Lyapunov-Based Two-Dimensional Cruise Control of Autonomous Vehicles on Lane-Free Roads

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Abstract— **In this paper, we design decentralized control strategies for the two- dimensional movement of autonomous vehicles on lane-free roads. The bicycle kinematic model is used to model the dynamics of the vehicles, and each vehicle determines its control input based only on its own speed and on the distance from other (adjacent) vehicles and the boundary of the road. Potential functions and Barbălat's lemma are employed to prove the following properties, which are ensured by the proposed controller: (i) the vehicles do not collide with each other or with the boundary of the road; (ii) the speeds of all vehicles are always positive, i.e., no vehicle moves backwards at any time; (iii) the speed of all vehicles remain below a given speed limit; (iv) all vehicle speeds converge to a given longitudinal speed set-point; and (v) the accelerations, lateral speeds, and orientations of all vehicles tend to zero. The efficiency of the proposed 2-D cruise controllers is illustrated by means of numerical examples.**

I. Introduction

Vehicle automation has made tremendous advances in the last decades, and the path to full automation of vehicles in a foreseeable future seems more than likely. An initial stage of vehicle automation is the standard cruise control system which maintains the speed of the vehicle at a desired value to assist the driver. These systems have meanwhile evolved to Adaptive Cruise Control (ACC) systems, which automatically adjust the speed to maintain certain distance from a front vehicle or to maintain a desired speed. Recent advances of communication technologies have also been used in vehicle automation to develop Cooperative ACC systems (CACC) so that vehicles can communicate wirelessly which may increase their safety, reduce congestion, and improve traffic flow on highways ([1], [15], [24]) Both ACC and CACC systems have been extensively studied in the literature (see for instance [8], [11], [15], [21], [25], [28]).

The vast majority of research effort is focused on studying lane-based traffic models, where vehicles abide to a lane discipline, which increases traffic safety, as it simplifies the task of manual driving. Indeed, all control strategies for ACC and CACC systems are developed based on information from the vehicle directly in front or behind (see for instance [8], [11], [15], [21], [25] and references therein). Apart from the car-following task, another necessary driving task is lane-changing, which is a more complex and riskier maneuver, since the driver needs to look for an available gap on the target lane and estimate the speeds of many adjacent vehicles quasi-simultaneously. Modeling lane changes and two-dimensional movement on lane-based roads is a complicated problem, and various approaches have been considered, see for instance [5], [22], [30].

Recently, launched by [19], new principles and research directions were proposed for autonomous vehicles operating on lane-free roads ([3], [16], [19]) that may improve traffic flow and increase capacity of highways. The vehicles move on the two-dimensional surface of the road without obeying to a lane discipline as in conventional traffic. Since connected and automated vehicles use sensors and can communicate their presence and state to other vehicles, they are suitable and more efficient in a lane-free environment where they can use their capabilities to their full extent. For the lane-free concept, only a few models have been proposed that can describe vehicle movement on lane-free roads, driven by human drivers; see [1], [9], [17]. These approaches are not suitable to describe autonomous vehicles since they are based on linear systems theory and traditional longitudinal car-following models, which, however, do not guarantee: (i) collision avoidance with other vehicles or the boundary of the road, (ii) positivity of speeds, and (iii) speeds within road speed limits. In addition to the lane-free traffic, another concept that can increase the flow of vehicles on a road is the associated concept of 'nudging' (see [19]). Nudging implies a virtual force that vehicles apply to the vehicles in front of them, and it has been shown that nudging can increase the flow in a ring-road and can have a strong stabilizing effect; see [10] and references therein.

In this paper, we consider identical autonomous vehicles described by the bicycle kinematic model, since, it is able to capture the non-holonomic constraints of the actual vehicle (see [20], [21]). We design a family of nonlinear decentralized controllers for the safe operation of the vehicles on lane-free roads. The main features of the proposed approach are:

(i) The proposed nonlinear controllers are fully decentralized, and each vehicle only has access to the distance from the boundaries of the road and the distance from adjacent vehicles and does not require any information or estimates of relative speeds or relative orientation;
(ii) the vehicles do not collide with each other or with the boundary of the road;
(iii) the speeds of all vehicles are always positive and remain below a given speed limit;

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(iv) all vehicle speeds converge to a given longitudinal speed set-point; and

(v) the accelerations, lateral speeds, rotation rates, and orientations of all vehicles tend to zero.

(vi) all the above features are valid globally, i.e., for all physically relevant initial conditions.

To our knowledge, a cruise controller for a two-dimensional lane-free road that captures all these properties simultaneously and globally, is not available in the literature. To avoid collisions between vehicles and with the boundary of the road, we employ potential functions, which have been extensively used to address a variety of problems see [4], [6], [13], [14], [18], [23], [29]. Finally, we combine Lyapunov functions with barrier functions (see [2], [11], [26]) to restrain the movement of the vehicles and exploit Barbălat's lemma [13] to globally address the objectives of speeds, acceleration, and orientation convergence as stated above. The main theoretical challenges stem from the fact that the nonlinear control system studied in the paper evolves on a specific open set, and, in addition, various objectives and constraints must be satisfied simultaneously and globally.

The structure of the paper is as follows. Section II is devoted to the presentation of the problem formulation and the objectives of the paper. Section III contains the main results. Section IV presents numerical examples to demonstrate the efficiency of the proposed decentralized cruise controllers. Finally, concluding remarks are given in Section V. Due to space constraints all proofs can be found in [12].

Notation. Throughout this paper, we adopt the following notation. $\mathbb{R}_+ := [0,+\infty)$ denotes the set of non-negative real numbers. By $|x|$ we denote both the Euclidean norm of a vector $x \in \mathbb{R}^n$ and the absolute value of a scalar $x \in \mathbb{R}$. By x' we denote the transpose of a vector $x \in \mathbb{R}^n$. By $|x|_\infty = \max\{|x_i|, i = 1,...,n\}$ we denote the infinity norm of a vector $x = (x_1, x_2, ..., x_n)' \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be an open set. By $C^0(A,\Omega)$, we denote the class of continuous functions on $A \subseteq \mathbb{R}^n$, which take values in $\Omega \subseteq \mathbb{R}^m$. By $C^k(A;\Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \mathbb{R}^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \mathbb{R}^m$. When $\Omega = \mathbb{R}$ the we write $C^0(A)$ or $C^k(A)$.

II. Problem Description

Consider n identical vehicles moving on a lane-free road of width $2a > 0$. The movement of the vehicles is described by the following set of ODEs:

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= \sigma^{-1} v_i \tan(\delta_i) \\ \dot{v}_i &= F_i \end{aligned} \tag{1}$$

for $i = 1,...,n$, where $\sigma > 0$ is the length of each vehicle (a constant). Here, (x_i, y_i) is the reference point of the i-th vehicle with $i \in \{1,...,n\}$ and is placed at the midpoint of the rear axle, with $x_i \in \mathbb{R}$ being the longitudinal position and $y_i \in (-a, a)$ being the lateral position of the vehicle; v_i is the speed of the i-th vehicle at the point (x_i, y_i), $\theta_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the angular orientation of the i-th vehicle, δ_i is the steering angle of the front wheels relative to the orientation θ_i of the i-th vehicle, and F_i is the acceleration of the i-th vehicle. Model (1) is known as the bicycle kinematic model (see Fig. 1) and has been widely used to represent vehicles due to its simplicity to capture vehicle motion in normal driving conditions, see ([14], [20], [21], [22]). To make the subsequent analysis less cumbersome, we define

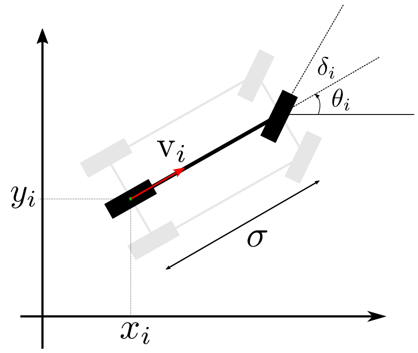

Fig. 1: Each vehicle is modeled by the bicycle kinematic model.

$$u_i = \sigma^{-1} v_i \tan(\delta_i), i = 1,...,n \tag{2}$$

Then, model (1) can be written in the form

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= u_i \\ \dot{v}_i &= F_i \end{aligned} \tag{3}$$

for $i = 1,...,n$, where u_i and F_i are the control inputs. Then, δ_i can be obtained directly from (2) as a function of u_i.

In what follows, we assume that there is no communication between the vehicles, and that the only available sensing among adjacent vehicles concerns the (elliptical) distance between vehicles, defined by

$$d_{i,j} := \sqrt{(x_i - x_j)^2 + p(y_i - y_j)^2}, \text{ for } i,j = 1,...,n \tag{4}$$

where $p > 0$ is a weighting factor. For $p = 1$ we obtain the standard Euclidean distance, while for larger values of $p > 1$, we have an "elliptical" metric which will allow to approximate more accurately the dimensions of a vehicle and to place more vehicles across the width of the road. The optimal selection of the constant $p \geq 1$ can be found in [12].

In what follows we use the notation

$$w = (x_1, ..., x_n, y_1, ..., y_n, \theta_1, ..., \theta_n, v_1, ..., v_n)' \in \mathbb{R}^{4n} \tag{5}$$

for the stack vector of longitudinal and lateral positions, orientations and speeds off all n vehicles. We assume that all vehicles operate on a lane-free road with speed limit $v_{\max} > 0$. Moreover, for any given constant $\varphi \in \left(0, \frac{\pi}{2}\right)$, we define the set

$$S := \mathbb{R}^n \times (-a,a)^n \times (-\varphi,\varphi)^n \times (0, v_{\max})^n. \tag{6}$$

The set S in (6) represents all possible states of the system of all n vehicles described by (3) and has the following interpretation. First, each vehicle should stay within the road, i.e., $(x_i, y_i) \in \mathbb{R} \times (-a,a)$ for $i = 1,...,n$. Moreover, with the given constant $\varphi \in \left(0, \frac{\pi}{2}\right)$, the vehicles should not be able to turn perpendicular to the road, as it should hold that $\theta_i \in (-\varphi, \varphi)$ for $i = 1, ..., n$. The constant φ can be understood as a safety constraint, which restricts the movement of a vehicle; for instance, for vehicles moving at high speed, φ should take values close to zero. Finally, the speeds of all vehicles should always be positive, i.e., no vehicle moves backwards at any time; and respect the road speed limits. One very important property, that is not captured by the set S, is that of collision avoidance between vehicles. This implies that the distance between the reference points of any pair of vehicles should always be greater than $L > 0$, which is a safety distance that prevents collisions, see [12].

Due to the various constraints explained above, we must consider system (3) on the open set $\Omega \subset \mathbb{R}^{4n}$ defined by:

$$\Omega := \left\{ w \in S \ : \ d_{i,j} > L, \ i, j = 1, ..., n \,, \ j \neq i \right\}. \quad (7)$$

The set Ω in (7) describes the state-space of the n vehicles operating on a lane-free road and acts as a basis for the problem formulation and for expressing the main objectives of the paper.

Problem Statement: For a group of n vehicles modeled by (3) and operating on a lane-free road of width $2a > 0$, design decentralized feedback laws for u_i and F_i such that the following objectives hold:

1) the vehicles do not collide with each other or with the boundary of the road, i.e., $d_{i,j}(t) > L$ for all $t \geq 0$, $i, j = 1, ..., n, \ j \neq i$, for a given constant $L > 0$, and $y_i(t) \in (-a, a)$ for all $t \geq 0$.
2) the speeds of all vehicles are always positive and remain below the given speed limit, i.e., $v_i(t) \in (0, v_{\max})$ for $t \geq 0$, and converge to a given longitudinal speed set-point $v^* \in (0, v_{\max})$, i.e., $\lim_{t\to+\infty} (v_i(t)) = v^*$, $i = 1, ..., n$.
3) the orientation of each vehicle is always bounded by the given value $\varphi \in \left(0, \frac{\pi}{2}\right)$, i.e., $\theta_i(t) \in (-\varphi, \varphi)$ for $t \geq 0$, and converges to zero, i.e., $\lim_{t\to+\infty} (\theta_i(t)) = 0$, $i = 1, ..., n$.
4) the accelerations, angular speeds, and lateral speeds of all vehicles tend to zero, i.e., $\lim_{t\to+\infty} (F_i(t)) = 0$, $\lim_{t\to+\infty} (u_i(t)) = 0$, and $\lim_{t\to+\infty} (\dot{y}_i(t)) = 0$, $i = 1, ..., n$.

It should be noted that, in mathematical terms, we require the closed-loop system to be well-posed on the state space $\Omega \subset \mathbb{R}^{4n}$ defined by (7), i.e., for every initial condition $w(0) \in \Omega$, the closed-loop system (3), under the feedback laws u_i and F_i for $i = 1, ..., n$, has a unique solution $w(t) \in \Omega$ defined for all $t \geq 0$. Moreover, we require that, for every initial condition $w(0) \in \Omega$, the solution $w(t) \in \Omega$ of the closed-loop system (3), under the effect of all feedback laws for u_i and F_i for $i = 1, ..., n$, satisfies $\lim_{t\to+\infty} (v_i(t)) = v^*$, $\lim_{t\to+\infty} (\theta_i(t)) = 0$, $\lim_{t\to+\infty} (F_i(t)) = 0$, $\lim_{t\to+\infty} (u_i(t)) = 0$ for all $i = 1, ..., n$. It should also be noticed that the lateral speed of each vehicle also tends to zero, i.e., $\lim_{t\to+\infty} (\dot{y}_i(t)) = 0$ for $i = 1, ..., n$.

III. Main Results

In this section, we design a novel decentralized control strategy in order to achieve the various objectives discussed in Section II. First and foremost, we want to design the control inputs u_i and F_i in such a way that vehicles operating on a lane-free road do not collide with each other or with the boundary of the road. A typical approach for collision avoidance between vehicles is the use of repulsive potential functions (see for instance [4], [6], [7], [14], [23], [27], [29]). Repulsive potential functions are continuously differentiable functions, which repel vehicles based on their distance, with the force of repulsion being stronger as the distance between two vehicles becomes smaller, while there is little or no repulsion when the vehicles are distant. To that end, let $V : (L, +\infty) \to \mathbb{R}_+$ be a C^2 function that satisfies:

$$\lim_{d\to L^+} (V(d)) = +\infty \quad (8)$$

$$V(d) = 0, \ \text{for all } d \geq \lambda \quad (9)$$

where $\lambda > L$ is a constant. Let also $U : (-a, a) \to \mathbb{R}_+$ be a C^2 function that satisfies:

$$\lim_{y\to(-a)^+} (U(y)) = +\infty, \lim_{y\to a^-} (U(y)) = +\infty \quad (10)$$

$$U(0) = 0. \quad (11)$$

The potential function $U(y)$ in (10), (11) is designed so as to exert a repulsive force when the vehicles approach the boundary of the road.

To design feedback control laws that address objectives (1)-(4) in the Problem Statement, we apply a control Lyapunov function methodology, where the feedback laws are selected appropriately to render the derivative of a Lyapunov function negative semi-definite. An appropriate function for this task is the following. Define, for all $w \in \Omega$,

$$\begin{aligned} H(w) :=& \frac{1}{2} \sum_{i=1}^{n} (v_i \cos(\theta_i) - v^*)^2 + \frac{1}{2} \sum_{i=1}^{n} v_i^2 \sin^2(\theta_i) \\ &+ \sum_{i=1}^{n} U(y_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j\neq i} V(d_{i,j}) \\ &+ A \sum_{i=1}^{n} \left(\frac{1}{\cos(\theta_i) - \cos(\varphi)} - \frac{1}{1 - \cos(\varphi)} \right) \end{aligned} \quad (12)$$

where $A > 0$ is a parameter of the controller and the Lyapunov function, $v^* \in (0, v_{\max})$ is a given longitudinal speed set-point, and $\varphi \in \left(0, \frac{\pi}{2}\right)$ is any constant that satisfies the inequality

$$\cos(\varphi) \geq \frac{v^*}{v_{\max}}. \quad (13)$$

The function H in (12), is inspired by the total energy of the system of n vehicles and will allow us to exploit certain properties of the state space Ω in (7). The first two terms ($\frac{1}{2} \sum_{i=1}^{n} (v_i \cos(\theta_i) - v^*)^2 + \frac{1}{2} \sum_{i=1}^{n} v_i^2 \sin^2(\theta_i)$) represent the kinetic energy of the system of n vehicles relative to an observer moving along the $x-$direction with speed equal to v^*. The sum of the third and fourth term ($\sum_{i=1}^{n} U(y_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j\neq i} V(d_{i,j})$), which are based on the potential functions (10) and (11), is the potential energy of the system. Finally, the last term of (12) ($A \sum_{i=1}^{n} \left(\frac{1}{\cos(\theta_i) - \cos(\varphi)} - \frac{1}{1-\cos(\varphi)} \right)$) is a penalty term that blows up when $\theta_i \to \pm\varphi$. Inequality (13) is a technical assumption that restricts the movement of the vehicle when the desired speed is close to the road speed limit. Notice also that H is not only a Lyapunov function, but possesses also certain characteristics of barrier functions, (see for instance [2], [11], [26]). Indeed, $H(w)$ grows unbounded on certain parts of the boundary of Ω in (7), i.e., when $y_i \to \pm a$ or $\theta_i \to \pm\varphi$ or $d_{i,j} \to L$ for some $i, j = 1, ..., n$ with $i \neq j$ (recall (8) and (10)).

Proposition 1: Let constants $A > 0$, $v_{\max} > 0$, $v^* \in (0, v_{\max})$, $\lambda > L > 0$, $\varphi \in \left(0, \frac{\pi}{2}\right)$ that satisfies (13), and define the function $H : \Omega \to \mathbb{R}_+$ by means of (12), where Ω is given by (7). Then, there exist a non-decreasing function $\kappa : \mathbb{R}_+ \to [0, a)$, a non-increasing function $\rho : \mathbb{R}_+ \to (L, \lambda]$ and a non-decreasing function $\omega : \mathbb{R}_+ \to [0, \varphi)$ such that the following implication holds:

$$w \in \Omega \Rightarrow |\theta_i| \leq \omega\left(H(w)\right), |y_i| \leq \kappa\left(H(w)\right), \quad d_{i,j} \geq \rho\left(H(w)\right), \text{ for } i, j = 1, ..., n\,,\ j \neq i. \tag{14}$$

Implication (14) suggests that for any $w \in \Omega$, the orientations θ_i and the lateral positions y_i of all vehicles $i = 1, ..., n$, as well as the distances $d_{i,j}$, $i, j = 1, ..., n\,,\ j \neq i$, are bounded by the energy of the system, see (12).

The feedback laws for each vehicle $i = 1, ..., n$ can be designed using (12), in terms of their own speed and orientation and the gradient of the potential functions V_i and U_i that satisfy (8), (9) and (10), (11), respectively:

$$u_i = -\left(v^* + \frac{A}{v_i\left(\cos(\theta_i) - \cos(\varphi)\right)^2}\right)^{-1} \times\left(\mu_1 v_i \sin(\theta_i) + U'(y_i) + p\sum_{j\neq i} V'(d_{i,j})\frac{(y_i - y_j)}{d_{i,j}} + \sin(\theta_i) F_i\right) \tag{15}$$

$$F_i = -\frac{k_i(w)}{\cos(\theta_i)}\left(v_i\cos(\theta_i) - v^*\right) - \frac{1}{\cos(\theta_i)}\sum_{j\neq i} V'(d_{i,j})\frac{(x_i - x_j)}{d_{i,j}} \tag{16}$$

$$k_i(w) = \mu_2 + \frac{1}{v^*}\sum_{j\neq i} V'(d_{i,j})\frac{(x_i - x_j)}{d_{i,j}} + \frac{v_{\max}\cos(\theta_i)}{v^*\left(v_{\max}\cos(\theta_i) - v^*\right)} f\left(-\sum_{j\neq i} V'(d_{i,j})\frac{(x_i - x_j)}{d_{i,j}}\right) \tag{17}$$

where $\mu_1, \mu_2 > 0$ are constants (controller gains) and $f \in C^1(\mathbb{R})$ is any function that satisfies

$$\max(x, 0) \leq f(x) \text{ for all } x \in \mathbb{R}. \tag{18}$$

The term $k_i(w)$ in the acceleration $F_i(t)$, given by (16), is a state-dependent controller gain which guarantees that the speed of each vehicle will remain positive and less than the speed limit. The second term that appears in (16), is the summation of repelling forces ($V'(d)$) from vehicles that are in close proximity to vehicle i. If V in (8), (9) is decreasing, then, the second term of (16) is positive if vehicle j is behind vehicle i, i.e., $(x_i - x_j) > 0$. Indeed, in this case, we have that $-V'(d_{i,j})\frac{(x_i - x_j)}{d_{i,j}} > 0$, which represents the effect of nudging, since vehicles that are close and behind vehicle i will exert a "pushing" force towards it that will increase its acceleration. It should be noticed that the control laws above are designed in such a way that the nudging force will not jeopardize traffic safety in terms of collisions, speeds exceeding desired bounds or vehicles departing from the road.

Remark 1: (i) Property (9) guarantees that the feedback laws (15), (16), (17) depend only on information from adjacent vehicles, namely from vehicles that are located at a distance less than $\lambda > 0$. Notice also that the control inputs (15), (16), (17) only require the distance from neighboring vehicles and not additional information, such as relative speeds $(v_i - v_j)$ or relative orientations $(\theta_i - \theta_j)$.

(ii) Any function $f \in C^1(\mathbb{R})$ that satisfies (18) can be used in (17). For example, the function $f(x) = \frac{\varepsilon}{2} + \frac{1}{2\varepsilon}x^2$ for every $\varepsilon > 0$ satisfies (18), since $\max(x, 0) \leq |x| \leq \frac{\varepsilon}{2} + \frac{1}{2\varepsilon}x^2$ for all $x \in \mathbb{R}$. Another function that satisfies (18) is the function

$$f(x) = \frac{1}{2\varepsilon}\begin{cases} 0 & if\ x \leq -\varepsilon \\ (x+\varepsilon)^2 & if\ -\varepsilon < x < 0 \\ \varepsilon^2 + 2\varepsilon x & if\ x \geq 0 \end{cases} \tag{19}$$

for every $\varepsilon > 0$. This generic design for the function f will allow to regulate the longitudinal acceleration as desired.

Let $p \geq 1$, and consider two concentric ellipses with semi-major axes L and λ, with $L < \lambda$, and semi-minor axes $\frac{L}{\sqrt{p}}$ and $\frac{\lambda}{\sqrt{p}}$, respectively. Let $m \geq 2$ be the maximum number of points that can be placed within the area bounded by the two concentric ellipses, so that each point has distance (in the metric given by (4)) at least L from every other point. The following proposition presents certain properties of the control laws (15), (16), (17).

Proposition 2: Let constants $\lambda > L > 0$, $a > 0$, $p \geq 1$, and let $V : (L, +\infty) \to \mathbb{R}_+$, $U : (-a, a) \to \mathbb{R}_+$ be C^2 functions that satisfy (8), (9) and (10), (11), respectively, and define

$$b_1(s) := \max\left\{\left|V'(d)\right| \,:\, s \leq d \leq \lambda\right\} \text{ for } s \in (L, \lambda] \tag{20}$$

$$b_2(s) := \max\left\{\left|U'(y)\right| \,:\, |y| \leq s\right\} \text{ for } s \in [0, a)\,. \tag{21}$$

Define the set Ω by means of (7). Then, for any $w \in \Omega$, there exist a non-decreasing function $\kappa : \mathbb{R}_+ \to [0, a)$ and a non-increasing function $\rho : \mathbb{R}_+ \to (L, \lambda]$ such that the functions u_i, F_i and k_i in (15), (16), and (17), respectively, satisfy the following inequalities

$$k_i(w)v^* \geq \sum_{j\neq i} V'(d_{i,j})\frac{(x_i - x_j)}{d_{i,j}} \geq -k_i(w)\left(v_{\max}\cos(\theta_i) - v^*\right), \quad i = 1, ..., n; \tag{22}$$

$$\mu_2 \leq k_i(w) \leq R(H(w)), i = 1, ..., n; \tag{23}$$

$$k_i(w)v_{\max} \geq k_i(w)\left(v_{\max} - v_i\right) \geq F_i \geq -k_i(w)v_i \geq -k_i(w)v_{\max}; \tag{24}$$

$$|u_i| \leq \frac{1}{v^*}\left(\left(\mu_1 + k_i(w)\right)v_{\max} + b_2\left(\kappa\left(H(w)\right)\right)\right) + \frac{m}{v^*}\sqrt{p}\, b_1\left(\rho\left(H(w)\right)\right) \tag{25}$$

where $R : \mathbb{R}_+ \to \mathbb{R}_+$ is the increasing function defined by

$$R(s) := \mu_2 + \frac{m}{v^*}b_1\left(\rho\left(s\right)\right) + v_{\max}\frac{\left(A + s\cos(\varphi)\left(1 - \cos(\varphi)\right)\right)\max\left\{f(z) : |z| \leq m b_1\left(\rho\left(s\right)\right)\right\}}{Av^*\left(v_{\max} - v^*\right) + v^*\left(v_{\max}\cos(\varphi) - v^*\right)\left(1 - \cos(\varphi)\right)s}. \tag{26}$$

Inequality (23) suggests that the magnitude of $k_i(w)$ depends on the "energy" of the system defined by the Lyapunov-like function H and the maximum number of neighboring vehicles $m \geq 2$. Moreover, $k_i(w)$ plays an important role, since it provides certain bounds on the acceleration F_i in (24) and the maximum "nudging" effect that each vehicle $i = 1, ..., n$ experiences from neighboring vehicles as described in (22). We are now in a position to state the main result.

Theorem 1: Suppose that there exist constants $a > 0, \lambda > L > 0$, $p \geq 1$ and C^2 functions $V : (L, +\infty) \to \mathbb{R}_+$, $U : (-a, a) \to \mathbb{R}_+$ that satisfy (8), (9), and (10), (11), respectively. In addition, for given constants $v_{\max} > 0$, $v^* \in (0, v_{\max})$, and $\varphi \in \left(0, \frac{\pi}{2}\right)$ that satisfies (13), define the function $H : \Omega \to \mathbb{R}_+$ by means of (12) where Ω is given by (7). Then, for every $w_0 \in \Omega$ there exists a unique solution $w(t) \in \Omega$ of the initial-value problem (3), (15), (16), (17) with initial condition $w(0) = w_0$. The solution $w(t) \in \Omega$ is defined for all $t \geq 0$ and satisfies for $i = 1, ..., n$

$$\lim_{t\to+\infty} (v_i(t)) = v^*, \lim_{t\to+\infty} (\theta_i(t)) = 0 \tag{27}$$

$$\lim_{t\to+\infty} (u_i(t)) = 0, \lim_{t\to+\infty} (F_i(t)) = 0. \tag{28}$$

Moreover, there exist a non-decreasing function $\kappa : \mathbb{R}_+ \to [0, a)$ and a non-increasing function $\rho : \mathbb{R}_+ \to (L, \lambda]$ such that

$$|F_i(t)| \leq R(H(w_0))v_{\max} \text{ for all } t \geq 0, \tag{29}$$

$$\begin{aligned} |u_i(t)| \leq & \frac{1}{v^*} \left((\mu_1 + R(H(w_0))) v_{\max} + b_2 (\kappa (H(w_0))) \right) \\ & + \frac{m\sqrt{p}}{v^*} b_1 (\rho (H(w_0))) \text{ for all } t \geq 0. \end{aligned} \tag{30}$$

where b_1, b_2, R are defined by (20), (21) and (26), respectively.

Remark 2: (i) It is important to notice that due to technical constraints, an inequality of the form $|F_i(t)| \leq K$ must be satisfied for all $t \geq 0$, where $K > 0$ is a constant that depends on the technical characteristics of the vehicles and the road. Inequality (29) allows us to determine the set of initial conditions $w_0 \in \Omega$ for which the inequality $|F_i(t)| \leq K$ holds: it includes the set of all $w_0 \in \Omega$ with $R(H(w_0))v_{\max} \leq K$.

(ii) Although we cannot predict the "ultimate" arrangement of the vehicles on the road (and we cannot even show that a final configuration of the vehicles on the road is attained; see remark below), the limits (27), (28) and definitions (16) allow us to predict that $\lim_{t\to+\infty} \left(\sum_{j\neq i} V'(d_{i,j}(t)) \frac{(x_i(t)-x_j(t))}{d_{i,j}(t)} \right) = \lim_{t\to+\infty} \left(U'(y_i(t)) + p \sum_{j\neq i} V'(d_{i,j}(t)) \frac{(y_i(t)-y_j(t))}{d_{i,j}(t)} \right) = 0$ for $i = 1, ..., n$. Consequently, the "ultimate" arrangement of the vehicles in the road (if such a thing exists) must satisfy the equations $\sum_{j\neq i} V'(d_{i,j}) \frac{(x_i-x_j)}{d_{i,j}} = U'(y_i) + p \sum_{j\neq i} V'(d_{i,j}) \frac{(y_i-y_j)}{d_{i,j}} = 0$ for $i = 1, ..., n$ as well as the constraints $|y_i| < a$, $d_{i,j} > L$ for $i, j = 1, ..., n$, $j \neq i$. Despite the fact that the constrained system of $2n$ equations has infinite solutions, not every arrangement of vehicles satisfies the aforementioned constrained system.

(iii) The proof of Theorem 1 relies on Barbălat's lemma [13] and does not use LaSalle's invariance principle. The reason that LaSalle's invariance principle cannot be used for the proof of Theorem 1 is the fact that the state components $x_i(t)$, $i = 1, ..., n$, do not take values in a bounded set. Moreover, we cannot show that the relative positions of the vehicles, i.e., the quantities $x_i(t) - x_j(t)$ for $i, j = 1, ..., n$, $j \neq i$, take values in a bounded set. Thus, we cannot show that the limits $\lim_{t\to+\infty} (y_i(t))$, $\lim_{t\to+\infty} \left(x_i(t) - x_j(t)\right)$, $\lim_{t\to+\infty} \left(d_{i,j}(t)\right)$ for $i, j = 1, ..., n$, $j \neq i$, exist. Consequently, we cannot ensure that a final configuration of the vehicles on the road will be attained. However, the proof of Theorem 1 shows that $\lim_{t\to+\infty} (\dot{y}_i(t)) = 0$, $\lim_{t\to+\infty} \left(\dot{x}_i(t) - \dot{x}_j(t)\right) = 0$, $\lim_{t\to+\infty} \left(\dot{d}_{i,j}(t)\right) = 0$ for $i, j = 1, ..., n$, $j \neq i$ (a consequence of (3), (27) and inequality (5.29) in the proof of Theorem 1, see [12]). Therefore, it is expected that the vehicles on the road will approach a final configuration.

IV. Illustrative Examples

In the simulation results below, we demonstrate the application and effectiveness of the proposed nonlinear decentralized cruise controllers for autonomous vehicles driving on lane-free roads. Specifically, we consider a group of $n = 10$ vehicles on a lane-free road of width $2a > 0$, modeled as in (2) with the feedback laws (15), (16), (17), and $f(x)$ given by means of (19). The vehicle-repulsive potential function V and the boundary-repulsive potential function U are given by

$$V(d) = \begin{cases} q\frac{(\lambda-d)^3}{d-L} & , L < d \leq \lambda \\ 0 & , d > \lambda \end{cases}, \tag{31}$$

$$U(y) = \begin{cases} \left(\frac{1}{a^2-y^2} - \frac{c}{a^2}\right)^4 & , \begin{cases} -a < y < -\frac{a\sqrt{c-1}}{\sqrt{c}} \text{ and} \\ \frac{a\sqrt{c-1}}{\sqrt{c}} < y < a \end{cases} \\ 0 & -\frac{a\sqrt{c-1}}{\sqrt{c}} \leq y \leq \frac{a\sqrt{c-1}}{\sqrt{c}} \end{cases}$$

where $c \geq 1, q > 0$ are design parameters. Notice that V and U above, satisfy (8), (9) and (10), (11), respectively. More specifically, for $c = 1$ we have that $U(y) = 0$ if $y = 0$, which will force the vehicles to form a single platoon in the middle of the road. For $c > 1$, we have that $U(y) = 0$ in a neighborhood around $y = 0$, and the vehicles' converged lateral positions in this case will be within the strip $-\frac{a\sqrt{c-1}}{\sqrt{c}} \leq y \leq \frac{a\sqrt{c-1}}{\sqrt{c}}$.

To verify numerically and illustrate the results of Theorem 1, we assume that all vehicles have length $\sigma = 5m$ and operate on a road with speed limit $v_{-\max} = 35m/s$ and width $2a = 14.4m$, which corresponds to a road with 4 conventional lanes of width $3.6m$. We set the longitudinal set-point $v^* = 30m/s$ and select $\varphi = 0.25$ in order to satisfy condition (13). Using [12], we select the optimal eccentricity and safety distance $p = 5.11$ and $L = 5.59m$, respectively. This choice allows us to effectively use the full width of the road and increases the lateral occupancy by 45%. We set $\varepsilon = 0.2$, $\mu_1 = 0.5$, $\mu_2 = 0.1$, $q = 3 * 10^{-3}$, $\lambda = 25m$, $A = 1$, and $c = 1.5$.

Fig. 2 displays the longitudinal speed $\dot{x}_i$ and acceleration F_i of each vehicle. The speeds of all vehicles are seen to

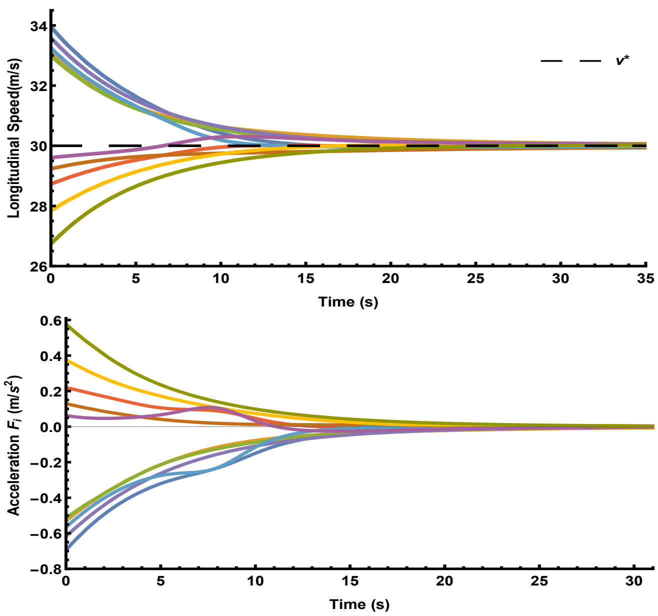

Fig. 2: The longitudinal speed of each vehicle where all speeds converge to the speed set-point v^* on the top; and the acceleration F_i of each vehicle on the bottom.

remain within the bounds $(0, v_{\max})$ and to converge to the longitudinal set-point v^*. It can be seen from Fig. 3, that the speed and acceleration of vehicle 9 is increased, as the distance from vehicle 7 decreases. This is exactly the effect of nudging, i.e., vehicle 7 exerts a pushing force on vehicle 9, which, as a result, increases its acceleration and speed. On the other hand, it can be seen in Fig. 3, that vehicle 7 decelerates (is repulsed) to avoid collision with vehicle 9.

The lateral speeds $\dot{y}_i$ and lateral accelerations $\ddot{y}_i$ of the vehicles are shown in Fig. 4; both converge to zero, indicating that eventually the vehicles move parallel to the road. Fig. 5 shows the rotation rates u_i and the orientations θ_i, all converging to zero as suggested by Theorem 1. Finally, Fig. 6 depicts the minimum inter-vehicle distance $d_{\min}(t) := \min\left\{d_{i,j}(t),\ i, j = 1, ..., n,\ i \neq j\right\}$ (blue line), showing that the vehicles do not collide with each other, since $d_{i,j}(t) > L$, $i, j = 1, ..., n,\ i \neq j$, at any time. Moreover, Fig. 6 also shows the minimum inter-vehicle distance using the same initial conditions with $\lambda = 40m$ (yellow line).

V. Concluding Remarks

The present work proposed decentralized control strategies for the two-dimensional movement of autonomous vehicles described by the bicycle kinematic model on lane-free roads. By leveraging appropriate tools, such as potential functions, Lyapunov functions, and barrier functions, we developed decentralized controllers that ensure that: the vehicles do not collide with each other or with the boundary of the road; the speeds of all vehicles are always positive and remain below a given speed limit; all vehicle speeds converge to a given longitudinal speed set-point; and, finally, the accelerations, lateral speeds, and orientations of all vehicles tend to zero. Future work will address the effects of nudging and appropriate notions of string-stability for vehicles operating on lane-free roads. We will also study the effect of different

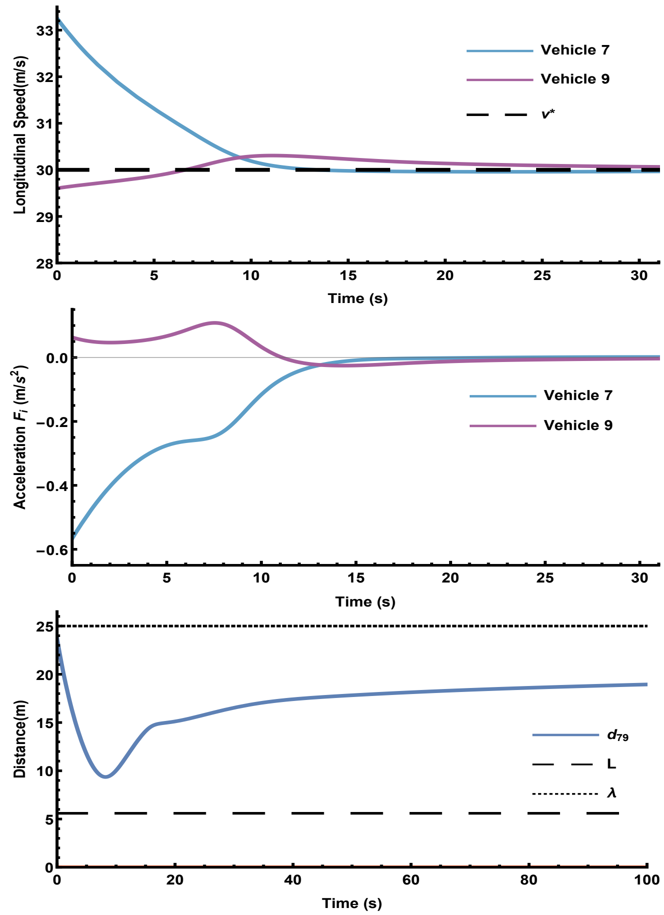

Fig. 3: The effect of nudging. Vehicle 9 accelerates, and vehicle 7 decelerates, as the (elliptical) distance between them decreases.

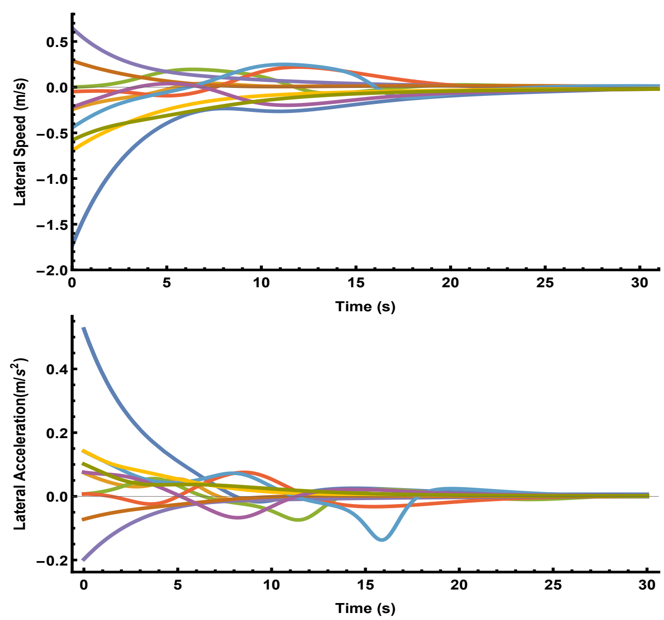

Fig. 4: The lateral speed (top) and lateral acceleration (bottom) of each vehicle.

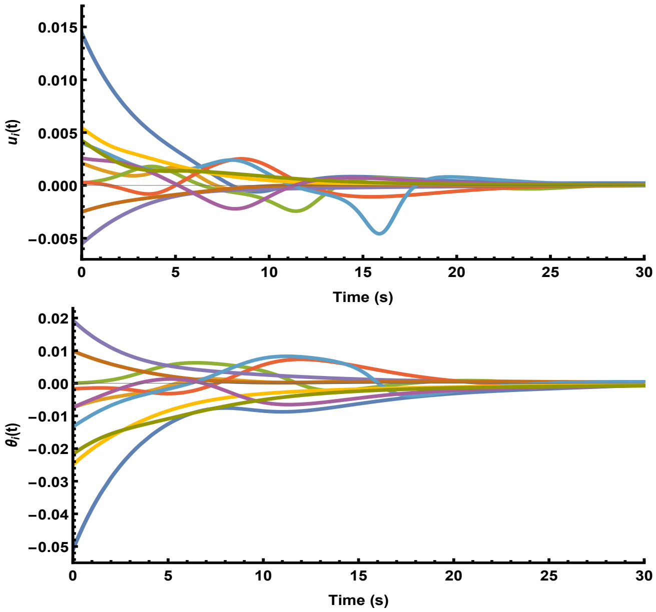

Fig. 5: The rotation rate u_i and orientation θ_i converge to zero as indicated by Theorem 1.

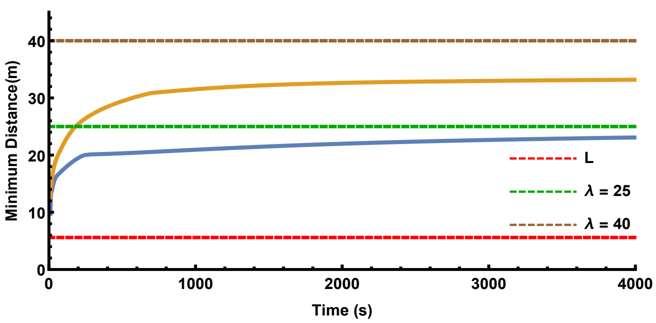

Fig. 6: The minimum inter-vehicle distance for $\lambda = 25m$ (blue) and for $\lambda = 40m$ (yellow), which verify that there are no collisions among vehicles.

potential functions and the possible use of non-monotone potential functions.

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