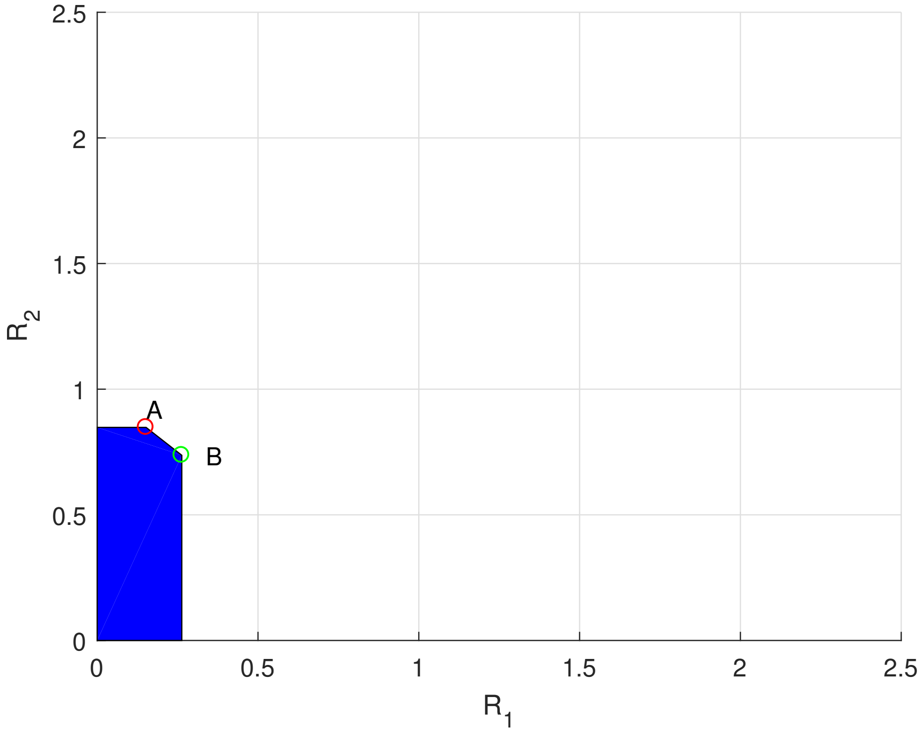
Figure 3.1: Capacity region of uplink channel.

We will depict the capacity region considering different power splits among the two users. We depict the union bound of these regions in order to calculate the symmetric capacity. Considering all possible power splits between the two users, we depict the pentagons in one single figure. All possible pentagons form an isosceles triangle. Since all possible rate pairs conclude to the same sum rate, these operation points lie on the same line segment, which is the triangle's hypotenuse. It's also clearly understood that the triangle is isosceles since the two users experience the same channel gain and consequently the rate of each user is the same when it transmits with all the available power.

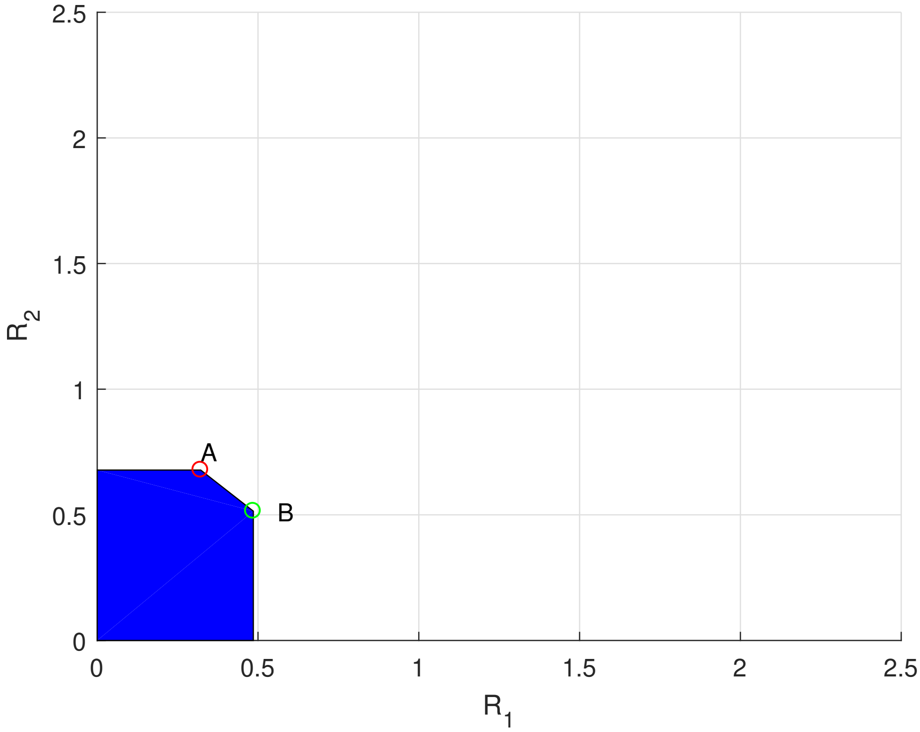
Now, we will calculate the symmetric capacity considering the capacity region. A line segment starting from the origin with slope 45° gives point C which refers to the common rate achieved. Since the outer bound of all possible capacity regions is formed by operation points A, B and the line segment which connects them, the maximum common rate is achieved when point C lies on the aforementioned line segment. All these cases conclude to the same result, regarding symmetric capacity. Considering the case where point C lies on point A, we get that the symmetric capacity is

$$C_{\text{symmetric}} = \log_2\left(1+\frac{P_1|h_1|^2}{P_2|h_2|^2+N}\right) = \log_2\left(1+\frac{P_2|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.24}$$

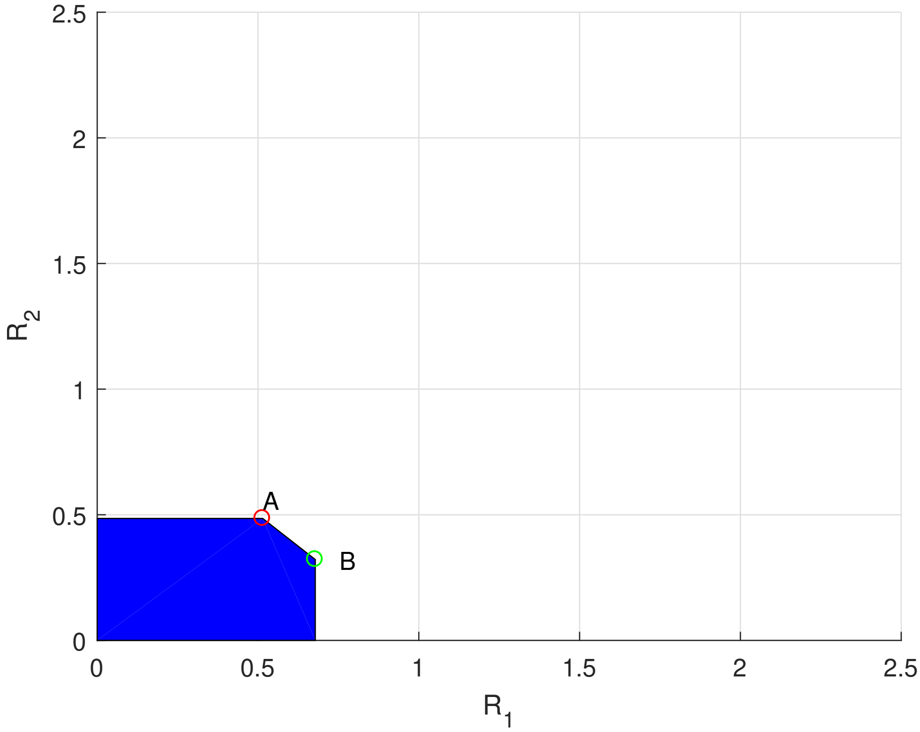
To calculate symmetric capacity, we have to calculate the power splits P_1 and P_2



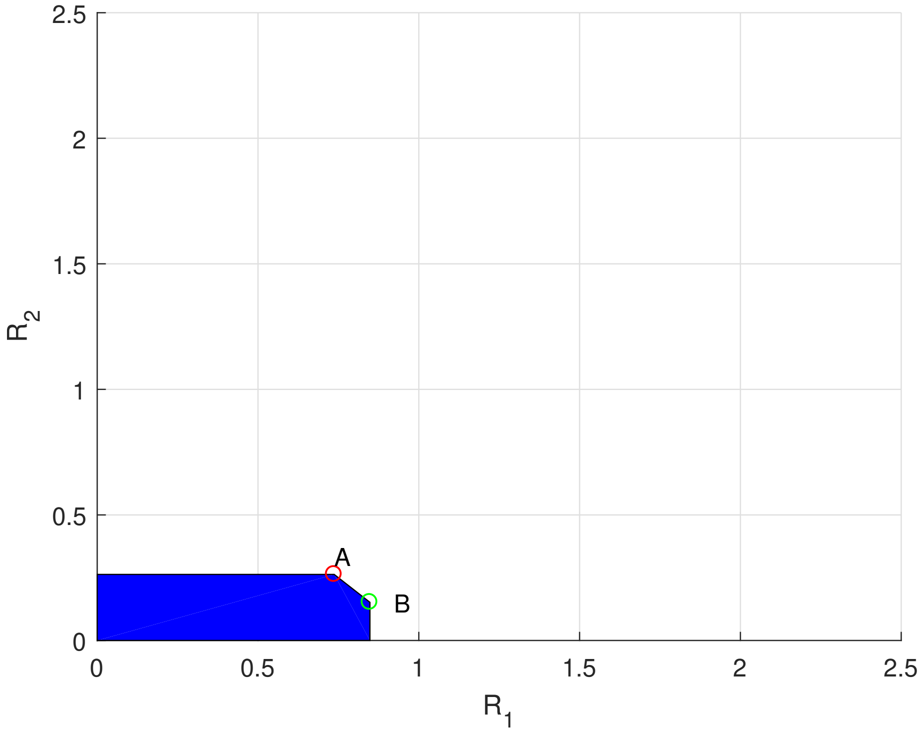
(a) $P_1 = 0.2 \cdot P$ and $P_2 = 0.8 \cdot P$.



(b) $P_1 = 0.4 \cdot P$ and $P_2 = 0.6 \cdot P$.



(a) $P_1 = 0.6 \cdot P$ and $P_2 = 0.4 \cdot P$.



(b) $P_1 = 0.8 \cdot P$ and $P_2 = 0.2 \cdot P$.

Figure 3.3: Capacity region of uplink channel (power splits).

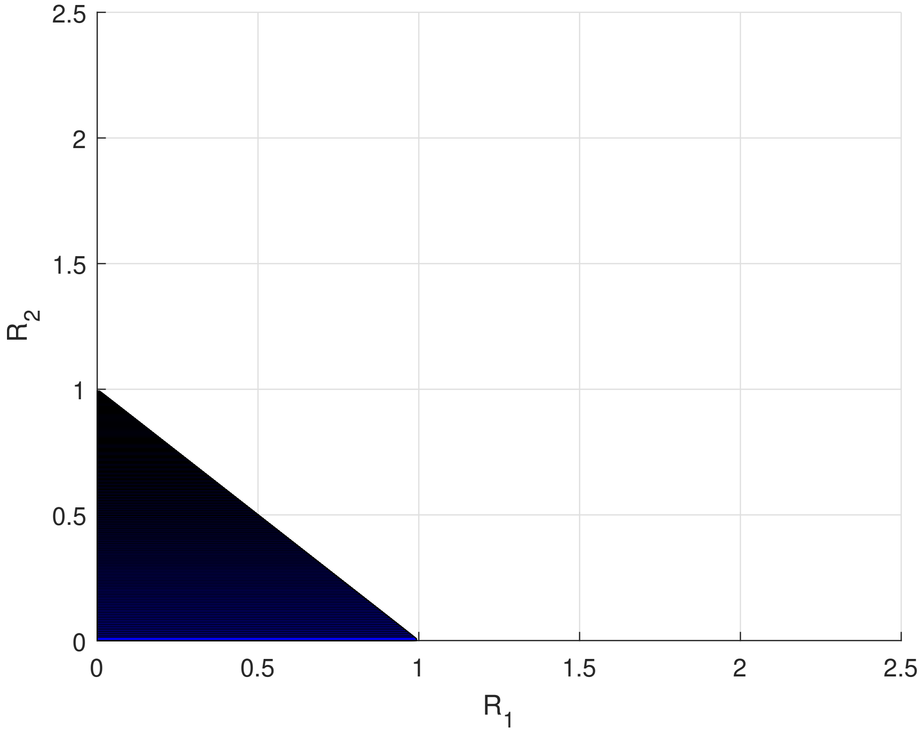


Figure 3.4: Union bound of uplink channel ($|h_1| = |h_2|$).

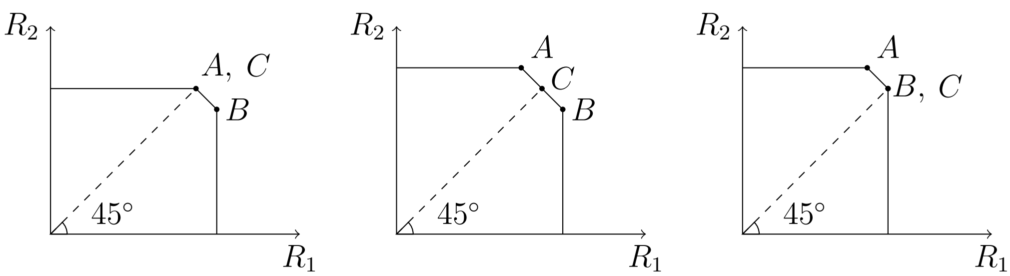


Figure 3.5: Symmetric capacity of uplink channel ($|h_1| = |h_2|$).

among the two users. Hence, we get

$$R_1 = R_2 \tag{3.25}$$
$$\implies \log_2\left(1 + \frac{P_1|h_1|^2}{P_2|h_2|^2 + N}\right) = \log_2\left(1 + \frac{P_2|h_2|^2}{N}\right) \tag{3.26}$$

which implies that

$$P_2 = \frac{-(|h_2|^2 + |h_1|^2)N + \sqrt{(|h_2|^4 + 2|h_2|^2|h_1|^2 + |h_1|^4)N^2 + 4P|h_2|^4|h_1|^2N}}{2|h_2|^4}, \tag{3.27}$$
$$P_1 = P - P_2. \tag{3.28}$$

The proof is given in Chapter 6. Considering the case where point C lies on point B, we get that the symmetric capacity is

$$C_{\text{symmetric}} = \log_2\left(1 + \frac{P_1|h_1|^2}{N}\right) = \log_2\left(1 + \frac{P_2|h_2|^2}{P_1|h_1|^2 + N}\right) \quad \text{bits/symbol.} \tag{3.29}$$

Working in the same way as before, we conclude that

$$P_2 = P - P_1, \tag{3.30}$$
$$P_2 = P - \frac{-(|h_1|^2 + |h_2|^2)N + \sqrt{(|h_1|^4 + 2|h_1|^2|h_2|^2 + |h_2|^4)N^2 + 4P|h_1|^4|h_2|^2N}}{2|h_1|^4}. \tag{3.31}$$

Any solution between those two is acceptable, i.e.,

$$P_2 \in \left[\frac{-(|h_2|^2 + |h_1|^2)N + \sqrt{(|h_2|^4 + 2|h_2|^2|h_1|^2 + |h_1|^4)N^2 + 4P|h_2|^4|h_1|^2N}}{2|h_2|^4},\right. \tag{3.32}$$
$$\left. P - \frac{-(|h_1|^2 + |h_2|^2)N + \sqrt{(|h_1|^4 + 2|h_1|^2|h_2|^2 + |h_2|^4)N^2 + 4P|h_1|^4|h_2|^2N}}{2|h_1|^4}\right].$$

One of these solutions is $P_1 = P_2 = P/2$. In that case, point C lies in the middle of the line segment AB.

Asymmetric case $|h_1| > |h_2|$

In this case, user 1 experiences a better channel gain than that of user 2. This time the power splits between the two users do not lead to the same sum rate. The sum rate is larger when user 1 gets more power in order to transmit in comparison with the power that user 2 gets. Obviously, the best we can do is to give all the available power to user 1. Hence, user 1 will be able to transmit at the maximum possible rate, whereas user 2 will not transmit at all.

$$\log_2\left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2}{N}\right) < \log_2\left(1 + \frac{P|h_1|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.33}$$

Hence, the sum capacity is

$$C_{\text{sum}} = \log_2\left(1 + \frac{P|h_1|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.34}$$

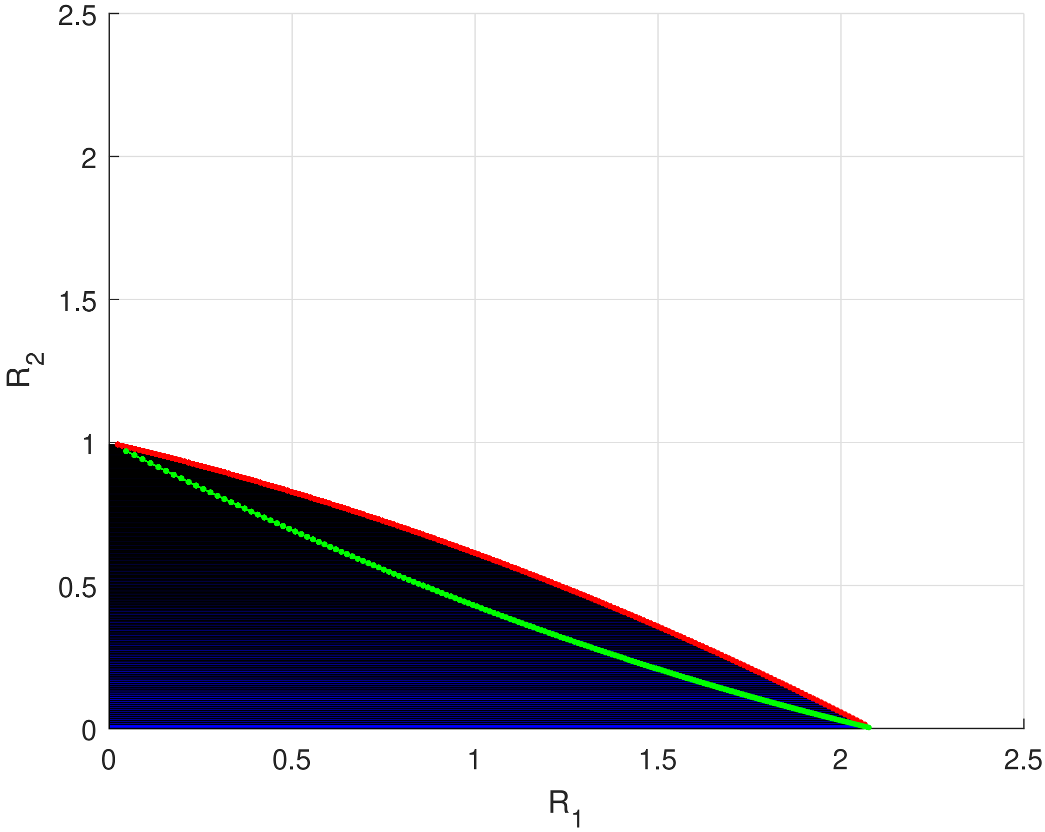


Figure 3.6: Union bound of uplink channel ($|h_1| > |h_2|$).

Considering all possible power splits between the two users, we depict the union bound of the pentagons created in one single figure. Since the outer bound is formed by operation point A, the maximum common rate is achieved when point C lies on that point. Therefore, the symmetric capacity is

$$C_{\text{symmetric}} = \log_2\left(1 + \frac{P_1|h_1|^2}{P_2|h_2|^2 + N}\right) = \log_2\left(1 + \frac{P_2|h_2|^2}{N}\right) \quad \text{bits/symbol} \tag{3.35}$$

where

$$P_2 = \frac{-(|h_2|^2 + |h_1|^2)N + \sqrt{(|h_2|^4 + 2|h_2|^2|h_1|^2 + |h_1|^4)N^2 + 4P|h_2|^4|h_1|^2N}}{2|h_2|^4}, \tag{3.36}$$

$$P_1 = P - P_2. \tag{3.37}$$

Asymmetric case $|h_1| < |h_2|$

Now, user 2 experiences a better channel gain than that of user 1. The best we can do is to give all the available power to user 2. Hence, user 2 will be able to transmit at the maximum possible rate, whereas user 1 will not transmit at all.

$$\log_2\left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2}{N}\right) < \log_2\left(1 + \frac{P|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.38}$$

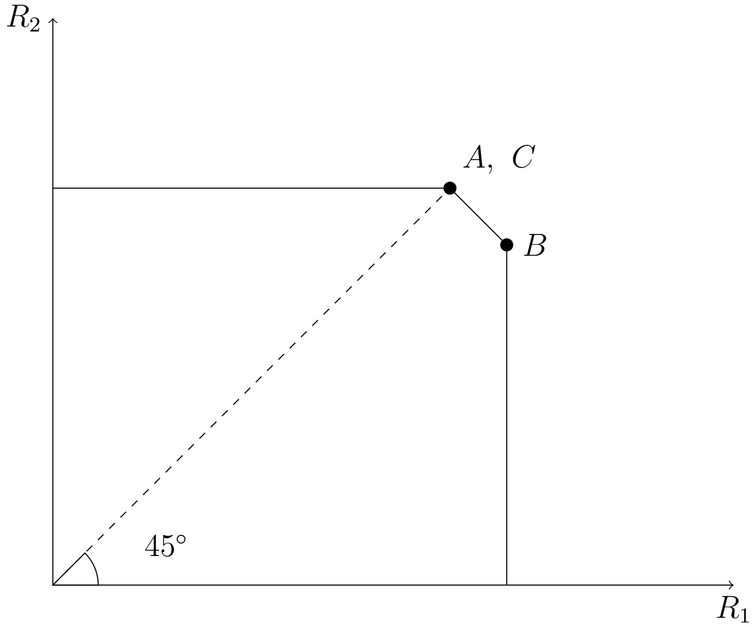


Figure 3.7: Symmetric capacity of uplink channel ($|h_1| > |h_2|$).

Hence, the sum capacity is

$$C_{\text{sum}} = \log_2\left(1 + \frac{P|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.39}$$

Considering all possible power splits between the two users, we depict the union bound of the pentagons created in one single figure. Since the outer bound is formed by operation point B, the maximum common rate is achieved when point C lies on that point. Therefore, the symmetric capacity is

$$C_{\text{symmetric}} = \log_2\left(1 + \frac{P_1|h_1|^2}{N}\right) = \log_2\left(1 + \frac{P_2|h_2|^2}{P_1|h_1|^2 + N}\right) \quad \text{bits/symbol} \tag{3.40}$$

where

$$P_2 = P - P_1, \tag{3.41}$$

$$P_2 = P - \frac{-(|h_1|^2 + |h_2|^2)N + \sqrt{(|h_1|^4 + 2|h_1|^2|h_2|^2 + |h_2|^4)N^2 + 4P|h_1|^4|h_2|^2N}}{2|h_1|^4}. \tag{3.42}$$

Fast-fading channel

In the fast-fading scenario, we are interested in calculating the sum capacity of the system. Considering that the channel state of each user varies during the transmission of information, we get the signal model

$$y[m] = h_1[m]x_1[m] + h_2[m]x_2[m] + w[m]. \tag{3.43}$$

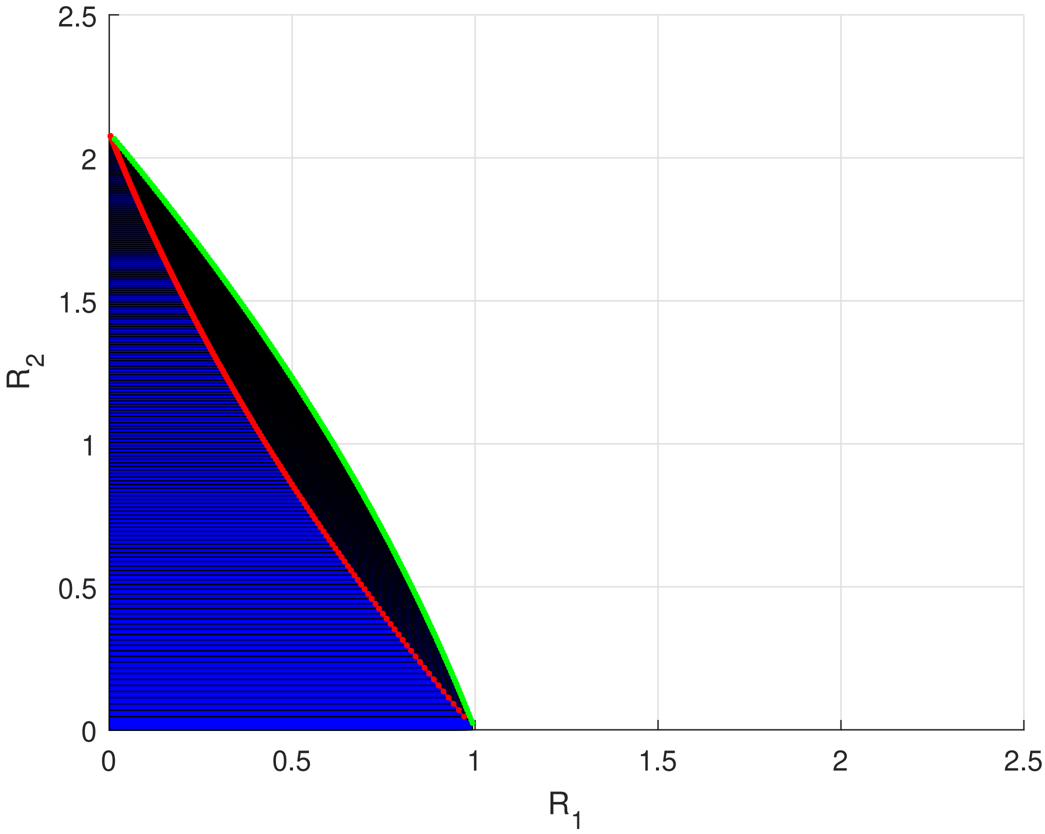


Figure 3.8: Union bound of uplink channel ($|h_1| < |h_2|$).

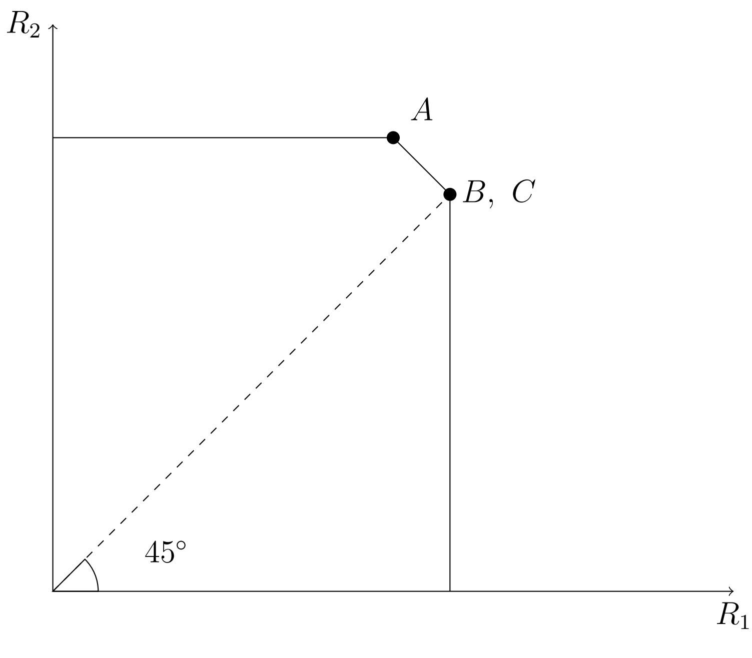


Figure 3.9: Symmetric capacity of uplink channel ($|h_1| < |h_2|$).

Tx CSI

Since the users can track the channel, we should have the user that experiences the better channel gain transmit at each time. As a result, we get a communication scenario identical to the single user scheme, i.e.,

$$C_{\text{sum}} = \frac{1}{L}\sum_{m=1}^{L}\log_2\left(1 + \frac{p_{k,m}|h_k[m]|^2}{N}\right) \quad \text{bits/symbol} \tag{3.44}$$

where k is selected to satisfy the following

$$|h_k[m]|^2 = \max_i |h_i[m]|^2, \quad i = 1, 2, \tag{3.45}$$

considering the power constraint

$$\sum_{m=1}^{L} p_{k,m} = LP, \qquad p_{k,m} \geq 0, \qquad k = 1, 2, \qquad m = 1, ..., L. \tag{3.46}$$

We note that the optimal power allocation is given by the waterfilling algorithm. The ergodic capacity of the fast-fading channel is

$$C_{\text{sum}} = \mathbb{E}\left[\log_2\left(1 + \frac{P_k(h_k)|h_k|^2}{N}\right)\right] \quad \text{bits/symbol} \tag{3.47}$$

where

$$P_k(h_k) = \left(\frac{1}{\lambda} - \frac{N}{|h_k|^2}\right)^+, \quad \mathbb{E}[P_k(h_k)] = P. \tag{3.48}$$

Note that the index k denotes the user with the stronger channel gain at each time, i.e.,

$$|h_k|^2 = \max_i |h_i|^2, \quad i = 1, 2. \tag{3.49}$$

No Tx CSI

In that case the users cannot track the channel. The ergodic capacity of the system is

$$C_{\text{sum}} = \mathbb{E}\left[\log_2\left(1 + \frac{P_1|h_1|^2 + P_2|h_2|^2}{N}\right)\right] \quad \text{bits/symbol} \tag{3.50}$$

where $P_1 = P_2 = P/2$. The best we can do is to have both users transmit with equal power and that is confirmed by Figure 3.10.

3.2 Downlink Channel

In the downlink channel, the base station transmits information to the users. We consider again the fading channel model with two users present. In this communication scenario, we first have to discriminate cases regarding the channel gains, to depict the capacity region.

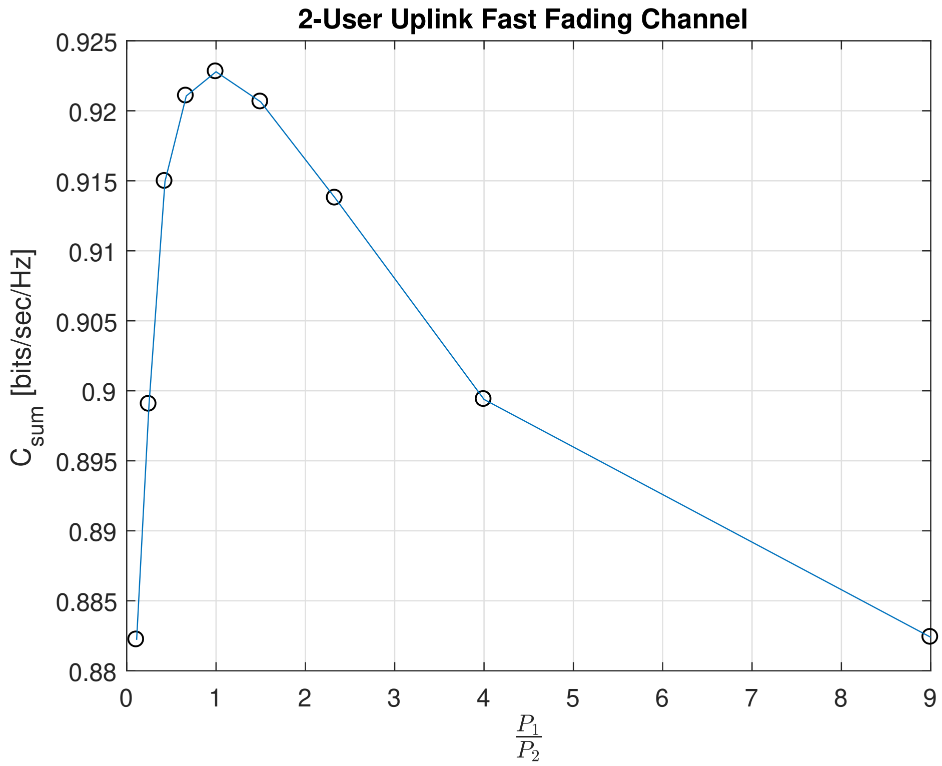


Figure 3.10: Sum capacity of fast-fading uplink channel (no Tx CSI).

Slow-fading channel

The base station transmits the sum of the two signals intended for user 1 and user 2, i.e., $x[m] = x_1[m] + x_2[m]$. We also assume an average power constraint of P_1 and P_2 per input symbol $x_1[m]$ and $x_2[m]$ respectively where $P_1 + P_2 = P$. The signal model is defined as

$$y_k[m] = h_k x[m] + w_k[m], \quad w_k[m] \sim \mathcal{CN}(0, N), \qquad k = 1, 2. \tag{3.51}$$

Asymmetric case $|h_1| \geq |h_2|$

Without loss of generality, we assume that user 1 experiences a better or equaly strong channel gain in comparison with that of user 2. In that case user 1 can always decode the data intended for user 2, subtract it from the received signal and then decode its own data. User 2 can only treat the signal intended for user 1 as interference so as to decode its signal. Using the signal model introduced before, we get

$$y_1[m] = h_1 x[m] + w_1[m] = h_1 x_1[m] + h_1 x_2[m] + w_1[m], \tag{3.52}$$
$$y_2[m] = h_2 x[m] + w_2[m] = h_2 x_1[m] + h_2 x_2[m] + w_2[m]. \tag{3.53}$$

Therefore, at first stage, the SNR of each user is calculated as

$$\mathrm{SNR}_{\mathrm{user}_1} = \frac{E\left\{|h_1 x_2[m]|^2\right\}}{E\left\{|h_1 x_1[m] + w_1[m]|^2\right\}} = \frac{P_2 |h_1|^2}{P_1 |h_1|^2 + N} = \frac{P_2}{P_1 + \frac{N}{|h_1|^2}}, \tag{3.54}$$

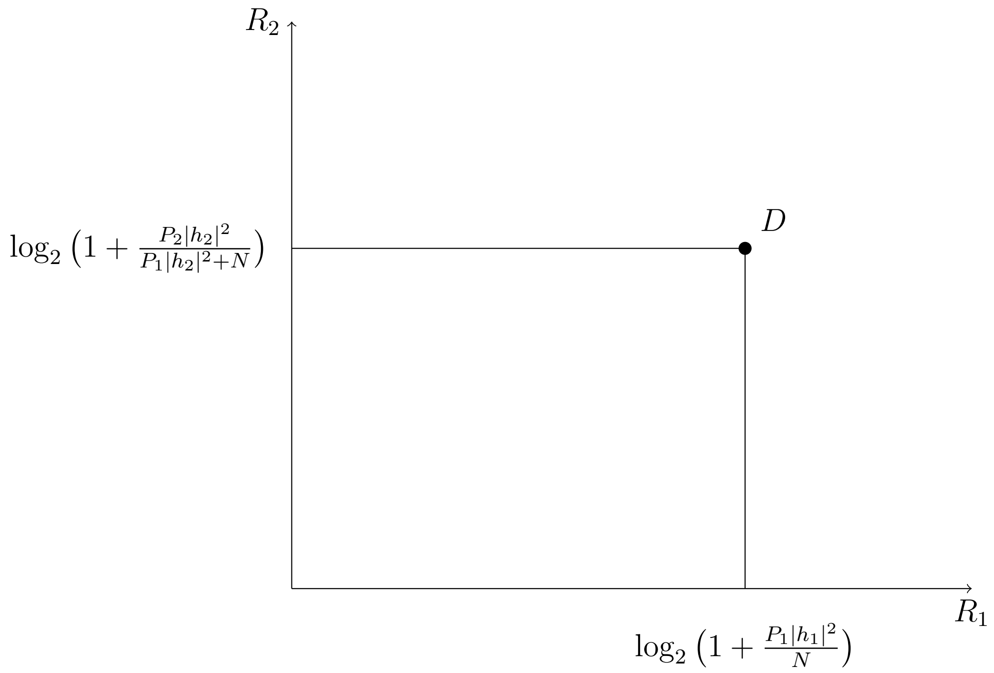


Figure 3.11: Capacity region of downlink channel ($|h_1| \geq |h_2|$).

$$\mathrm{SNR}_{\mathrm{user}_2} = \frac{E\left\{|h_2 x_2[m]|^2\right\}}{E\left\{|h_2 x_1[m] + w_1[m]|^2\right\}} = \frac{P_2|h_2|^2}{P_1|h_2|^2 + N} = \frac{P_2}{P_1 + \frac{N}{|h_2|^2}}. \tag{3.55}$$

Since the SNR of user 1 is greater or equal than that of user 2, we proved that only user 1 can perform successive interference cancellation. Therefore user 1 can eventually have an SNR of

$$\mathrm{SNR}_{\mathrm{user}_1} = \frac{E\left\{|h_1 x_1[m]|^2\right\}}{E\left\{|w_1[m]|^2\right\}} = \frac{P_1|h_1|^2}{N}. \tag{3.56}$$

In any case, i.e., $|h_1| > |h_2|$ or $|h_1| = |h_2|$, the achievable rate pairs are

$$R_1 \leq \log_2\left(1 + \frac{P_1|h_1|^2}{N}\right) \quad \text{bits/symbol,} \tag{3.57}$$

$$R_2 \leq \log_2\left(1 + \frac{P_2|h_2|^2}{P_1|h_2|^2 + N}\right) \quad \text{bits/symbol.} \tag{3.58}$$

When $|h_1| = |h_2|$, all possible rate pairs lead to the same sum rate, because at this scenario we get the same rate pairs as in the symmetric case of the uplink channel. However, when $|h_1| > |h_2|$, it's obvious that the best we can do is to have the base station transmit only to the user which experiences the better channel gain. In any case the sum capacity is

$$C_{\mathrm{sum}} = \log_2\left(1 + \frac{P|h_1|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.59}$$

Depending on the power splits related to the symbols transmitted by the base station, we can define various forms of the rectangle. The same approach was applied at the uplink channel as well. Regarding symmetric capacity, a line segment starting from the origin with slope 45° gives point C which refers to the common rate achieved. Obviously the outer bound is formed by point D, and the maximum common rate is achieved when point C lies on point D.

$$C_{\text{symmetric}} = \log_2\left(1 + \frac{P_1|h_1|^2}{N}\right) = \log_2\left(1 + \frac{P_2|h_2|^2}{P_1|h_2|^2 + N}\right) \quad \text{bits/symbol.} \tag{3.60}$$

In order to calculate this quantity we have to calculate the power splits P_1 and P_2, where $P_1 + P_2 = P$, among the two users in the same way as we did in the uplink channel.

$$P_1 = \frac{-(|h_1|^2 + |h_2|^2)N + \sqrt{(|h_1|^4 + 2|h_1|^2|h_2|^2 + |h_2|^4)N^2 + 4P|h_1|^4|h_2|^2N}}{2|h_1|^2|h_2|^2}, \tag{3.61}$$

$$P_2 = P - P_1. \tag{3.62}$$

Asymmetric case $|h_1| < |h_2|$

In that case, user 2 can perform successive interference cancellation in order to decode its own signal. We work in the same way as we did before. Hence, the achievable rate pairs of the two users are

$$R_1 \le \log_2\left(1 + \frac{P_1|h_1|^2}{P_2|h_1|^2 + N}\right) \quad \text{bits/symbol,} \tag{3.63}$$

$$R_2 \le \log_2\left(1 + \frac{P_2|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.64}$$

Obviously, in order to maximize the sum rate we have the base station transmit to user 2 only. Hence, the sum capacity is

$$C_{\text{sum}} = \log_2\left(1 + \frac{P|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.65}$$

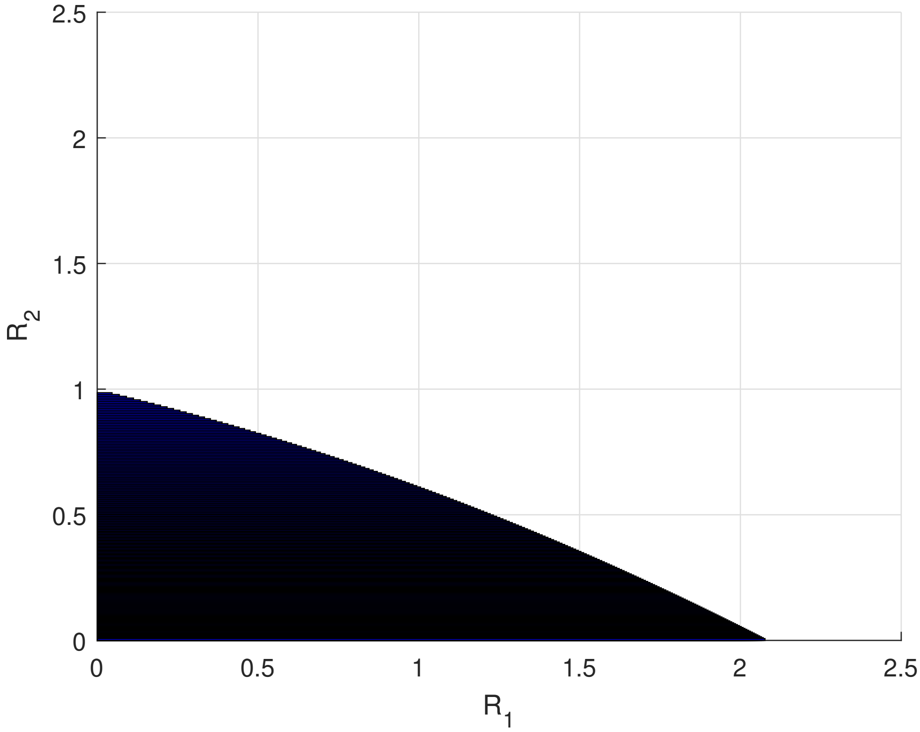
Considering different power splits among the two users, we depict the union bound of the rectangles created. Regarding symmetric capacity, the maximum common rate is achieved when point C lies on point E and equals

$$C_{\text{symmetric}} = \log_2\left(1 + \frac{P_1|h_1|^2}{P_2|h_1|^2 + N}\right) = \log_2\left(1 + \frac{P_2|h_2|^2}{N}\right) \quad \text{bits/symbol.} \tag{3.66}$$

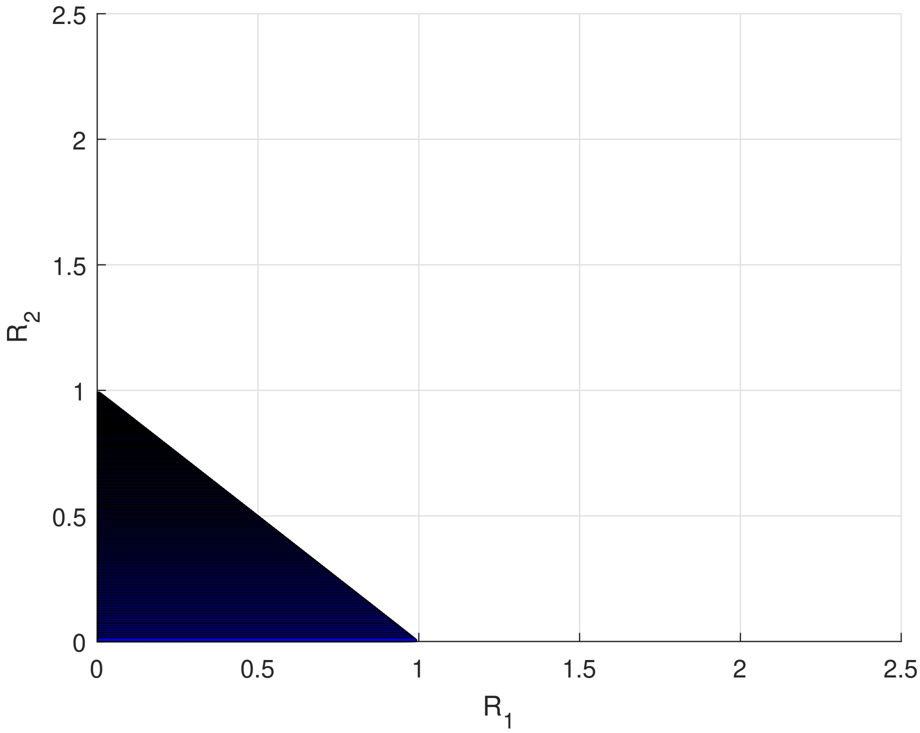
Then, we have to equalize the rate pairs in order to calculate the power splits P_1 and P_2, which lead to symmetric capacity.

$$P_2 = \frac{-(|h_2|^2 + |h_1|^2)N + \sqrt{(|h_2|^4 + 2|h_2|^2|h_1|^2 + |h_1|^4)N^2 + 4P|h_1|^4|h_2|^2N}}{2|h_1|^2|h_2|^2}, \tag{3.67}$$

$$P_1 = P - P_2. \tag{3.68}$$



(a) $|h_1| > |h_2|$



(b) $|h_1| = |h_2|$

Figure 3.12: Union bound of downlink channel ($|h_1| \geq |h_2|$).

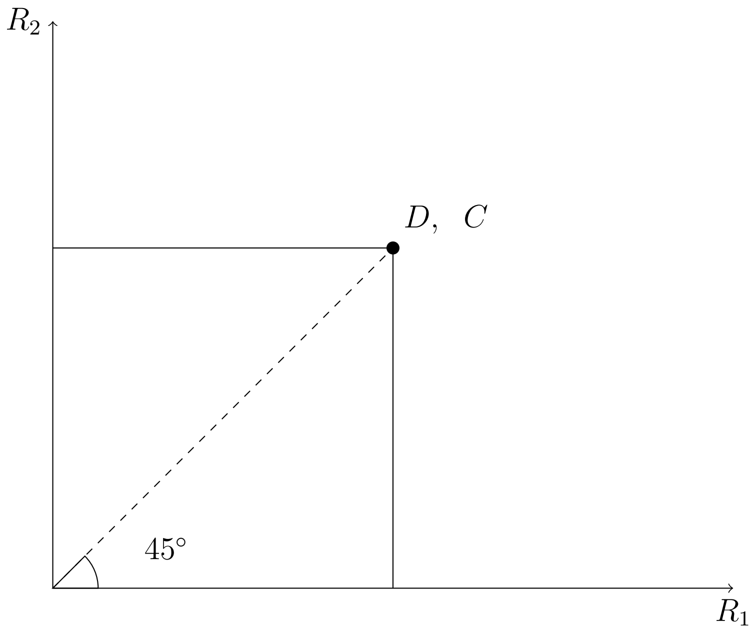


Figure 3.13: Symmetric capacity of downlink channel ($|h_1| \geq |h_2|$).

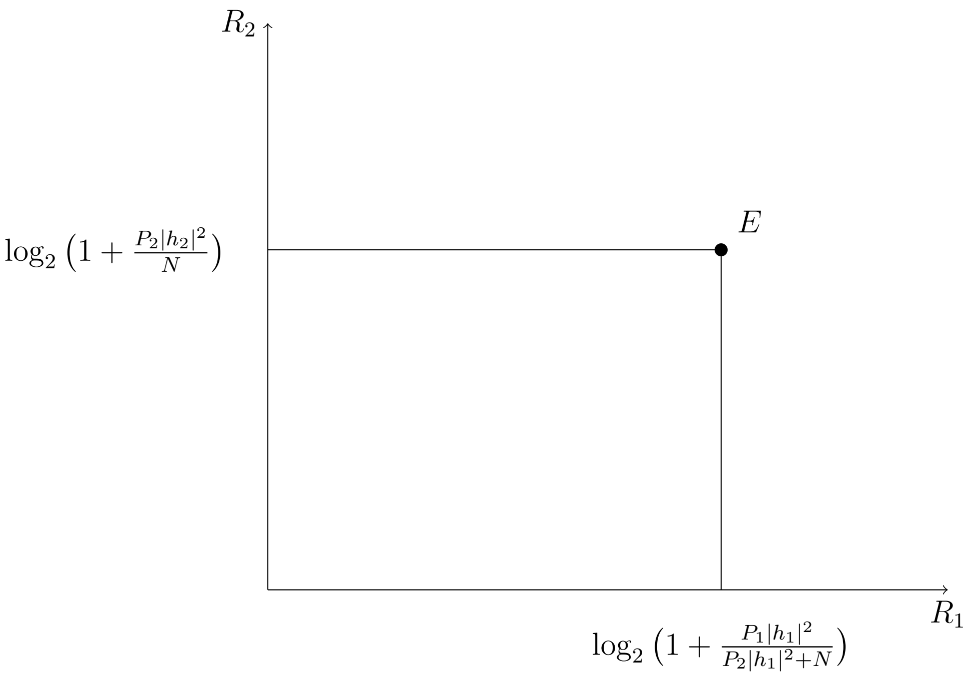


Figure 3.14: Capacity region of downlink channel ($|h_1| < |h_2|$).

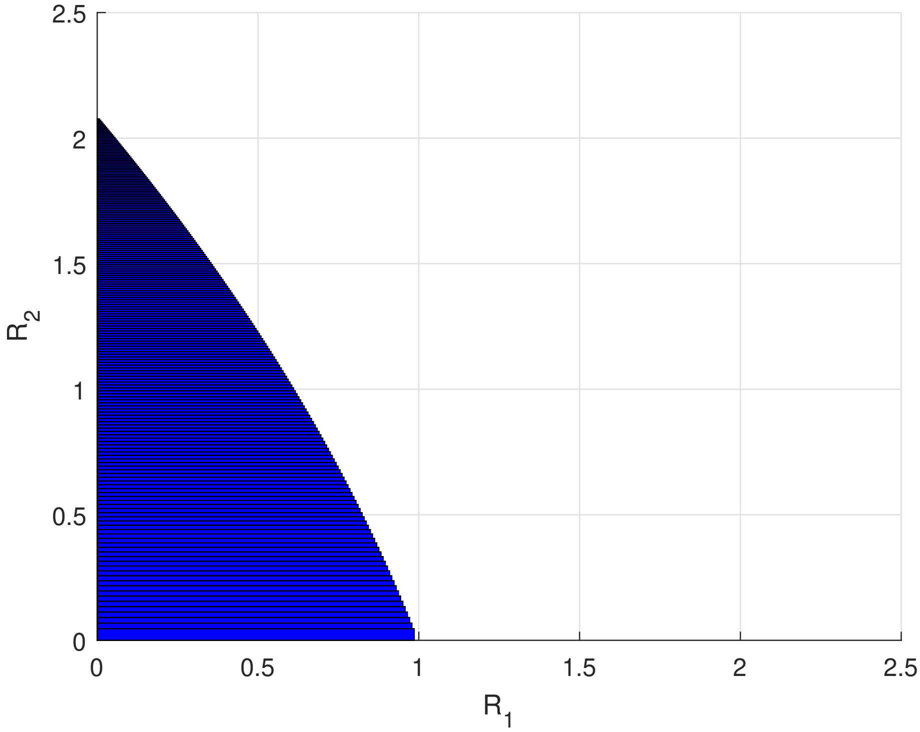


Figure 3.15: Union bound of downlink channel ($|h_1| < |h_2|$).

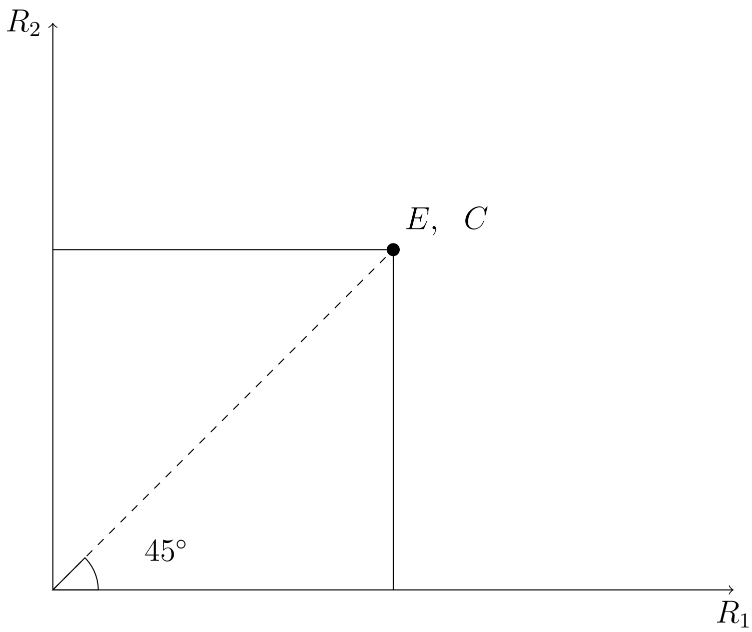


Figure 3.16: Symmetric capacity of downlink channel ($|h_1| < |h_2|$).

Fast-fading channel

In that case we calculate the sum capacity of the system. Since the channel does not remain constant during the transmission of information, the signal model is defined as

$$y_k[m] = h_k[m]x[m] + w_k[m], \quad w_k[m] \sim \mathcal{CN}(0,\, N), \qquad k = 1, 2. \tag{3.69}$$

Tx CSI

Since the base station can track the channel, the best we can do is to have the base station transmit only to the user which experiences the stronger channel gain. Hence, this communication scenario is identical to the single-user case and the sum capacity is

$$C_{\text{sum}} = \frac{1}{L}\sum_{m=1}^{L} \log_2\left(1 + \frac{p_m|h_k[m]|^2}{N}\right) \quad \text{bits/symbol} \tag{3.70}$$

where k is selected to satisfy the following

$$|h_k[m]|^2 = \max_i |h_i[m]|^2, \;\; i = 1, 2, \tag{3.71}$$

subject to the power constraint

$$\sum_{m=1}^{L} p_m = LP, \quad p_m \geq 0, \quad m = 1, ..., L. \tag{3.72}$$

We note that the optimal power allocation is given by the waterfilling algorithm. We also define the ergodic capacity of the fast fading channel as

$$C_{\text{sum}} = \mathbb{E}\left[\log_2\left(1 + \frac{P(h_k)|h_k|^2}{N}\right)\right] \quad \text{bits/symbol,} \tag{3.73}$$

where

$$P(h_k) = \left(\frac{1}{\lambda} - \frac{N}{|h_k|^2}\right)^+, \quad \mathbb{E}[P(h_k)] = P. \tag{3.74}$$

Note that the index k denotes the user with the stronger channel gain at each time, i.e.,

$$|h_k|^2 = \max_i |h_i|^2, \;\; i = 1, 2. \tag{3.75}$$

No Tx CSI

In that case the transmitter can not track the channel. The capacity formula as described in the corresponding chapter of "Fundamentals of Wireless Communication" by David Tse and Pramod Viswanath is considered to be

$$C_{\text{sum}} = \mathbb{E}\left[\log\left(1 + \frac{P|h|^2}{N}\right)\right]. \tag{3.76}$$

Chapter 4

MIMObit

Neben's MIMObit is a software tool, appropriate for analyzing MIMO signal processing algorithms. It models the propagation of electromagnetic waves under various types of environments. Therefore, we are able to model the fading channel introduced in Chapter 2 where the channel coefficients vary according to a distribution depending on the propagation environment and the antenna system used in MIMObit. We simulated the communication between one transmitter and one receiver where each of them is equipped with a dipole antenna. The dipoles were designed in MIMObit and Microwave Studio where the latter is an electromagnetics tool by Computer Simulation Technology (CST). We evaluated the accuracy of simulation results provided by MIMObit, when using dipoles with the same technical characteristics designed in both tools. We also calculated the capacity regarding SISO, SIMO and MIMO systems using the channel coefficients produced by MIMObit and Matlab.

4.1 Communication Setup

We need to define five components (i.e., transmitter, propagation environment, receiver, frequency, time) in order to run a simulation in MIMObit. Any parameter that is not mentioned in each of these components is considered to have its default value depicted on the corresponding figures. For further information about the exact functionality of certain parameters, refer to the software's manual.

Transmitter

This component refers to the transmitter's specifications. The location of the transmitter in the three dimensional space is specified by the notation $(0, 0, 10)$ where (x, y, z) refers to length, width, and height, respectively, in meters. The total available power of the transmitted signal is 30 dBm and uniformly distributed over the signal bandwidth which is 1 MHz. The source impedance of the antenna $Ant.\ Z_{in}$ is defined at the dipole's creation, while the transmitter's impedance is fixed properly so as to achieve the maximum radiation efficiency depending on the antenna used.

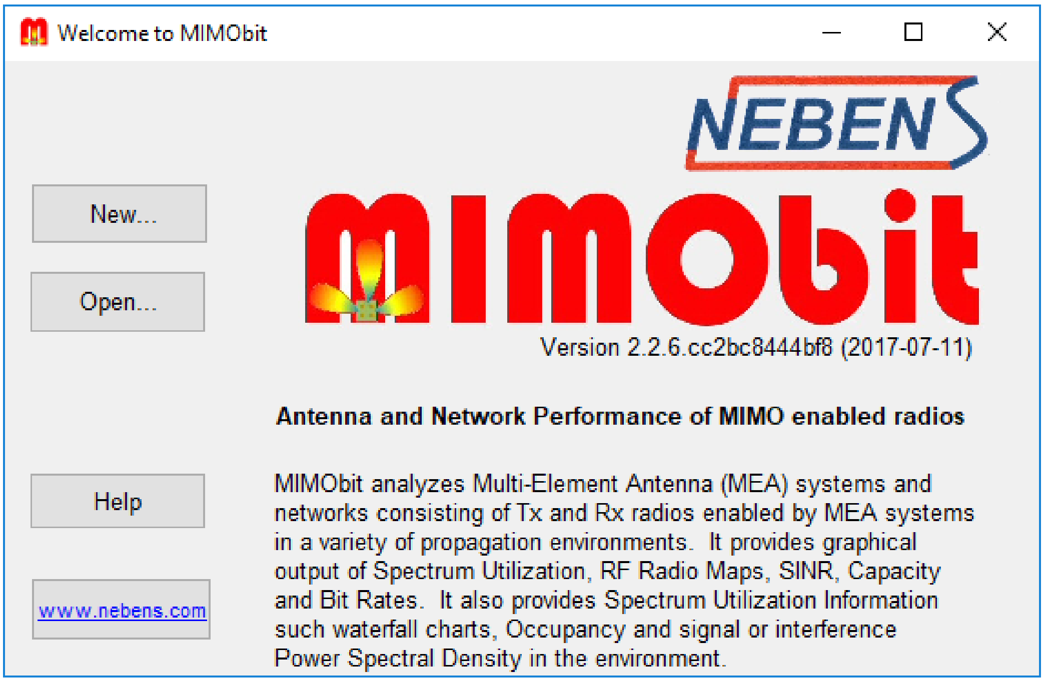


Figure 4.1: Application.

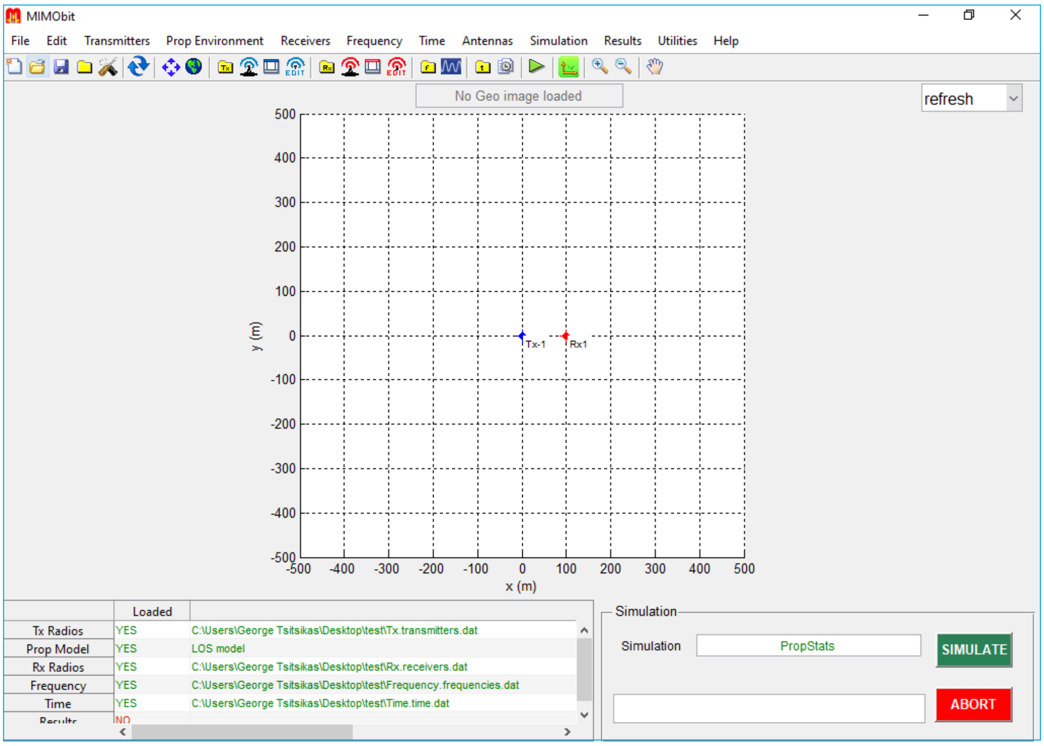


Figure 4.2: Main graphical user interface.

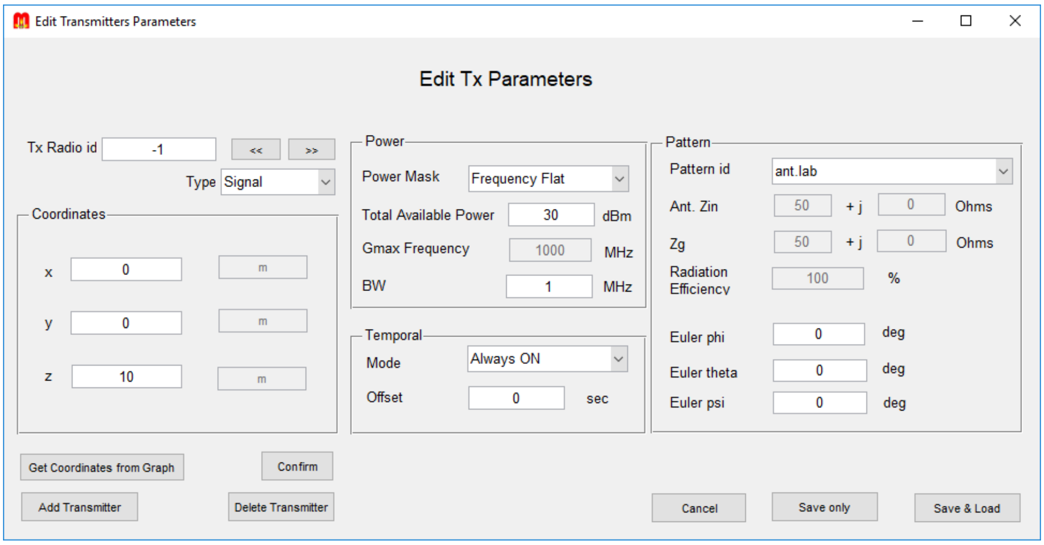


Figure 4.3: Transmitter.

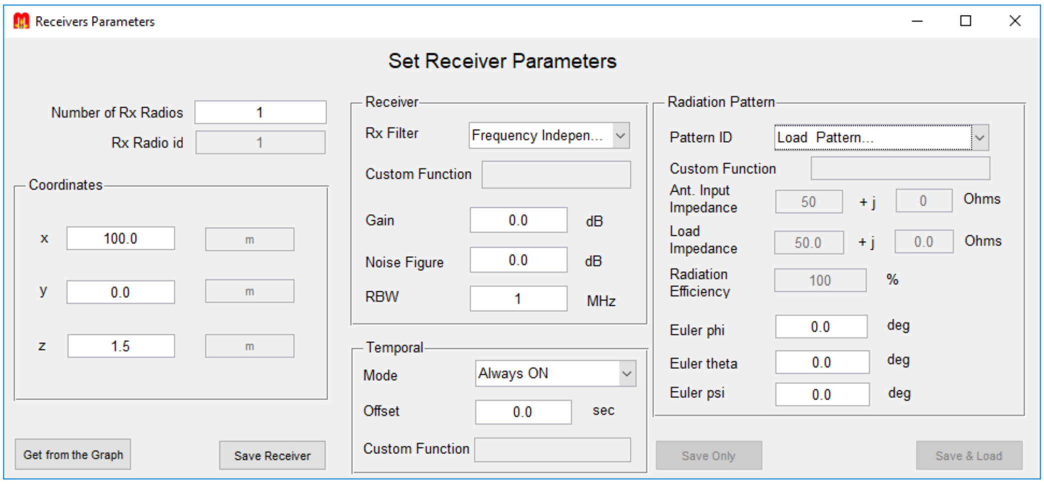


Figure 4.4: Receiver.

Propagation environment

MIMObit offers a variety of propagation environments to choose where only two of them are deterministic. These are the Line of Sight and the 2-Ray model. All other propagation environments are random, which means that we produce different channel coefficients every time we run a simulation under the same communication setup.

Receiver

The receiver's specifications are almost identical to those of the transmitter. The receiver is considered to be 100 m away of the transmitter and 1.5 m above the ground.

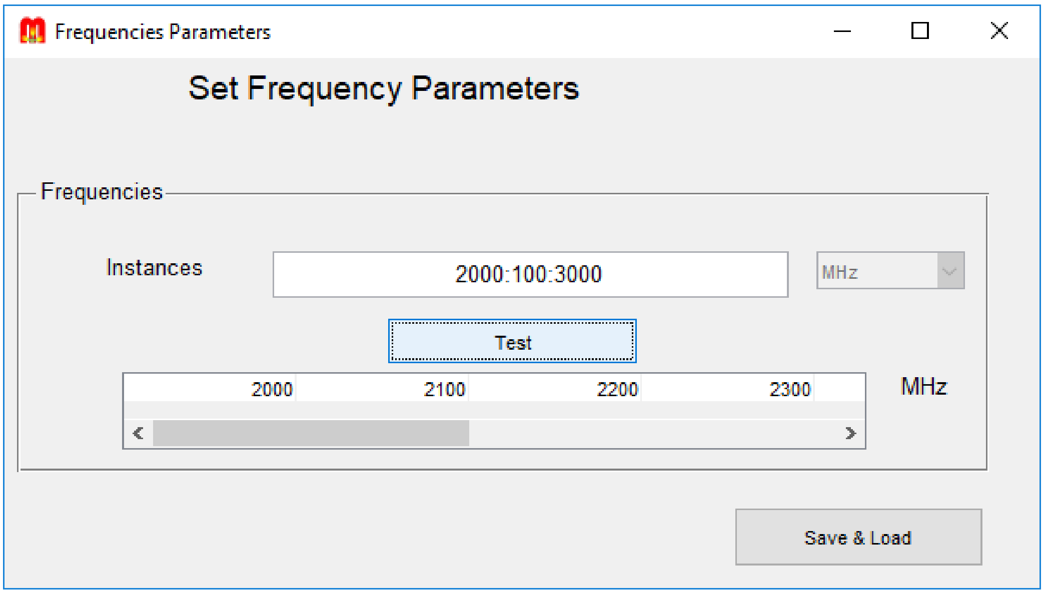


Figure 4.5: Frequency.

Frequency

This component refers to the frequencies at which we simulate our communication. Hence, we get a channel coefficient for every simulation frequency and for every combination of antennas at the transmitter and the receiver, regarding the deterministic propagation environments. For random evironments, we have the option to choose the number of channel coefficients in order to best approach the distribution that describes those environments.

Time

The time component is useful in order to generate different time instances regarding the function of the transmitter and the receiver at the corresponding temporal section. We can relate the time instances to different experiments incorporated in one single simulation. Note that these instances are not related to the behaviour of the random propagation environments, so we did not take into account this option for our simulations.

4.2 Dipole Antennas

MIMObit offers the ability to design custom dipoles and use them at the transmitter and the receiver. The dipole's characteristics are depicted in the corresponding figure. Note that the dipole's length specifies its operating frequency. To make this clear, we use the fundamental equation

$$c = \lambda f \tag{4.1}$$

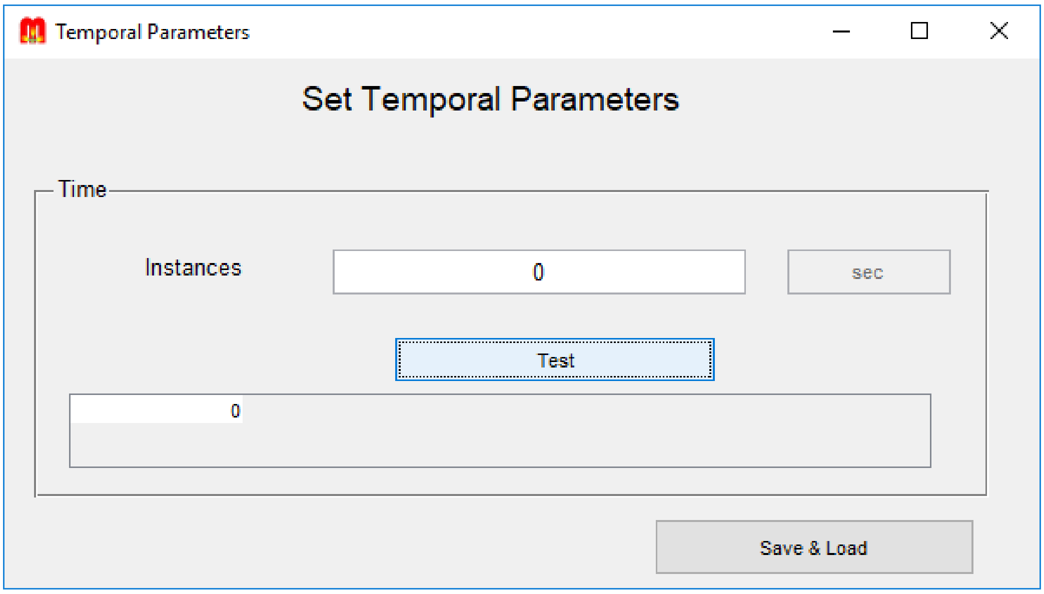


Figure 4.6: Time.

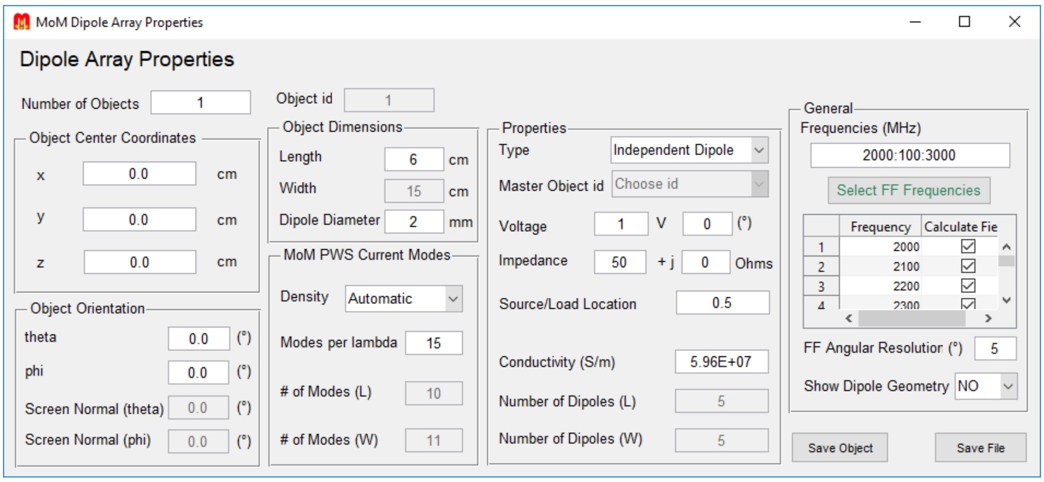


Figure 4.7: Dipole.

where $c = 3 \cdot 10^8$ meters/second and refers to the speed of light while λ refers to the wavelength which is the distance from crest to crest or trough to trough on the propagating wave. To get the maximum of the total radiated power for a specific frequency, we need to have the dipole's length half-wavelength long. For instance, to design a dipole for the frequency of 2500 MHz, we get

$$c = \lambda f \implies \lambda = \frac{c}{f} \implies \lambda = \frac{3 \cdot 10^8 \text{ meters / second}}{25 \cdot 10^8 \text{ second}^{-1}} = 0.12 \text{ meters.} \tag{4.2}$$

As a result, we need to use a dipole of 6 cm long. However, simulation results indicate that the desired resonant frequency is not exactly that calculated with the aforementioned formula. The dipole's diameter is 2 mm and its source (or load) impedance is 50 Ohms. Note that we also define the frequencies at which we simulate the dipole's performance. These frequencies should be exactly the same with those specified in the frequency component.

Dipole's comparison

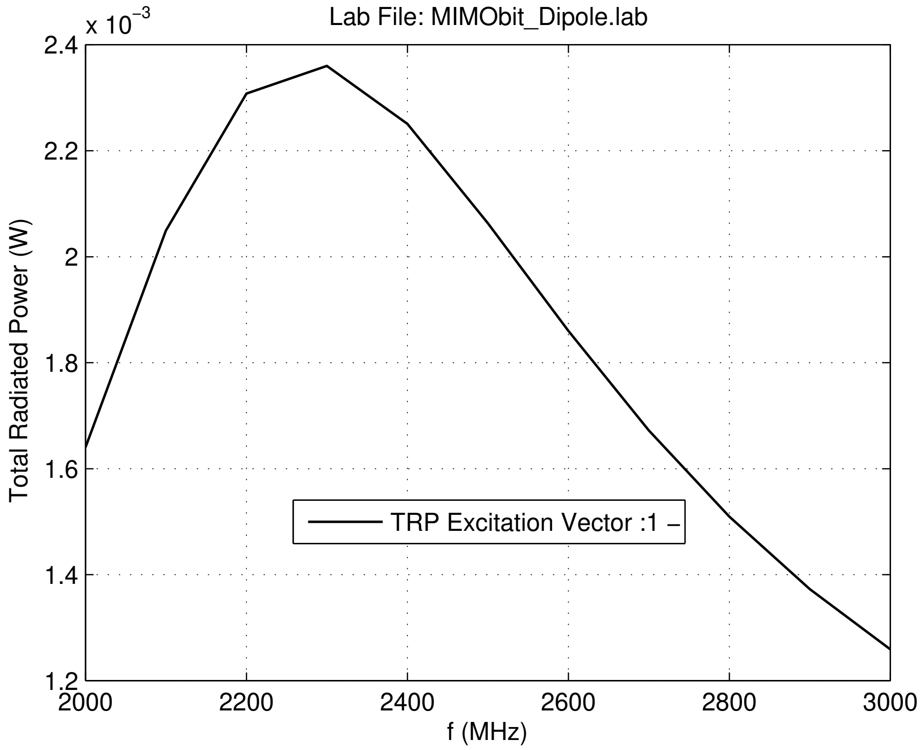
The antenna comparison mainly involves a single dipole antenna designed in MIMObit and CST Microwave Studio. The antenna's technical characteristics used are the following.

- Length = 6cm.
- Diameter = 2mm.
- Conductivity = 5.96×10^7 Siemens/meter.
- Center-fed dipole.
- Source voltage = 1 Volt.
- Source impedance = 50 Ohms.
- Simulation frequencies = 2000 : 100 : 3000 MHz.

We mention that, to get the best performance matching between the two dipoles, the same orientation needs to be applied at the design stage. In addition, the circuit of the feeder does not have to be considered at the appropriate stage in CST Microwave Studio, because in MIMObit only the dipole itself is designed. In order to get the appropriate files from CST-MWS and import them to MIMObit, there is a certain procedure that has to be applied which is mentioned in section 10.2.1 (Exporting From CST-MWS) and 10.2.2 (Importing from CST-MWS) of MIMObit 2 User's Manual. The files that need to be imported in MIMObit are the s-parameters file and the far-field radiation files. Regarding the latter, they need to have the following name notation "*AnyName_P1_F2000.txt*" for port 1 at 2000 MHz. In the case where we tested one dipole antenna, we needed eleven such files where each of them corresponds to the combination between port 1 and each simulation frequency. MIMObit uses these files in order to produce another one with a "*.lab*" extension, which refers to the antenna finally attached to the transmitter and the receiver. Note that this file is also created when we design dipoles in MIMObit. The figures referring to the dipole's performance are produced by MIMObit using the "*.lab*" file at the "Plot Lab file" option in the "Antennas" menu, whereas those referring to the channel coefficients are produced by MIMObit in the "*.Htotalresults.dat*" file considering the communication setup already discussed. For experimental reasons, we also tested the channel coefficients produced using a dipole array including two dipoles at a distance of 96 cm along the x-axis and a single dipole with the aforementioned technical characteristics designed in both MIMObit and CST-MWS. These cases involve MISO and SIMO systems.

4.3 Simulation Results

Besides the dipole's comparison, we also calculated the capacity of SISO, SIMO and MIMO systems using the channel coefficients produced by MIMObit. When multiple dipole antennas are used for the SIMO or MIMO cases, they were designed in MIMObit and the distance between them is considered to be $\lambda/4 = 3$ cm and $\lambda/2 = 6$ cm along the x-axis. The propagation environment used is the independent isotropic distribution (i.i.d. 3D), which involves a lot of scattering and the power arrives at the receiver uniformly from all angles and polarizations. However, the channel coefficients do not follow the Rayleigh fading distribution, because in MIMObit the antenna system and the propagation environments are physically connected. This is not a MIMObit's shortcoming, but rather an approach to real-world channels. Taking into account only the frequency of 2500MHz and assuming $L = 500$ transmissions, we computed the capacity of the system applying the waterfilling algorithm using $\lambda(h)$. We estimated the SNR of each system separately, considering the channel gains produced by MIMObit, to estimate the overall SNR by averaging the individual SNR values, Figure 4.20. In terms of comparisons, we also evaluated the capacity performance considering the Rayleigh fading model in Matlab, using the SNR estimation referring to the case where the distance between the dipole antennas is 6 cm, Figure 4.21.



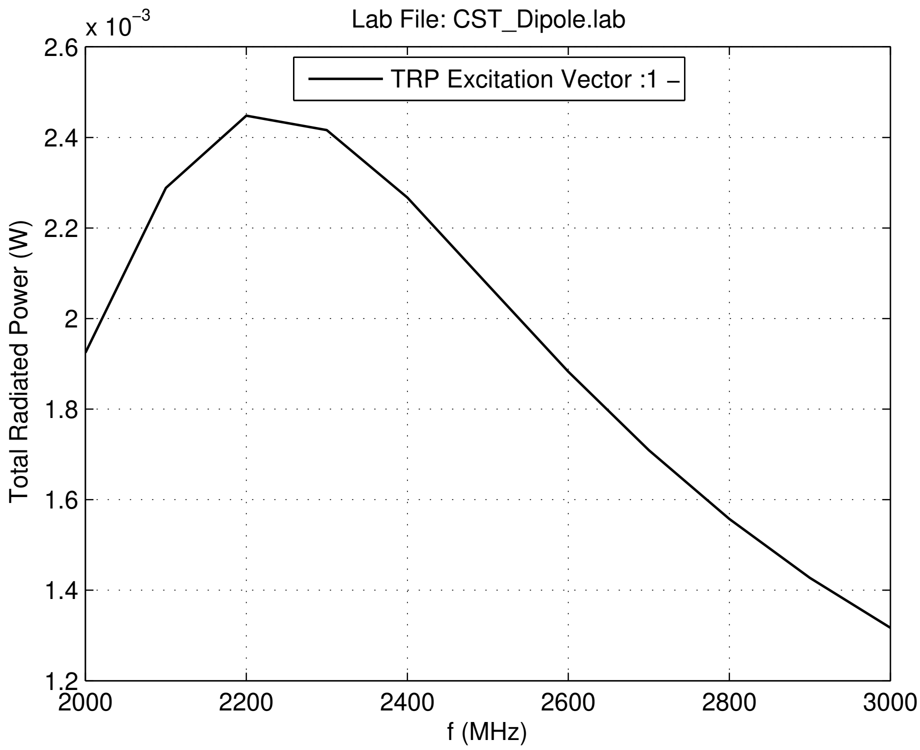
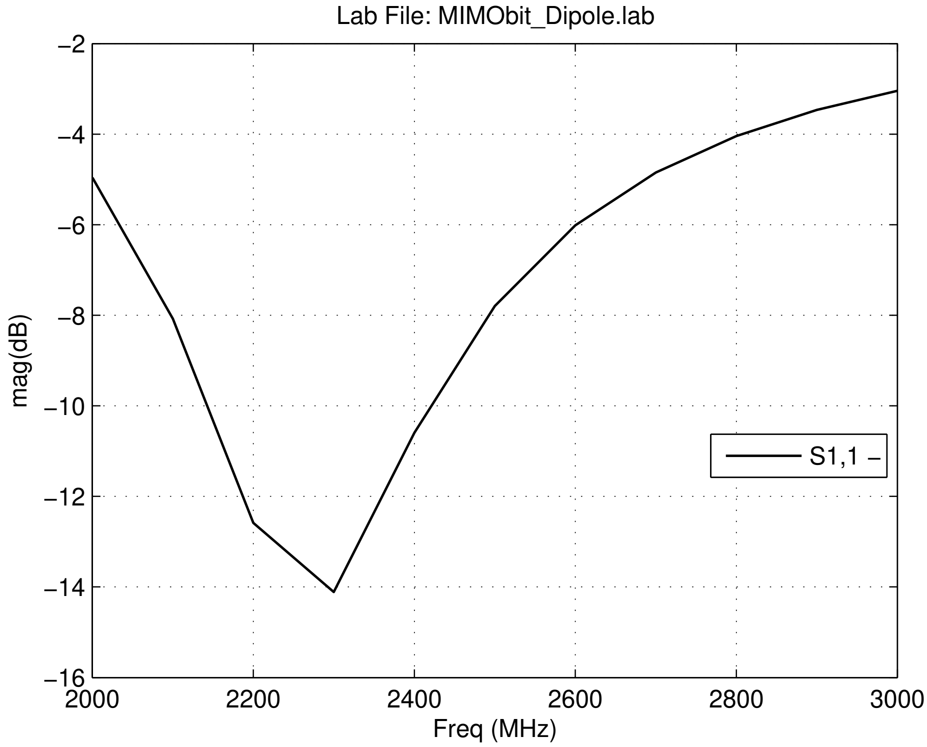
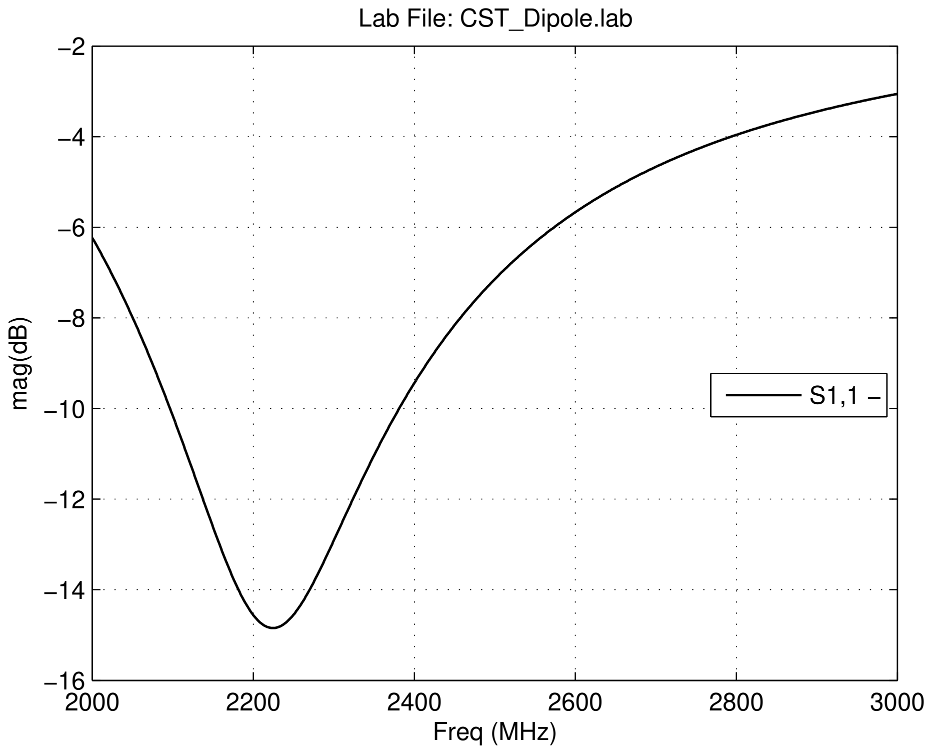


Figure 4.8: Dipole's total radiated power.


Lab File: MIMObit_Dipole.lab
mag(dB)
−2
−4
−6
−8
−10
−12
−14
−16
2000
2200
2400
2600
2800
3000
Freq (MHz)
S1,1 –


Lab File: CST_Dipole.lab
mag(dB)
−2
−4
−6
−8
−10
−12
−14
−16
2000
2200
2400
2600
2800
3000
Freq (MHz)
S1,1 –

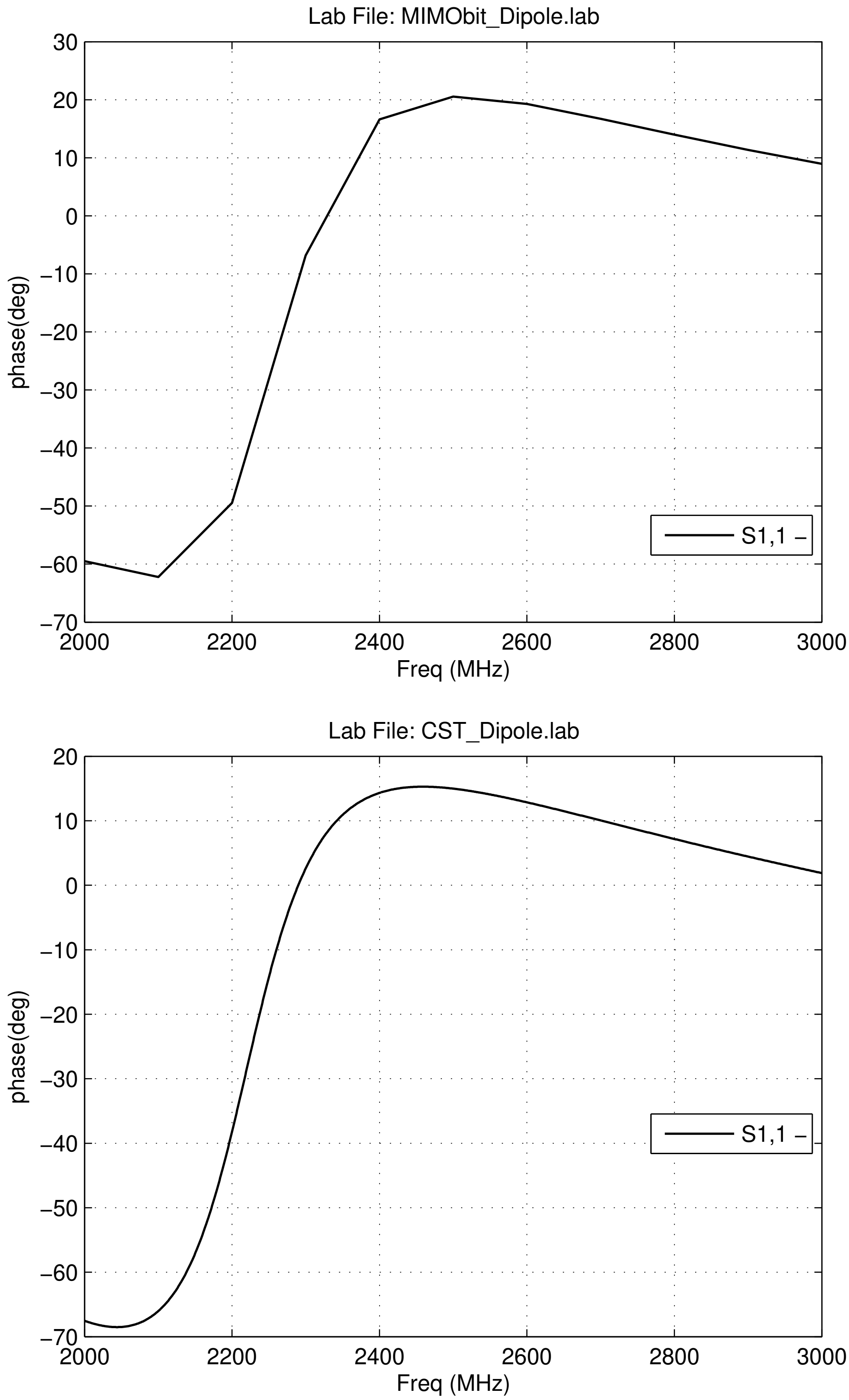
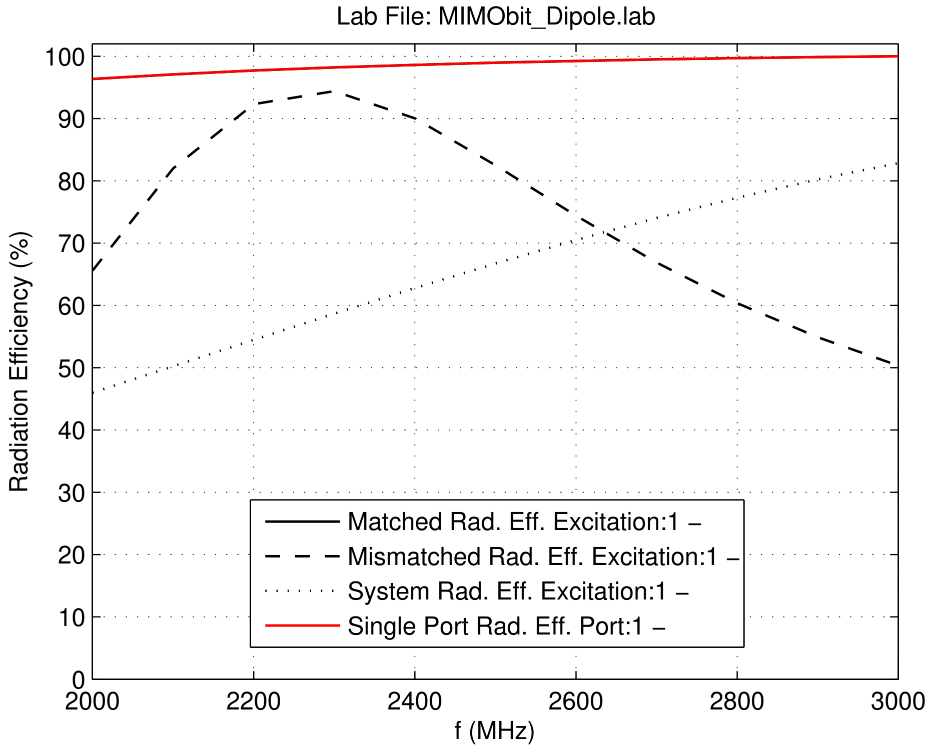


Figure 4.9: Dipole's s-parameters.



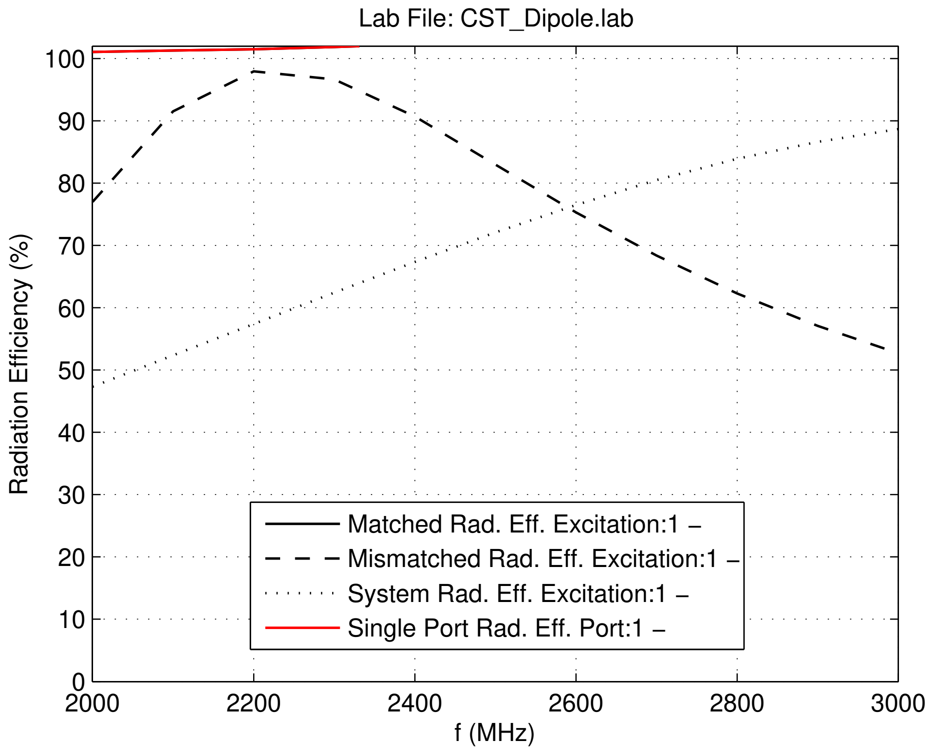


Figure 4.10: Dipole's radiation efficiency.

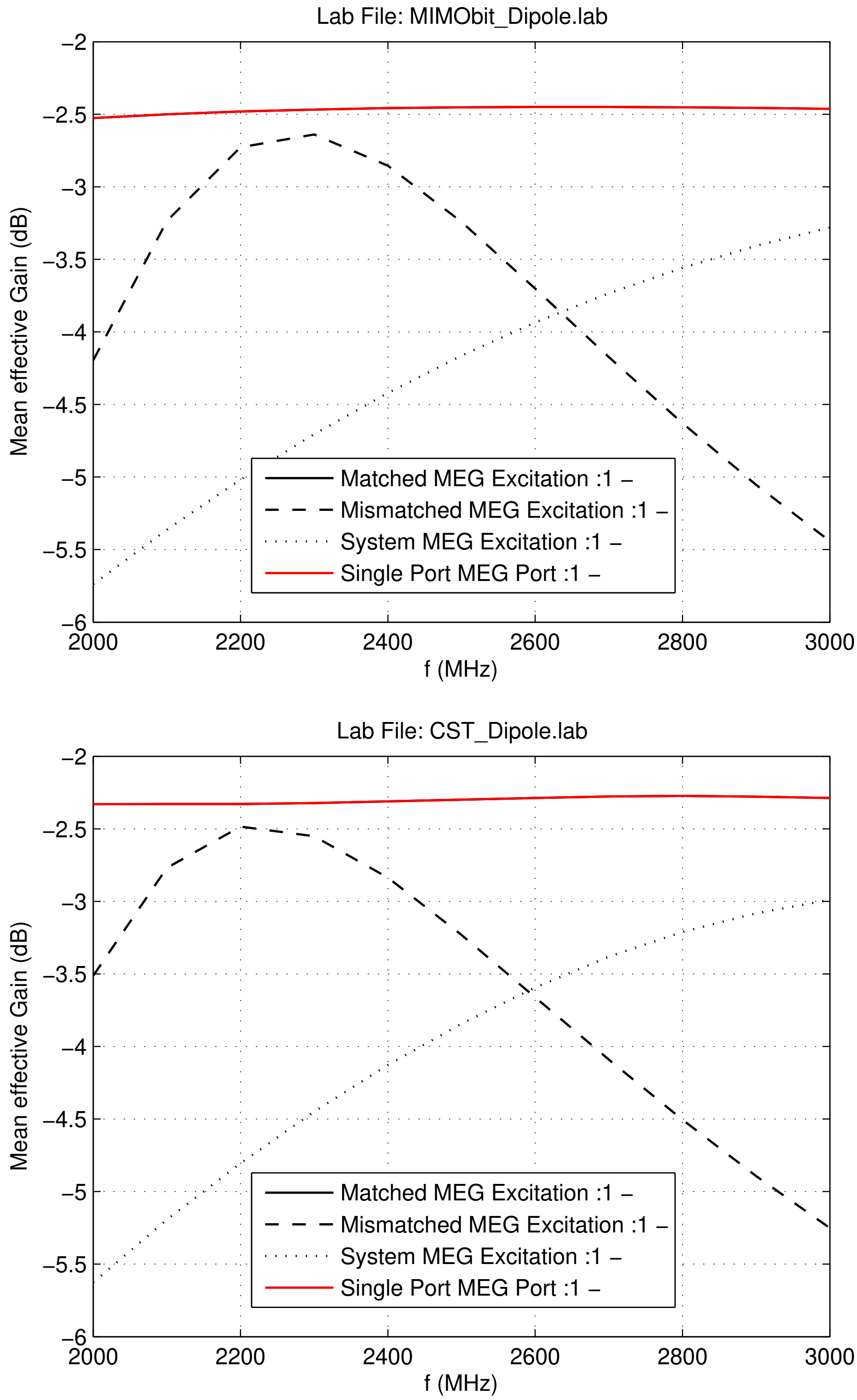


Figure 4.11: Dipole's mean effective gain.

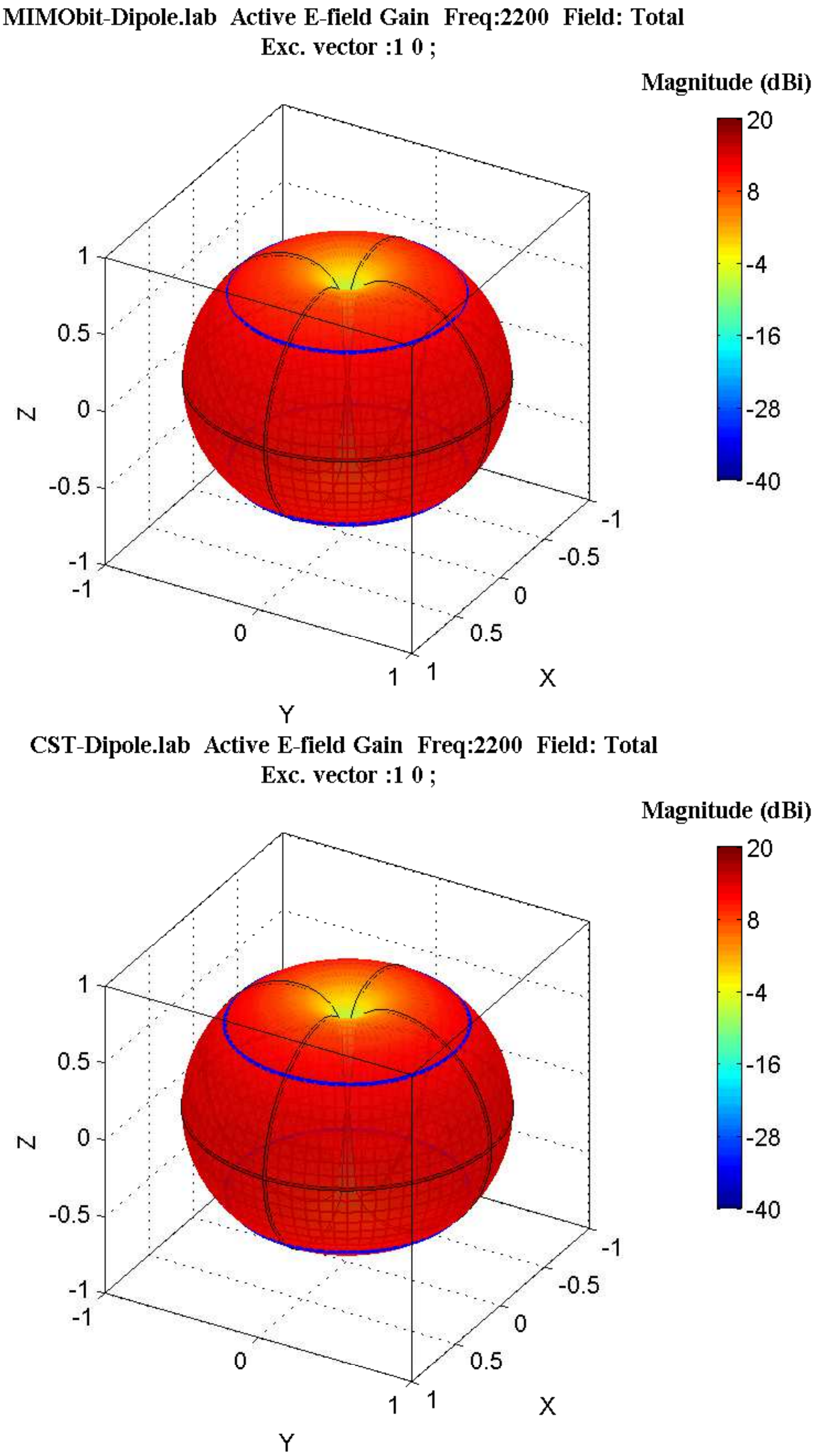


Figure 4.12: Dipole's active E-field gain (2200 MHz).

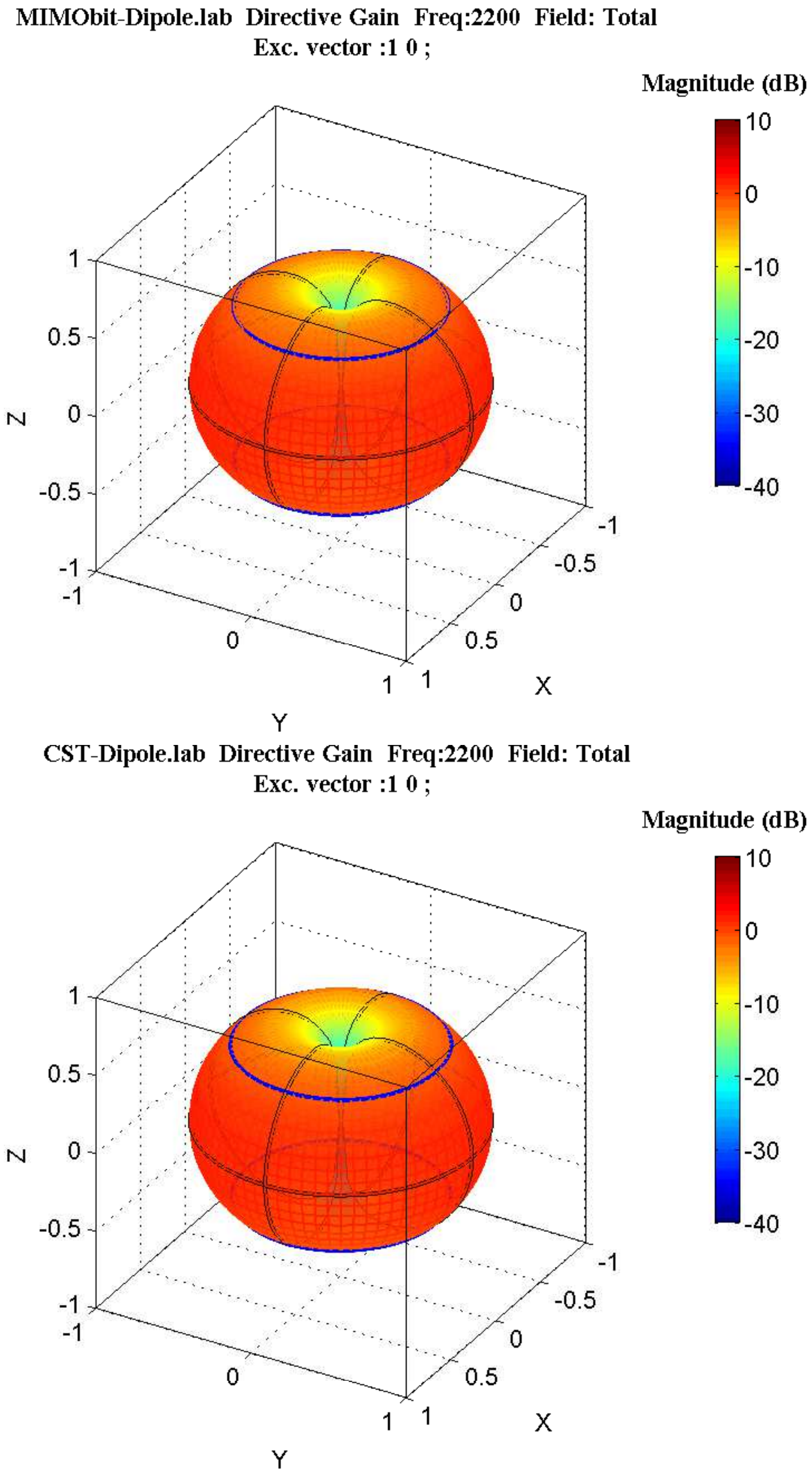
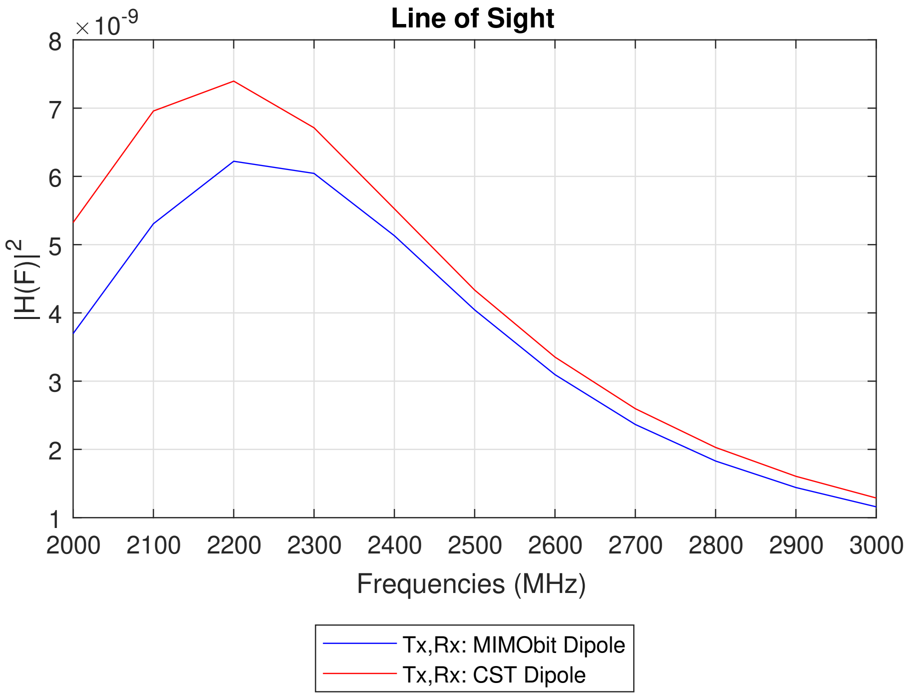
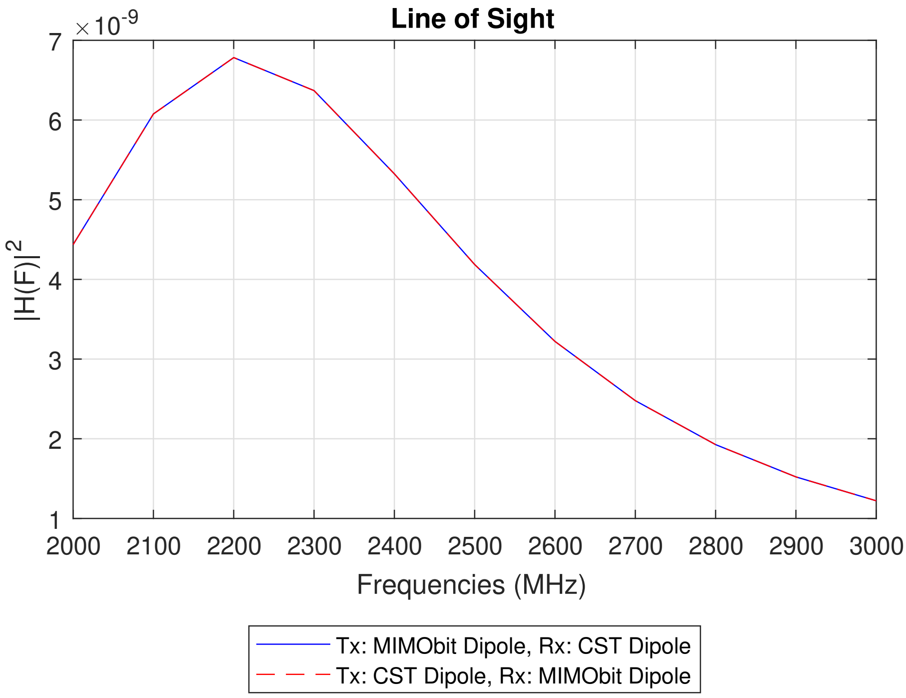
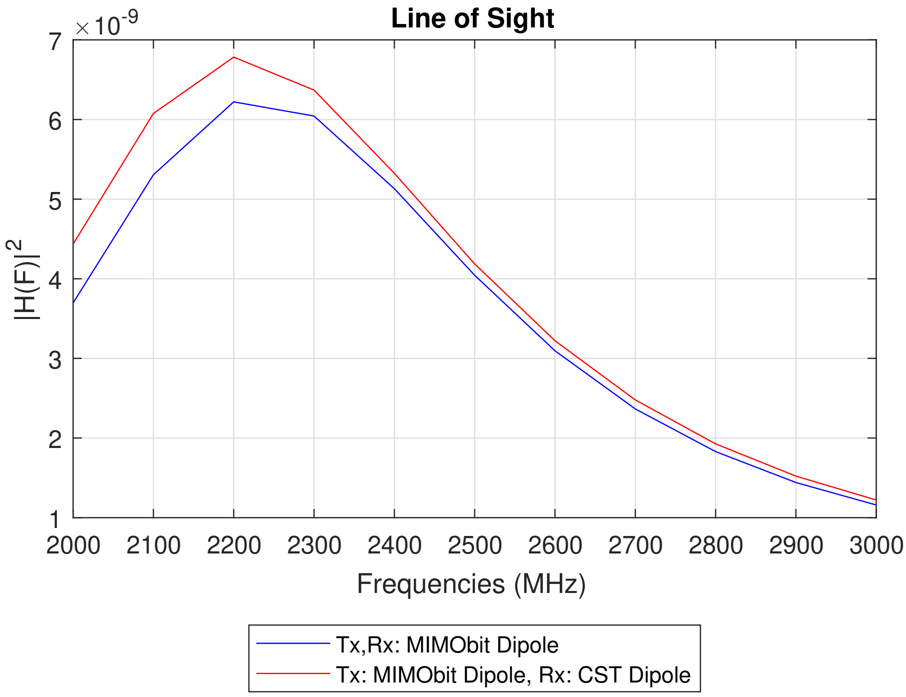
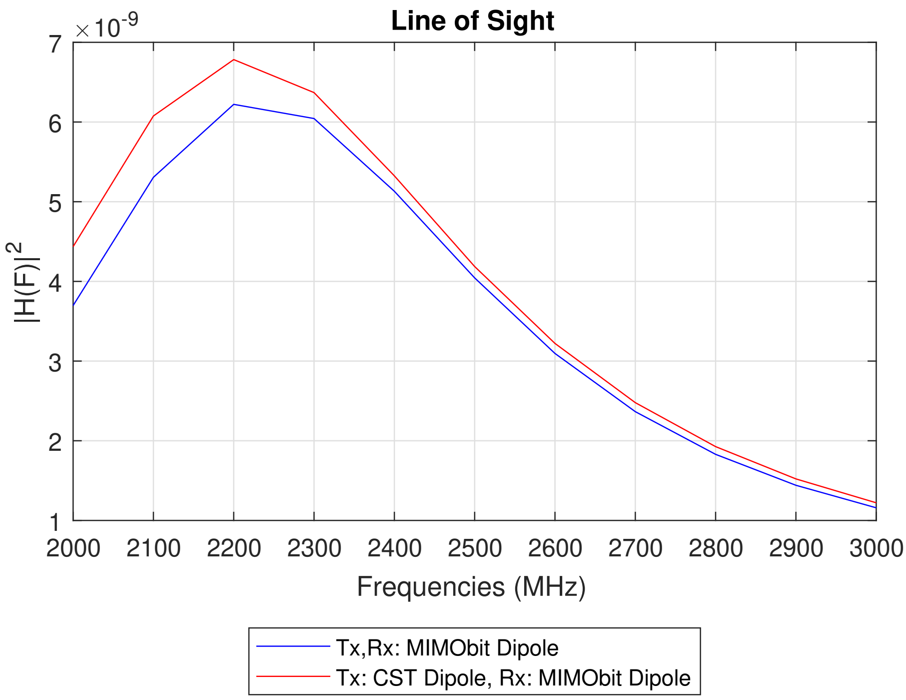


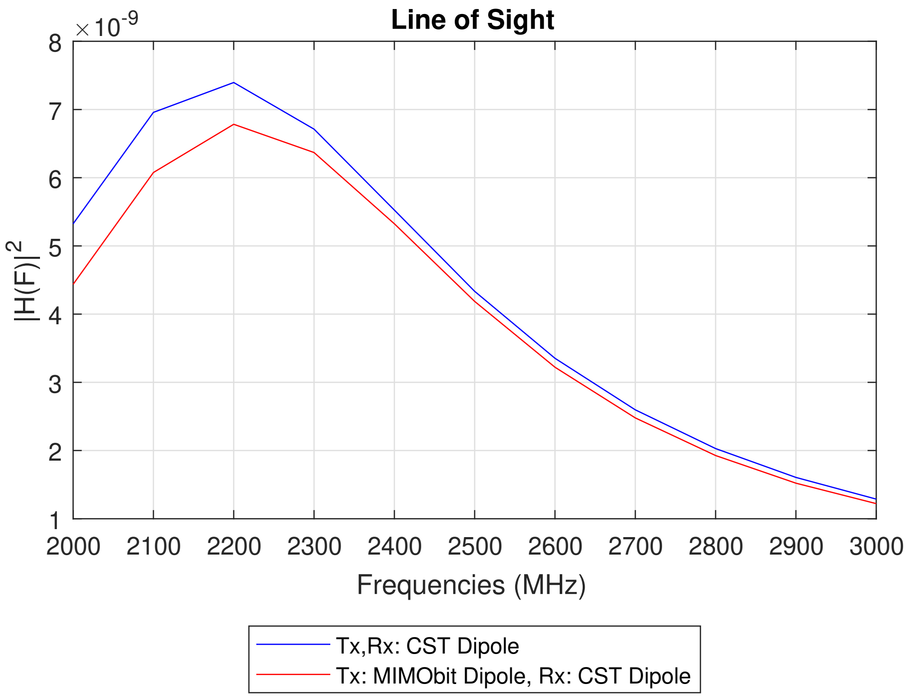
Figure 4.13: Dipole's directive gain (2200 MHz).


Line of Sight
×10^-9
8
7
6
5
4
3
2
1
|H(F)|^2
2000 2100 2200 2300 2400 2500 2600 2700 2800 2900 3000
Frequencies (MHz)
Tx,Rx: MIMObit Dipole
Tx,Rx: CST Dipole


Line of Sight
×10^-9
7
6
5
4
3
2
1
|H(F)|^2
2000 2100 2200 2300 2400 2500 2600 2700 2800 2900 3000
Frequencies (MHz)
Tx: MIMObit Dipole, Rx: CST Dipole
Tx: CST Dipole, Rx: MIMObit Dipole


Line of Sight
×10^-9
7
6
5
4
3
2
1
|H(F)|^2
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
Frequencies (MHz)
Tx,Rx: MIMObit Dipole
Tx: MIMObit Dipole, Rx: CST Dipole


Line of Sight
×10^-9
7
6
5
4
3
2
1
|H(F)|^2
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
Frequencies (MHz)
Tx,Rx: MIMObit Dipole
Tx: CST Dipole, Rx: MIMObit Dipole



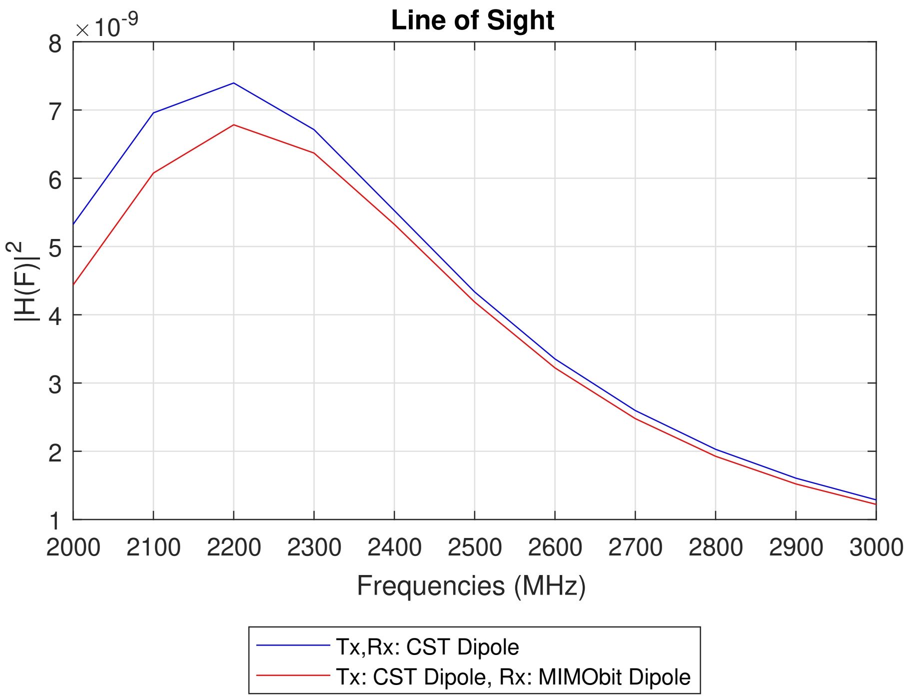


Figure 4.14: Channel gain vs frequency (line of sight - SISO).

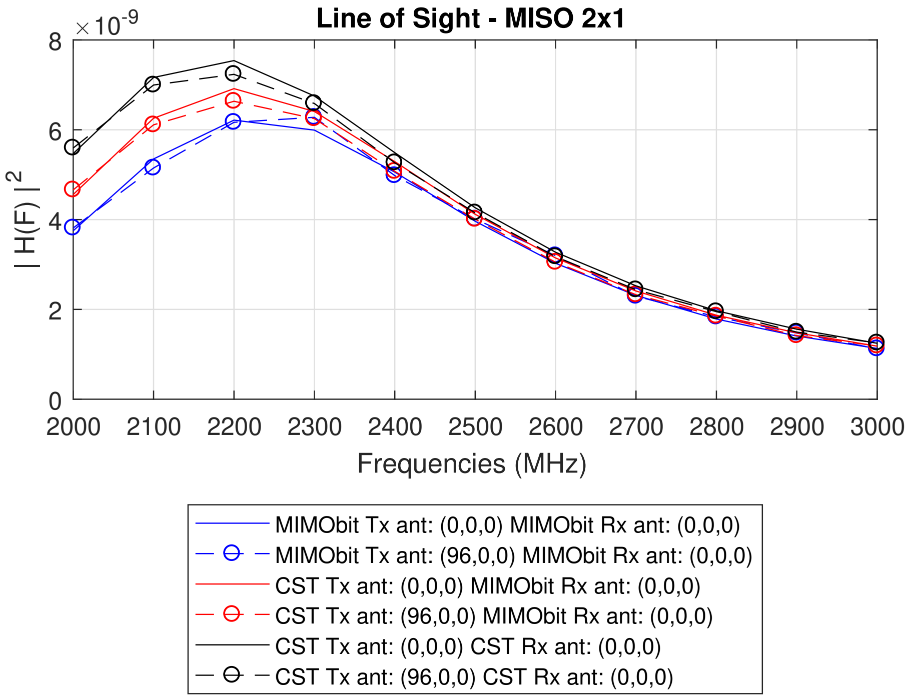


Figure 4.15: Channel gain vs frequency (line of sight - MISO).

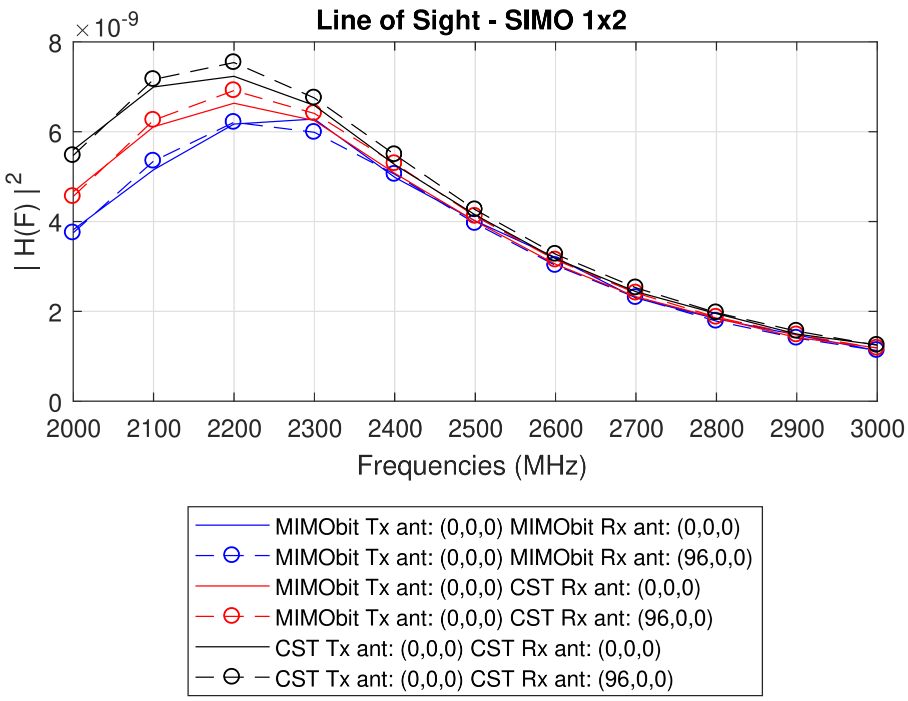
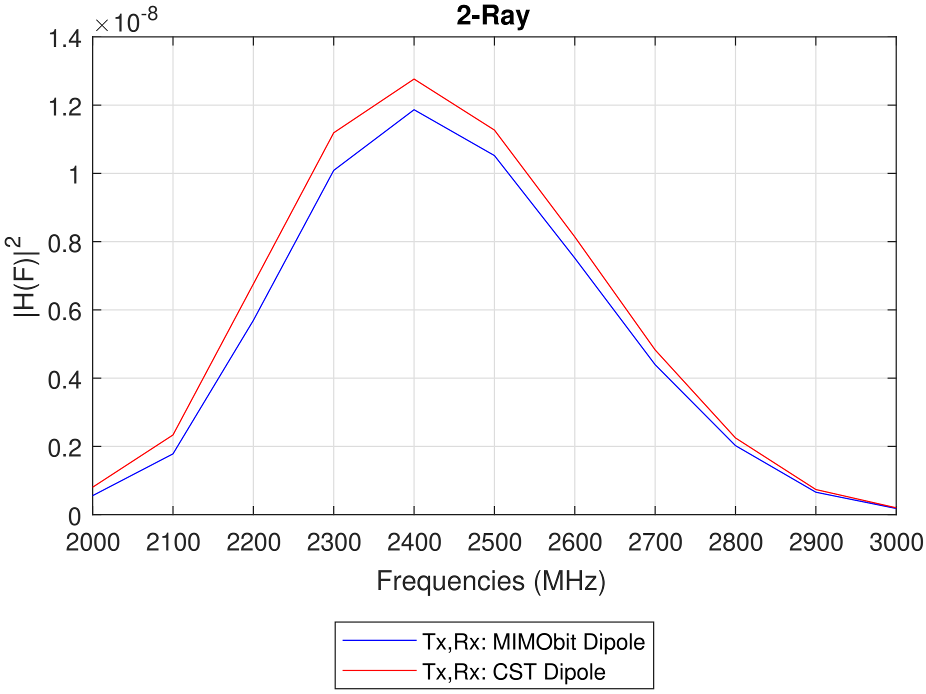
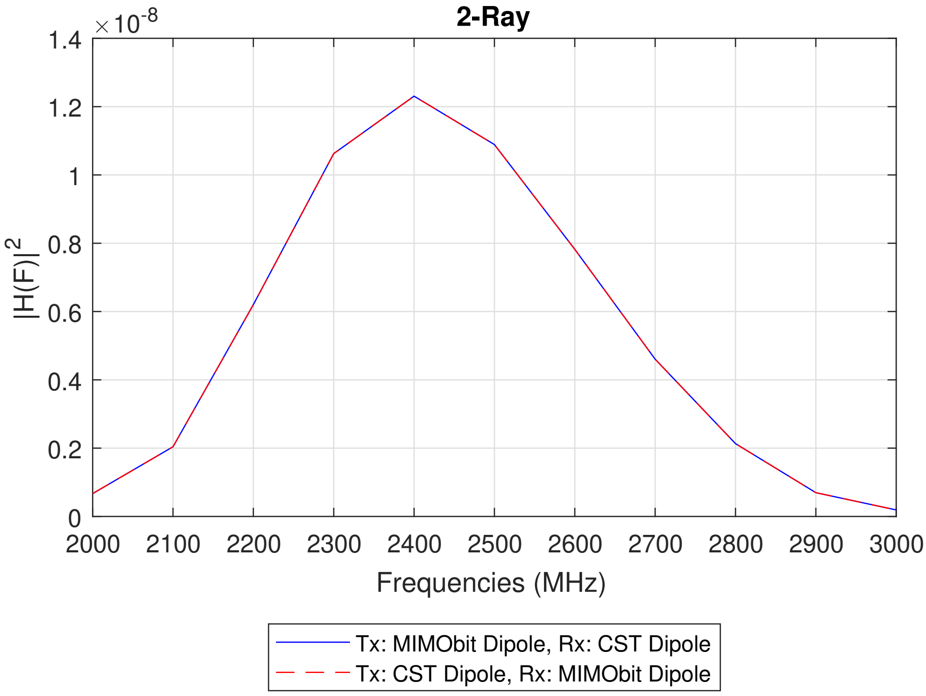
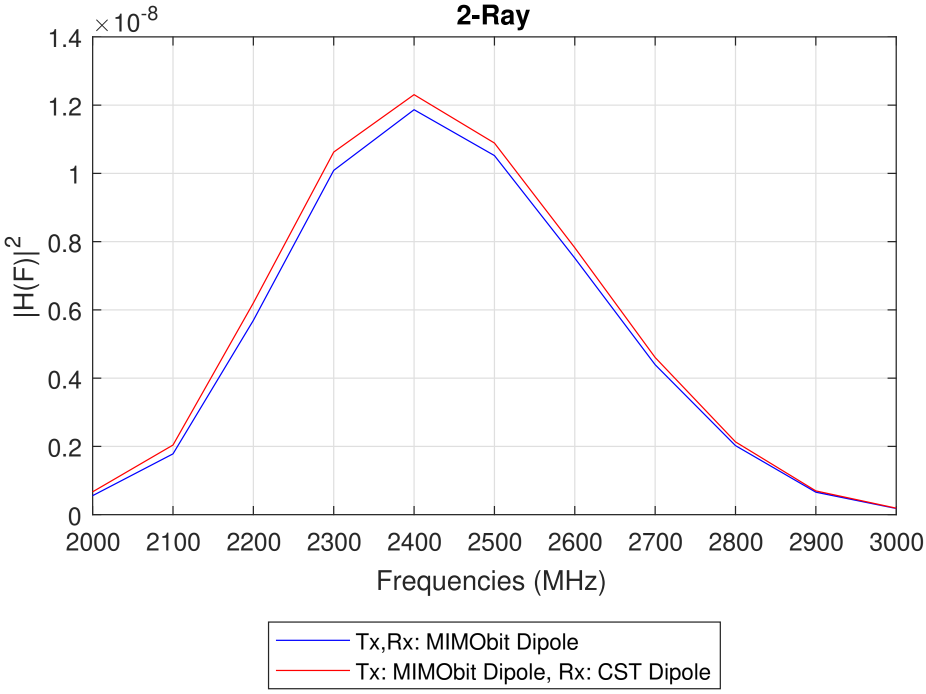
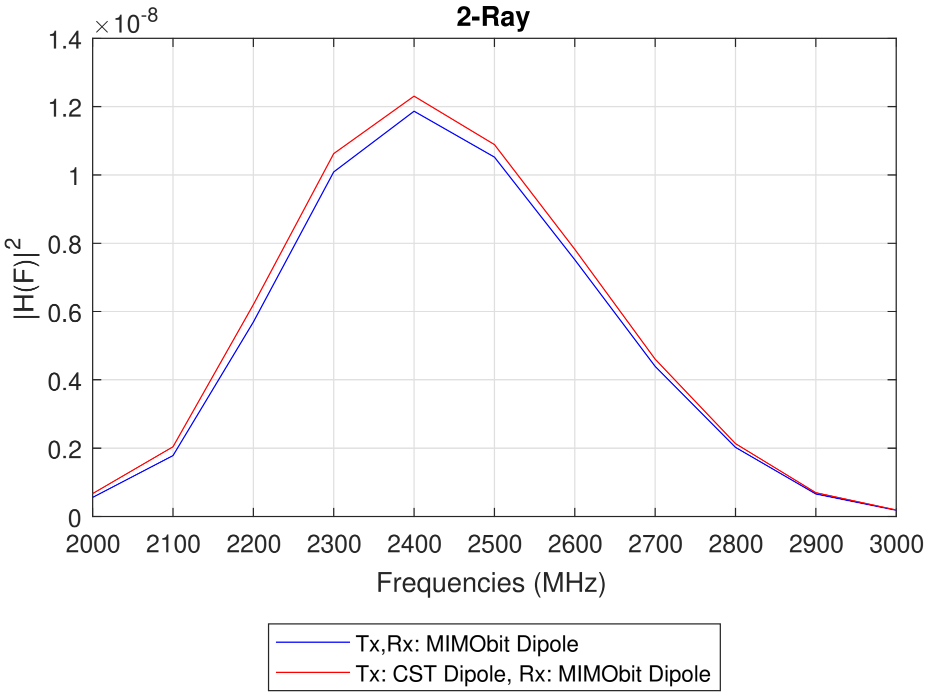


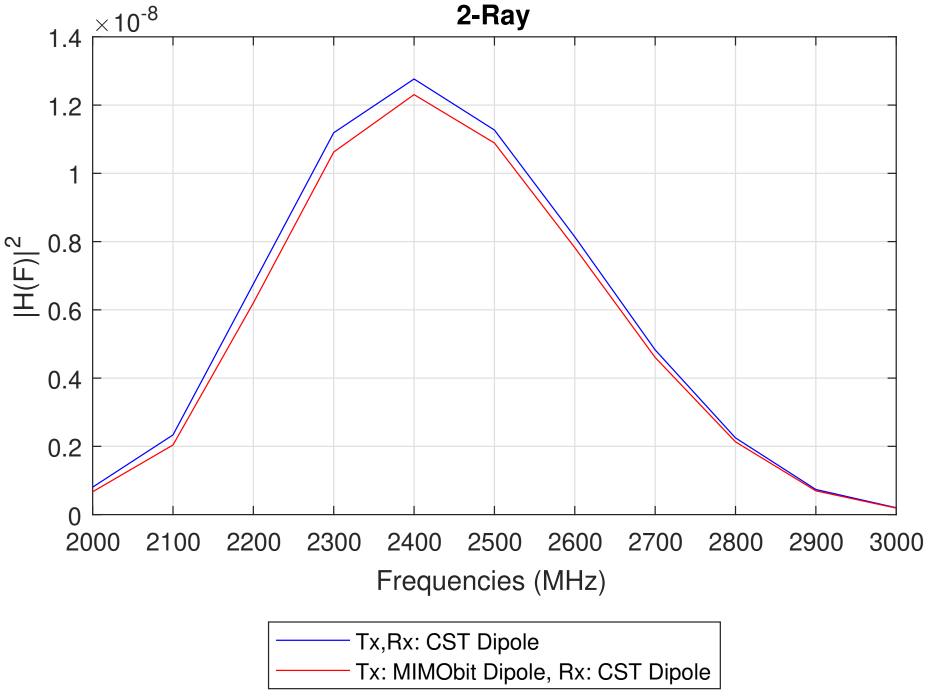
Figure 4.16: Channel gain vs frequency (line of sight - SIMO).


2-Ray
$\times 10^{-8}$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$|H(F)|^2$
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
Frequencies (MHz)
Tx,Rx: MIMObit Dipole
Tx,Rx: CST Dipole


2-Ray
$\times 10^{-8}$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$|H(F)|^2$
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
Frequencies (MHz)
Tx: MIMObit Dipole, Rx: CST Dipole
Tx: CST Dipole, Rx: MIMObit Dipole


2-Ray
×10^-8
1.4
1.2
1
0.8
0.6
0.4
0.2
0
|H(F)|^2
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
Frequencies (MHz)
Tx,Rx: MIMObit Dipole
Tx: MIMObit Dipole, Rx: CST Dipole


2-Ray
×10^-8
1.4
1.2
1
0.8
0.6
0.4
0.2
0
|H(F)|^2
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
Frequencies (MHz)
Tx,Rx: MIMObit Dipole
Tx: CST Dipole, Rx: MIMObit Dipole



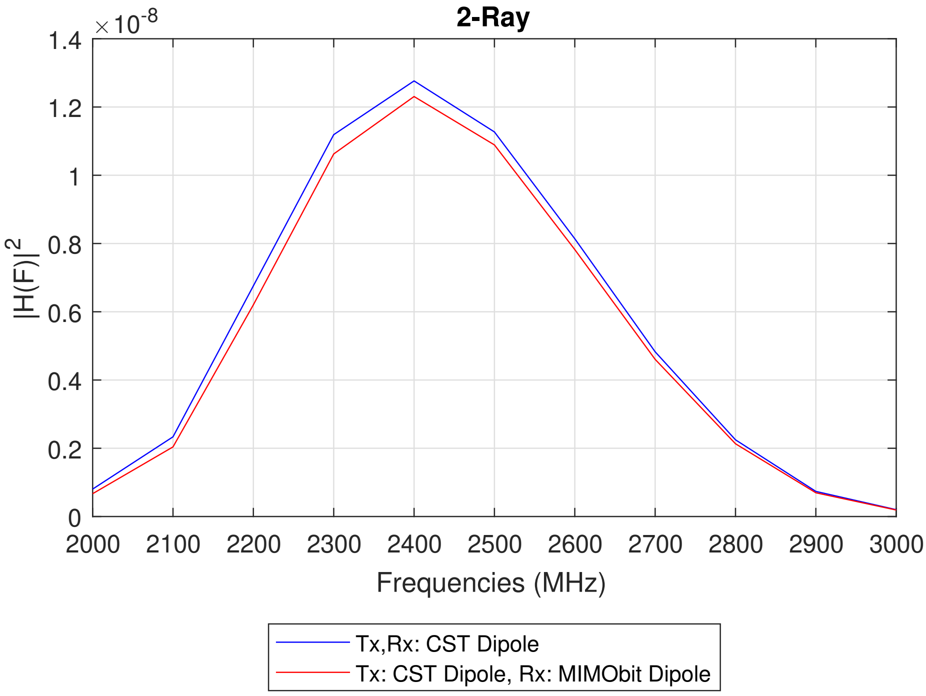


Figure 4.17: Channel gain vs frequency (2 Ray - SISO).

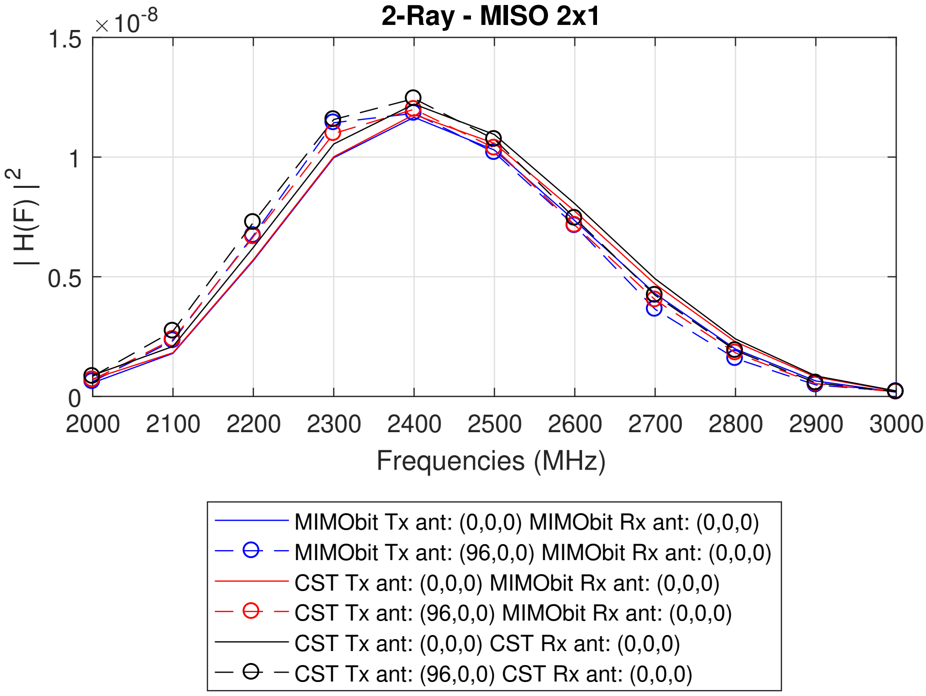


Figure 4.18: Channel gain vs frequency (2 Ray - MISO).

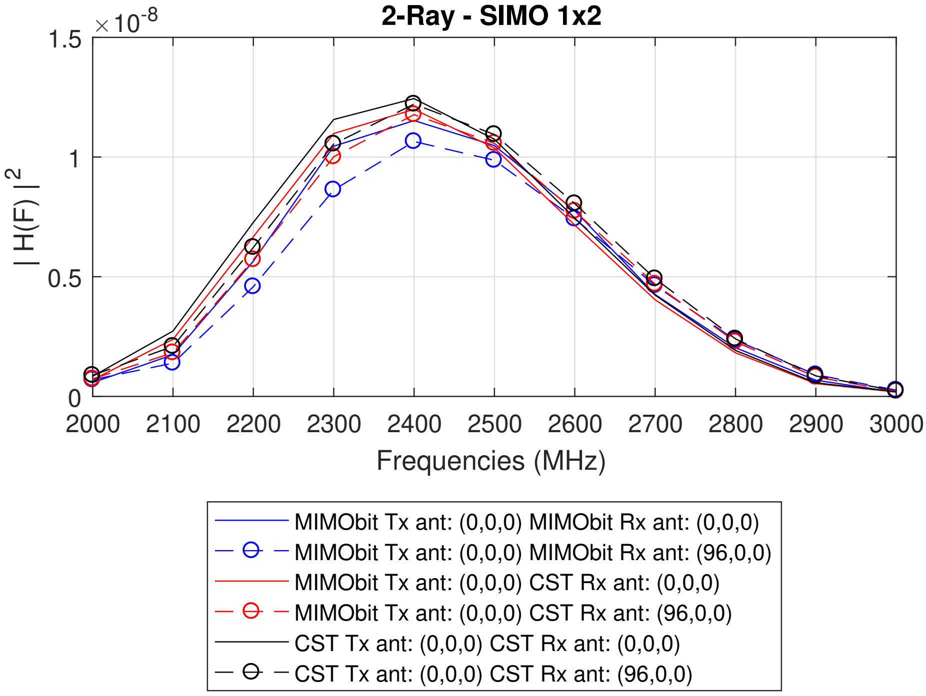
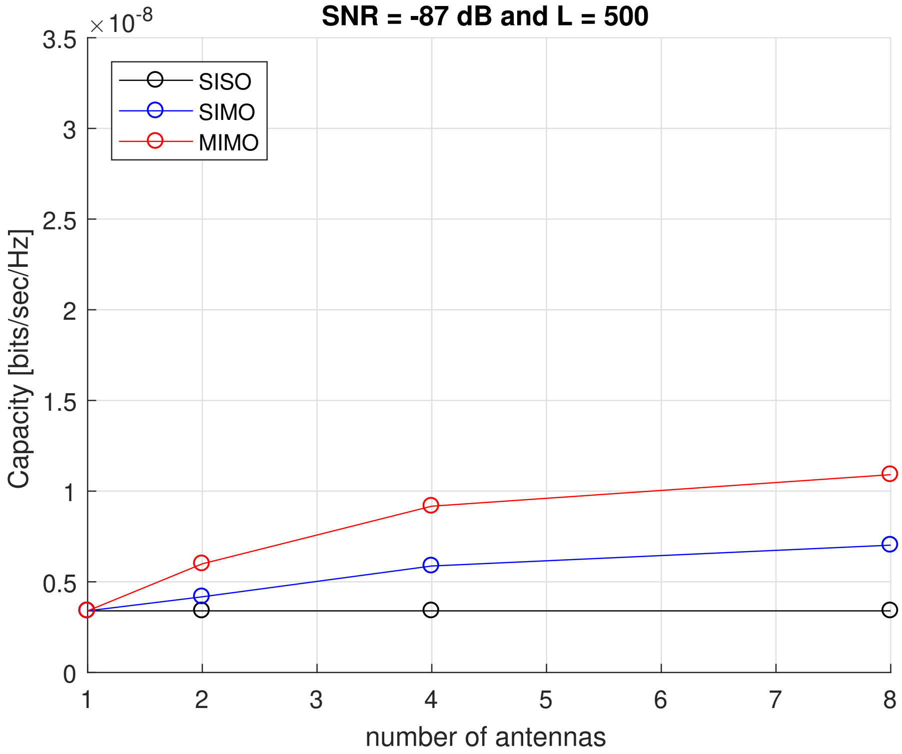
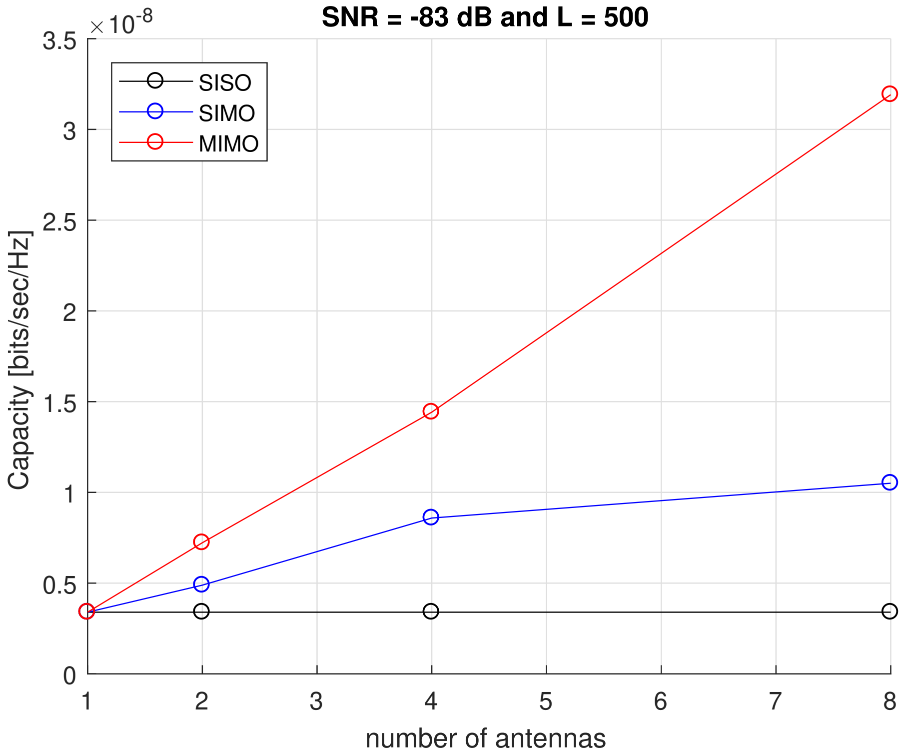


Figure 4.19: Channel gain vs frequency (2 Ray - SIMO).



(a) Antenna distance $\frac{\lambda}{4} = 3$ cm



(b) Antenna distance $\frac{\lambda}{2} = 6$ cm

Figure 4.20: Capacity vs number of antennas (MIMObit).

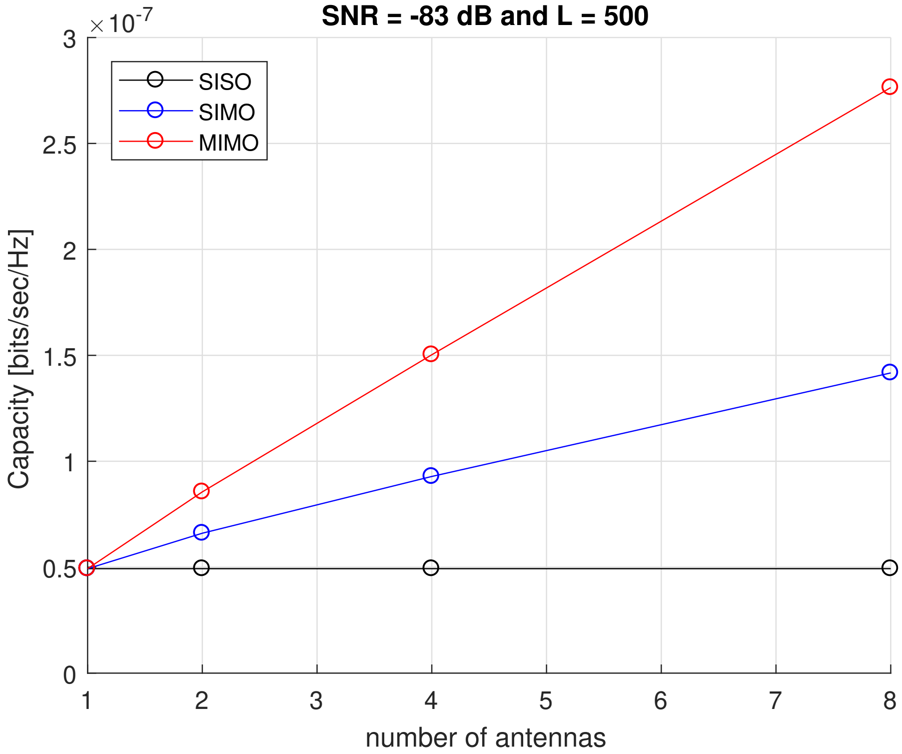


Figure 4.21: Mean capacity vs number of antennas (Matlab).

Chapter 5

Conclusions and Future Work

In this work, we mainly focused on the signal models of wireless communication scenarios in order to define the maximum achievable rate of reliable communication, that is, the system's capacity. Information theory is a fundamental part of this analysis, so we refered to all important quantities that are needed as background knowledge. Taking into account the communication scenario where the transmission channel is known to both the transmitter and the receiver, we optimized the capacity formula using the waterfilling power allocation. We evaluated the capacity performance of a MIMO system, where multiple antennas are used at both the transmitter and the receiver regarding the single-user case. In addition, the multiuser scenario with two users present was analyzed. To this end, we assumed a single antenna available at the users and the base station. In the multiuser case, we defined the capacity region, the sum capacity, and the symmetric capacity of the system.

We also evaluated MIMObit's capabilities regarding the accuracy of simulation results when using dipoles designed in its graphical user interface and in CST Microwave Studio. In addition, we made an effort to define the fading channel model which involves the communication between one transmitter and one receiver, using the channel coefficients produced by MIMObit. The capacity considering SISO, SIMO, and MIMO systems was calculated taking into account a real communication system.

However, there is still work to be done including the communication scenarios at which the transmission channel and the channel fading distribution are not known at the transmitter. Obviously, that involves a degrade in capacity performance as the waterfilling power allocation can not be applied. In addition a different receiver architecture has to be considered in MIMO cases as long as with the optimal preprocessing and post-processing done by the transmitter and the receiver, respectively. We also mention that the multiuser scenario can be further extended including the cases at which there are multiple antennas available at both the users and the base station.

Chapter 6

Appendix

Proposition 1

$$
\begin{aligned}
h(X) &= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \log_2\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx \\
&= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[\log_2\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \log_2\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)\right] dx \\
&= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \log_2\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) dx - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \log_2\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx \\
&= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \log_2\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) dx - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{\ln\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)}{\ln 2} dx \\
&= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \ \log_2(2\pi\sigma^2)^{-\frac{1}{2}} + \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{(x-\mu)^2}{2\sigma^2 \ln 2} dx \\
&= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \ \log_2(2\pi\sigma^2)^{-\frac{1}{2}} + \frac{1}{2\sigma^2 \ln 2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)^2 \ dx \\
&= \frac{1}{2} \log_2(2\pi\sigma^2) \int_{-\infty}^{+\infty} f_X(x) \ dx \ + \frac{1}{2\sigma^2 \ln 2} \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x) \ dx \\
&= \frac{1}{2} \log_2(2\pi\sigma^2) \cdot 1 + \frac{1}{2\sigma^2 \ln 2} E\left\{(X-\mu)^2\right\} \\
&= \frac{1}{2} \log_2(2\pi\sigma^2) + \frac{\sigma^2}{2\sigma^2 \ln 2} \\
&= \frac{1}{2} \log_2(2\pi\sigma^2) + \frac{1}{2} \log_2(e) \\
&= \frac{1}{2} \log_2(2\pi e\sigma^2)
\end{aligned}
$$

Proposition 2

$$
\begin{aligned}
h(X_1,...,X_n) &= -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \log_2\left(\frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}}\right) dx \\
&= -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \Big[\log_2 \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} + \log_2 e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}\Big]\, dx \\
&= -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \log_2 \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}}\, dx \\
&\quad -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \log_2 e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}\, dx \\
&= -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \log_2 \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}}\, dx \\
&\quad -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \frac{\ln e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\ln 2}\, dx \\
&= -\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}}\, dx \log_2 \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \\
&\quad +\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}}{\sqrt{(2\pi)^n \det(\boldsymbol{K_x})}} \frac{(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}{2\ln 2}\, dx \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] \int_{-\infty}^{+\infty} f_X(x)\, dx \;+ \frac{1}{2\ln 2}\int_{-\infty}^{+\infty} f_X(x)(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\, dx \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})]\cdot 1 + \frac{1}{2\ln 2} E\left\{(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right\} \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \frac{1}{2\ln 2} E\left\{tr\big[(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\big]\right\} \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \frac{1}{2\ln 2} E\left\{tr\big[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T \boldsymbol{K_x}^{-1}\big]\right\} \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \frac{1}{2\ln 2} tr\big[E\left\{(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T\right\} \boldsymbol{K_x}^{-1}\big] \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \frac{1}{2\ln 2} tr\big[\boldsymbol{K_x}\boldsymbol{K_x}^{-1}\big] \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \frac{1}{2\ln 2} tr\big[\boldsymbol{I}\big] \\
&= \frac{1}{2}\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \frac{1}{2\ln 2} n \\
&= \frac{1}{2}\big[\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + n\log_2 e\big] \\
&= \frac{1}{2}\big[\log_2[(2\pi)^n \det(\boldsymbol{K_x})] + \log_2 e^n\big] \\
&= \frac{1}{2}\log_2\big[(2\pi e)^n \det(\boldsymbol{K_x})\big]
\end{aligned}
$$

Proposition 3

$$
\begin{aligned}
&E[Z^2] = 0\\
&E[(X + jY)^2] = 0\\
&E[X^2 + jXY + jYX - Y^2] = 0\\
&E[X^2] + jE[XY] + jE[YX] - E[Y^2] = 0\\
&E[X^2] + jE[X]E[Y] + jE[Y]E[X] - E[Y^2] = 0\\
&E[X^2] = E[Y^2]
\end{aligned}
$$

Proposition 4

$$
\begin{aligned}
D_{KL}(f\|g) &= \int_{-\infty}^{+\infty} f_X \log_2 \frac{f_X}{g_X}\ dx \geq 0\\
&= \int_{-\infty}^{+\infty} f_X \log_2 f_X\ dx - \int_{-\infty}^{+\infty} f_X \log_2 g_X\ dx \geq 0\\
&= -h(f_X) - \int_{-\infty}^{+\infty} f_X \log_2 g_X\ dx \geq 0\\
&= -h(f_X) - \int_{-\infty}^{+\infty} f_X \log_2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx \geq 0\\
&= -h(f_X) - \int_{-\infty}^{+\infty} f_X \log_2 \frac{1}{\sqrt{2\pi\sigma^2}}\ dx - \int_{-\infty}^{+\infty} f_X \log_2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx \geq 0\\
&= -h(f_X) + \frac{1}{2}\log_2(2\pi\sigma^2) \int_{-\infty}^{+\infty} f_X\ dx + \frac{\log_2(e)}{2\sigma^2} \int_{-\infty}^{+\infty} f_X (x-\mu)^2\ dx \geq 0\\
&= -h(f_X) + \frac{1}{2}\log_2(2\pi\sigma^2) + \frac{\log_2(e)}{2\sigma^2}\sigma^2 \geq 0\\
&= -h(f_X) + \frac{1}{2}\log_2(2\pi\sigma^2) + \frac{1}{2}\log_2(e) \geq 0\\
&= -h(f_X) + \frac{1}{2}\log_2(2\pi e\sigma^2) \geq 0\\
&= -h(f_X) + h(g_X) \geq 0
\end{aligned}
$$

Proposition 5

$$\begin{aligned}
I(X;Y) &= \sum_x \sum_y p_{X,Y} \log_2 \frac{p_{X,Y}}{p_X p_Y} \\
&= \sum_x \sum_y p_{X,Y} \log_2 \frac{p_{X,Y}}{p_X} - \sum_x \sum_y p_{X,Y} \log_2 p_Y \\
&= \sum_x \sum_y p_{Y|X}\ p_X \log_2 \frac{p_{Y|X}\ p_X}{p_X} - \sum_x \sum_y p_{X,Y} \log_2 p_Y \\
&= \sum_x \sum_y p_{Y|X}\ p_X \log_2 p_{Y|X} - \sum_x \sum_y p_{X,Y} \log_2 p_Y \\
&= \sum_x p_X \sum_y p_{Y|X} \log_2 p_{Y|X} - \sum_y \log_2 p_Y \sum_x p_{X,Y} \\
&= -\sum_x p_X H(Y|X=x) - \sum_y p_Y \log_2 p_Y \\
&= -H(Y|X) + H(Y)
\end{aligned}$$

Proposition 6

Expanding the formula of the mutual information

$$\begin{aligned}
I(x;y) &= h(y) - h(y|x) \\
&= h(y) - h(x+w|x) \\
&= h(y) - h(w|x) \\
&= h(y) - h(w).
\end{aligned}$$

The output y has a constraint of

$$\begin{aligned}
E\left\{y^2\right\} &\leq E\left\{(x+w)^2\right\} \\
&\leq E\left\{x^2 + 2xw + w^2\right\} \\
&\leq E\left\{x^2\right\} + 2E\left\{x\right\} E\left\{w\right\} + E\left\{w^2\right\} \\
&\leq E\left\{x^2\right\} + E\left\{w^2\right\} \\
&\leq P + N.
\end{aligned}$$

Hence, $h(y)$ is maximized when y is chosen to be $\mathcal{N}(0,\ P+N)$. That is achieved by choosing the input x to be $\mathcal{N}(0,\ P)$. Finally we get

$$\begin{aligned}
\max_{E\left\{x^2\right\} \leq P} I(x;y) &= h(y) - h(w) \\
&= \frac{1}{2} \log_2(2\pi e(P+N)) - \frac{1}{2} \log_2(2\pi e N) \\
&= \frac{1}{2} \log_2 \left(1 + \frac{P}{N}\right).
\end{aligned}$$

Proposition 7

Expanding the formula of the mutual information

$$\begin{aligned} I(x;y) &= h(y) - h(y|x) \\ &= h(y) - h(x+w|x) \\ &= h(y) - h(w|x) \\ &= h(y) - h(w). \end{aligned}$$

The output y has a constraint of

$$\begin{aligned} E\left\{|y|^2\right\} &\leq E\left\{(x+w)(x+w)^*\right\} \\ &\leq E\left\{(x+w)(x^*+w^*)\right\} \\ &\leq E\left\{xx^* + xw^* + wx^* + ww^*\right\} \\ &\leq E\left\{xx^*\right\} + E\left\{x\right\}E\left\{w\right\}^* + E\left\{w\right\}E\left\{x\right\}^* + E\left\{ww^*\right\} \\ &\leq E\left\{|x|^2\right\} + E\left\{|w|^2\right\} \\ &\leq P + N. \end{aligned}$$

Hence, $h(y)$ is maximized when y is chosen to be $\mathcal{CN}(0,\, P+N)$. Finally we get

$$\begin{aligned} \max_{E\left\{|x|^2\right\}\leq P} I(x;y) &= h(y) - h(w) \\ &= \log_2(\pi e(P+N)) - \log_2(\pi e N) \\ &= \log_2\left(1 + \frac{P}{N}\right). \end{aligned}$$

Proposition 8

Expanding the formula of the mutual information

$$\begin{aligned} I(x;y) &= h(y) - h(y|x) \\ &= h(y) - h(x+w|x) \\ &= h(y) - h(w|x) \\ &= h(y) - h(w). \end{aligned}$$

The output y has a constraint of

$$\begin{aligned} E\left\{|y|^2\right\} &\leq E\left\{(hx+w)(hx+w)^*\right\} \\ &\leq E\left\{(hx+w)(x^*h^*+w^*)\right\} \\ &\leq E\left\{hxx^*h^* + hxw^* + wx^*h^* + ww^*\right\} \\ &\leq hE\left\{xx^*\right\}h^* + hE\left\{x\right\}E\left\{w\right\}^* + E\left\{w\right\}E\left\{x\right\}^*h^* + E\left\{ww^*\right\} \\ &\leq E\left\{|x|^2\right\}|h|^2 + E\left\{|w|^2\right\} \\ &\leq P|h|^2 + N. \end{aligned}$$

Hence, $h(y)$ is maximized when y is chosen to be $\mathcal{CN}(0,\, P+N)$. Finally we get

$$\begin{aligned}
\max_{E\left\{|x|^2\right\}\le P} I(x;y) &= h(y) - h(w) \\
&= \log_2(\pi e(P|h|^2+N)) - \log_2(\pi e N) \\
&= \log_2\left(1+\frac{P|h|^2}{N}\right).
\end{aligned}$$

Proposition 9

$$\begin{aligned}
SNR_{user1} &= \frac{E\left\{|h_1x_1|^2\right\}}{E\left\{|h_2x_2+w|^2\right\}} \\
&= \frac{E\left\{(h_1x_1)(x_1^*h_1^*)\right\}}{E\left\{(h_2x_2+w)(x_2^*h_2^*+w^*)\right\}} \\
&= \frac{E\left\{h_1x_1x_1^*h_1^*\right\}}{E\left\{h_2x_2x_2^*h_2^*+h_2x_2w^*+wx_2^*h_2^*+ww^*\right\}} \\
&= \frac{h_1E\left\{x_1x_1^*\right\}h_1^*}{h_2E\left\{x_2x_2^*\right\}h_2^*+h_2E\left\{x_2\right\}E\left\{w\right\}^*+E\left\{w\right\}E\left\{x_2\right\}^*h_2^*+E\left\{ww\right\}^*} \\
&= \frac{E\left\{|x_1|^2\right\}|h_1|^2}{E\left\{|x_2|^2\right\}|h_2|^2+E\left\{|w|^2\right\}} \\
&= \frac{P_1|h_1|^2}{P_2|h_2|^2+N}
\end{aligned}$$

Proposition 10

$$\begin{aligned}
&R_1 = R_2 \\
&\log_2\left(1+\frac{P_1|h_1|^2}{P_2|h_2|^2+N}\right) = \log_2\left(1+\frac{P_2|h_2|^2}{N}\right) \\
&\frac{P_1|h_1|^2}{P_2|h_2|^2+N} = \frac{P_2|h_2|^2}{N} \\
&P_1|h_1|^2N = P_2^2|h_2|^4 + P_2|h_2|^2N \\
&P_2^2|h_2|^4 + P_2|h_2|^2N - (P-P_2)|h_1|^2N = 0 \\
&P_2^2|h_2|^4 + P_2(|h_2|^2N+|h_1|^2N) - P|h_1|^2N = 0
\end{aligned}$$

The aforementioned formula refers to a quadratic equation, where the only unknown variable is P_2. The discriminant is the following

$$\begin{aligned}
&\Delta = (|h_2|^2N+|h_1|^2N)^2 - 4[|h_2|^4(-P|h_1|^2N)] \\
&\Delta = |h_2|^4N^2 + 2|h_2|^2N|h_1|^2N + |h_1|^4N^2 + 4P|h_2|^4|h_1|^2N.
\end{aligned}$$

We get two solutions where only one is acceptable since the other is negative. The acceptable one is the following

$$P_2 = \frac{-(|h_2|^2 N + |h_1|^2 N) + \sqrt{|h_2|^4 N^2 + 2|h_2|^2 N |h_1|^2 N + |h_1|^4 N^2 + 4P|h_2|^4 |h_1|^2 N}}{2|h_2|^4},$$

$$P_2 = \frac{-(|h_2|^2 + |h_1|^2) N + \sqrt{(|h_2|^4 + 2|h_2|^2 |h_1|^2 + |h_1|^4) N^2 + 4P|h_2|^4 |h_1|^2 N}}{2|h_2|^4}.$$

Hence, the power that user 1 gets is considered to be

$$P_1 = P - P_2.$$

Bibliography

[1] D. Tse and P. Viswanath. *Fundamentals of Wireless Communication*. Cambridge University Press, 2005.

[2] Andrea Goldsmith. *Wireless Communications*. 2004.

[3] G.J. Foschini and M.J. Gans. "On Limits of Wireless Communications in a Fading Environment when Using Multiple Antennas". In: *Wireless Personal Communications* 6 (1998), pp. 311-335

[4] Georgios Tsitsikas. *Wireless Channel Modeling using MIMObit*. 2017.

[5] LLC NEBENS. *MIMObit 2 User's Manual*. 2018.

[6] Peter Joseph Bevelacqua. *Antennas and Antenna Theory*. 2009. URL: `http://www.antenna-theory.com`