Technical University of Crete

Department of Electronic and Computer Engineering

Low-complexity Methods for Signal Detection in Wireless Communications Systems

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by
Dimitris S. Papailiopoulos
E-Mail: `papailiopoulos@telecom.tuc.gr`

Advisor
Assistant Prof. George N. Karystinos
E-Mail: `karystinos@telecom.tuc.gr`

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"I am interested in mathematics only as a creative art."

\- G. H. Hardy

Abstract

We prove that maximum-likelihood (ML) noncoherent sequence detection of orthogonal space-time block coded signals can be performed in polynomial time with respect to the sequence length for Rayleigh or Ricean distributed, correlated (in general) channel coefficients. We consider the case of time-varying Ricean fading and, using recent results on efficient maximization of rank-deficient quadratic forms over finite alphabets, we develop a novel algorithm that performs ML noncoherent sequence detection with polynomial complexity. The order of the polynomial complexity of the proposed receiver equals twice the rank of the covariance matrix of the vectorized channel matrix if the latter is Rayleigh distributed. Therefore, the lower the Rayleigh channel covariance rank the lower the receiver complexity. Instead, for Ricean channel distribution, we prove that polynomial complexity is attained through the proposed receiver as long as the mean channel vector belongs to the range of the covariance matrix of the vectorized channel matrix. Hence, full-rank channel correlation is desired to guarantee polynomial ML noncoherent detection complexity for the case of time-invariant Ricean fading.

Contents

This thesis is dedicated to $\Gamma\varepsilon\omega\varrho\gamma\acute{\iota}\alpha$ $\mu o\upsilon$.

1 Introduction

Present and next generation wireless standards aim at high data rates and reliable communications, features afforded by multiple antenna systems that are proven to attain higher channel capacity than single antenna setups while lowering the error probability [1]-[7]. Elaborate information-theoretic results tailored to Rayleigh fading [4] prove that channel capacity actually grows linearly when the number of receive and transmit antennas (simultaneously) increases.

It is, however, natural that antenna arrays are costly and space demanding, thus being a more plausible setup at base stations rather than remote terminals. As a result, transmit diversity techniques have enjoyed primary focus, with the first pioneering work coming from Alamouti [8] that delivered the first full-diversity, full-rate space-time block code (STBC) for two transmit antennas. Tarokh, Jafarkhani, and Calderbank generalized the design of the work in [8] to more than two transmit antennas introducing a paradigm for the construction of space-time block codes based on orthogonal designs [9]. The so-called orthogonal STBCs (OSTBCs) are proven to achieve full antenna diversity gain with linear-complexity single-symbol maximum-likelihood (ML) coherent detection [9], [10]. OSTBCs outperform nonorthogonal designs in terms of error rate; rate-one full-diversity OSTBCs' error-rate provides a lower bound on the one of QSTBCs due to lack of intersymbol interference [11].

Such an error rate is attainable with linear complexity, if the channel state information (CSI) is available at the receiver. However, the very nature of wireless channels suggests rapidly varying channel conditions that render channel estimation inadequate and inefficient. Even when the fading channel coefficients are not fast varying, channel estimation requires transmission of long pilot symbol sequences especially for the cases where large antenna arrays are used [3], with the direct implication of reduced effective transmission rate. Interestingly, the ergodic capacity promised by multiple antenna systems is attained even when CSI is not available to either transmitter or receiver. The work of Zheng and Tse [12] shows that when CSI is not available the capacity of multi-antenna systems with full CSI knowledge at the receiver under Rayleigh fading is approached at the high-SNR regime, if one transmits equal-energy symbols and utilizes space-time codes that are mutually orthogonal during each coherence time interval.

Certainly, when OSTBCs are used and the receiver has no CSI, ML noncoherent sequence detection has to be performed on the entire coherence interval for best performance [5], [10], [13]-[15]. However, if sequence detection is performed through exhaustive search among all possible data sequences [5], [10], [13], [15], then exponential computational complexity is required. The problem of ML noncoherent OSTBC detection under time-invariant independent and identically distributed (i.i.d.) Rayleigh fading was originally expressed as a trace maximization [5] and later proven [16]-[19] to also take the form of a binary quadratic form maximization problem that in the general case is NP-hard [20], [21]. In [17], [19] it was shown that the ML noncoherent OSTBC detection problem can be solved optimally by the sphere decoder, certainly an exponential worst and average case complexity approach at any SNR regime [22]. To avoid the exponential complexity of the optimal receiver many suboptimal schemes have been proposed in the literature such as differential detection schemes [23]-[28], the cyclic ML receiver proposed in [15] based on alternating optimization, and semidefine relaxation approaches [16]-[19]. To combat the exponential cost of the optimal noncoherent receiver the aid of pilot symbols was considered in [29]-[30] at the expense of information rate.

In this work, we consider the case of time-varying Ricean fading and prove that ML noncoherent OSTBC detection can be performed in polynomial time whose order is completely determined by the rank of the covariance matrix of the vectorized channel matrix, provided that the mean channel vector belongs to the range of the channel covariance matrix. Furthermore, motivated by the works in [31]-[36] which treat the problem of rank-deficient quadratic form maximization, we provide an algorithm that solves the ML noncoherent OSTBC detection problem in polynomial time. We tailor to our detection problem the algorithm originally introduced in [33]-[36] and observe that the polynomial in time solution lies in the utilization of multiple auxiliary spherical variables. The optimal data sequence is proven to belong to a polynomial in size set of binary vectors that is built in polynomial time, altogether resulting in an efficient, fixed-complexity algorithm.

Especially for the time-invariant Rayleigh fading channel, the channel mean is zero and the channel covariance matrix rank is always less than or equal to the product of the numbers of transmit and receive antennas, hence polynomial-complexity detection is always guaranteed. In contrast to Rayleigh fading, full-rank channel correlation is desired to guarantee polynomial-complexity ML noncoherent detection upon time-invariant Ricean fading "channel processing." For illustration purposes, we operate the proposed receiver for sequence lengths up to 106 bits in the context of plain 2×2 Alamouti transmissions in unknown Rayleigh or Ricean fading channel environments. Even for a length-106 bit sequence, the polynomial-complexity feature of our algorithm allows ML noncoherent detection without a prohibitive computational cost.

2 System Model and Problem Statement

We consider a multiple-input multiple-output (MIMO) system with M_{t} transmit and M_{r} receive antennas that employs orthogonal space-time coded transmission of size $M_{\mathrm{t}} \times T$ and rate $R = \frac{N}{T}$, $N \leq T$. We assume transmission of binary data that are split into vectors of N bits. Each bit vector forms a corresponding space-time block (matrix) of size $M_{\mathrm{t}} \times T$. The $M_{\mathrm{t}} \times T$ space-time block $\mathbf{C}(\mathbf{s}) \in \mathbb{C}^{M_{\mathrm{t}} \times T}$ that corresponds to the $N \times 1$ data vector $\mathbf{s} \in \{\pm 1\}^N$ is given by

$$\mathbf{C}(\mathbf{s}) = \sum_{n=1}^{N} \mathbf{X}_n s_n \tag{1}$$

where $s_n = \pm 1$ denotes the nth element (bit) of $\mathbf{s}$, $n = 1, 2, \ldots, N$, and $\mathbf{X}_n \in \mathbb{C}^{M_{\mathrm{t}} \times T}$, $n = 1, 2, \ldots, N$, are orthogonal space-time codes that satisfy the property

$$\mathbf{C}(\mathbf{s})\mathbf{C}^H(\mathbf{s}) = \|\mathbf{s}\|^2 \mathbf{I}_{M_{\mathrm{t}}} = T\mathbf{I}_{M_{\mathrm{t}}}, \tag{2}$$

for any $\mathbf{s} \in \{\pm 1\}^N$. Eq. (2) denotes orthogonality and leads to maximum spatial diversity gain [9].

Let $\mathbf{s}^{(p)} = \left[s_1^{(p)}\ s_2^{(p)}\ \ldots\ s_N^{(p)}\right]^T$ denote the data vector contained in the pth transmitted code block, $p = 1, 2, 3, \ldots$. The downconverted and pulse-matched equivalent pth received block of size $M_\mathrm{r} \times T$ is

$$\mathbf{Y}^{(p)} = \mathbf{H}^{(p)}\mathbf{C}\left(\mathbf{s}^{(p)}\right) + \mathbf{V}^{(p)}. \tag{3}$$

In (3), $\mathbf{H}^{(p)} \in \mathbb{C}^{M_\mathrm{r} \times M_\mathrm{t}}$ refers to the pth transmission and represents the channel matrix between the M_t transmit and M_r receive antennas. In general, $\mathbf{H}^{(p)}$ consists of correlated coefficients that are modeled as circular complex Gaussian random variables and account for flat fading. We assume that all collected energy is absorbed by the channel matrix $\mathbf{H}^{(p)}$. In addition, $\mathbf{V}^{(p)} \in \mathbb{C}^{M_\mathrm{r} \times T}$ denotes zero-mean additive spatially and temporally white circular complex Gaussian noise with variance σ_v^2. The channel and noise matrices $\mathbf{H}^{(p)}$ and $\mathbf{V}^{(p)}$, respectively, $p = 1, 2, 3, \ldots$, are independent of each other.

If the receiver has knowledge of the channel matrix, then coherent ML detection simplifies to one-shot block decisions according to

$$\hat{\mathbf{s}}^{(p)} = \arg \min_{\mathbf{s}^{(p)} \in \{\pm 1\}^N} \left\|\mathbf{Y}^{(p)} - \mathbf{H}^{(p)}\mathbf{C}\left(\mathbf{s}^{(p)}\right)\right\|_\mathrm{F}^2, \ p = 1, 2, 3, \ldots. \tag{4}$$

Since orthogonal space-time codes are utilized, exhaustive search among the 2^N possible bit vectors $\mathbf{s}^{(p)} \in \{\pm 1\}^N$ need not be performed because the detector in (4) is equivalent to linear-complexity single-bit decisions of the form

$$\hat{s}_n^{(p)} = \operatorname{sign}\left(\Re\left\{\operatorname{tr}\left\{\mathbf{Y}^{(p)}\mathbf{X}_n^H\left(\mathbf{H}^{(p)}\right)^H\right\}\right\}\right), \ n = 1, 2, \ldots, N, \ p = 1, 2, 3, \ldots. \tag{5}$$

In this work, we assume that the channel matrices $\mathbf{H}^{(p)}$, $p = 1, 2, 3, \ldots$, are not available to the receiver. Hence, coherent detection in (5) cannot be utilized and the ML receiver takes the form of a sequence detector. We consider a sequence of P space-time blocks consecutively transmitted by the source and collected by the receiver, say $\mathbf{Y}^{(1)}, \ \ldots\ , \mathbf{Y}^{(P)}$, and form the $M_\mathrm{r} \times TP$ observation matrix

$$\mathbf{Y} \triangleq \left[\mathbf{Y}^{(1)} \ldots \mathbf{Y}^{(P)}\right] = \left[\mathbf{H}^{(1)}\mathbf{C}\left(\mathbf{s}^{(1)}\right) \ldots \mathbf{H}^{(P)}\mathbf{C}\left(\mathbf{s}^{(P)}\right)\right] + \left[\mathbf{V}^{(1)} \ldots \mathbf{V}^{(P)}\right]. \tag{6}$$

In the sequel, based on the observation of P blocks at the receiver we present ML noncoherent detection developments.

3 Maximum-likelihood Noncoherent Detection

We consider a time-varying Ricean fading MIMO channel, derive an efficient algorithm for the implementation of the ML noncoherent receiver, and prove that the complexity of the proposed ML receiver implementation is polynomial in the sequence length P if the mean channel vector belongs to the range of the channel covariance matrix whose rank is not a function of the sequence length. Thus, full-rank channel correlation is desired to guarantee polynomial ML noncoherent detection complexity independently of the mean channel vector for the case of time-invariant Ricean fading. Interestingly, the order of the polynomial complexity depends strictly on the rank of the channel covariance matrix.

We assume that the channel matrix $\mathbf{H}^{(p)}$ changes during different transmissions and define the concatenated channel matrix $\mathbf{H} \triangleq \left[\mathbf{H}^{(1)} \ldots \mathbf{H}^{(P)}\right] \in \mathbb{C}^{M_{\rm r} \times PM_{\rm t}}$. Due to Ricean fading, the vectorized[1] channel matrix $\mathbf{h} \triangleq \mathrm{vec}(\mathbf{H})$ is a circular complex Gaussian vector of length $M_{\rm t}M_{\rm r}P$ with mean vector $\boldsymbol{\mu} \in \mathbb{C}^{M_{\rm t}M_{\rm r}P}$ and covariance matrix $\mathbf{C}_h = \mathrm{E}\left\{(\mathbf{h}-\boldsymbol{\mu})(\mathbf{h}-\boldsymbol{\mu})^H\right\} = \mathbf{Q}\,\mathbf{Q}^H \in \mathbb{C}^{M_{\rm t}M_{\rm r}P \times M_{\rm t}M_{\rm r}P}$ where $\mathbf{Q} \in \mathbb{C}^{M_{\rm t}M_{\rm r}P \times D}$ consists of orthogonal columns and $D \leq M_{\rm t}M_{\rm r}P$. Given the $M_{\rm r} \times TP$ observation matrix $\mathbf{Y}$, the ML detector for the bit sequence $\mathbf{s} = \left[\left(\mathbf{s}^{(1)}\right)^T \ldots \left(\mathbf{s}^{(P)}\right)^T\right]^T \in \{\pm 1\}^{NP}$ maximizes the conditional probability density function (pdf) of $\mathbf{Y}$ given $\mathbf{s}$. Thus, the optimal decision is given by

$$\hat{\mathbf{s}}_{\rm opt} = \arg \max_{\mathbf{s} \in \{\pm 1\}^{NP}} f(\mathbf{Y}|\mathbf{s}) = \arg \max_{\mathbf{s} \in \{\pm 1\}^{NP}} f\left(\mathrm{vec}(\mathbf{Y})|\,\mathbf{s}\right) = \arg \max_{\mathbf{s} \in \{\pm 1\}^{NP}} f(\mathbf{y}|\mathbf{s}) \tag{7}$$

where $\mathbf{y} \triangleq \mathrm{vec}(\mathbf{Y}) \in \mathbb{C}^{M_{\rm r}TP}$ and $f(\cdot|\cdot)$ represents the pertinent matrix/vector probability density function of the channel output conditioned on a bit sequence.

We define the block-diagonal matrix $\mathbf{D}(\mathbf{s}) \triangleq \mathrm{diag}\left(\left[\mathbf{C}\left(\mathbf{s}^{(1)}\right), \ldots, \mathbf{C}\left(\mathbf{s}^{(P)}\right)\right]\right) \in \mathbb{C}^{M_{\rm t}P \times TP}$ and note that it satisfies the orthogonality property, since

$$\begin{aligned}\mathbf{D}(\mathbf{s})\mathbf{D}^H(\mathbf{s}) &= \mathrm{diag}\left(\left[\mathbf{C}\left(\mathbf{s}^{(1)}\right), \ldots, \mathbf{C}\left(\mathbf{s}^{(P)}\right)\right]\right)\mathrm{diag}\left(\left[\mathbf{C}^H\left(\mathbf{s}^{(1)}\right), \ldots, \mathbf{C}^H\left(\mathbf{s}^{(P)}\right)\right]\right) \\ &= \mathrm{diag}\left(\left[\mathbf{C}\left(\mathbf{s}^{(1)}\right)\mathbf{C}^H\left(\mathbf{s}^{(1)}\right), \ldots, \mathbf{C}\left(\mathbf{s}^{(P)}\right)\mathbf{C}^H\left(\mathbf{s}^{(P)}\right)\right]\right) = \mathrm{diag}(T\mathbf{I}_{M_{\rm t}}, \ldots, T\mathbf{I}_{M_{\rm t}}) = T\mathbf{I}_{M_{\rm t}P}.\end{aligned} \tag{8}$$

Then, the received matrix in (6) becomes $\mathbf{Y} = \mathbf{H}\mathbf{D}(\mathbf{s}) + \mathbf{V}$ where $\mathbf{V} \triangleq \left[\mathbf{V}^{(1)} \ldots \mathbf{V}^{(P)}\right] \in \mathbb{C}^{M_{\rm r} \times TP}$. Due to [37]

$$\mathrm{vec}(\mathbf{ABC}) = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\mathrm{vec}(\mathbf{B}), \tag{9}$$

we obtain

$$\mathbf{y} = \mathrm{vec}(\mathbf{H}\mathbf{D}(\mathbf{s}) + \mathbf{V}) = \mathrm{vec}(\mathbf{I}_{M_{\rm r}}\mathbf{H}\mathbf{D}(\mathbf{s})) + \mathrm{vec}(\mathbf{V}) = \left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)\mathbf{h} + \mathbf{v} \tag{10}$$

where $\mathbf{v} = \mathrm{vec}(\mathbf{V}) \in \mathbb{C}^{M_{\rm r}TP}$ and operator $\otimes$ denotes Kronecker tensor product. Then, it can be proven that $\mathbf{y}$ given $\mathbf{s}$ is a complex Gaussian vector with mean $E\{\mathbf{y}|\mathbf{s}\} = \mathrm{E}\left\{\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)\mathbf{h} + \mathbf{v}\middle|\,\mathbf{s}\right\} = \left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)E\{\mathbf{h}\} + E\{\mathbf{v}\} = \left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)\boldsymbol{\mu}$ and covariance matrix

$$\begin{aligned}\mathbf{C}_y(\mathbf{s}) &= E\left\{\left(\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)(\mathbf{h}-\boldsymbol{\mu}) + \mathbf{v}\right)\left(\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)(\mathbf{h}-\boldsymbol{\mu}) + \mathbf{v}\right)^H\middle|\,\mathbf{s}\right\} \\ &= E\left\{\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)(\mathbf{h}-\boldsymbol{\mu})(\mathbf{h}-\boldsymbol{\mu})^H\left(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)\middle|\,\mathbf{s}\right\} + E\{\mathbf{v}\mathbf{v}^H|s\} \\ &= \left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)\mathbf{C}_h\left(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right) + \sigma_v^2\mathbf{I}_{M_{\rm r}TP} \\ &= \left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right)\mathbf{Q}\,\mathbf{Q}^H\left(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_{\rm r}}\right) + \sigma_v^2\mathbf{I}_{M_{\rm r}TP}.\end{aligned} \tag{11}$$

Therefore, the optimization problem in (7) is rewritten as

$$\hat{\mathbf{s}}_{\rm opt} = \arg \max_{\mathbf{s} \in \{\pm 1\}^{NP}} \frac{1}{\pi^{M_{\rm r}TP}|\mathbf{C}_y(\mathbf{s})|}\exp\left\{-\left(\mathbf{y}-\left(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_{\rm r}}\right)\boldsymbol{\mu}\right)^H\mathbf{C}_y^{-1}(\mathbf{s})\left(\mathbf{y}-\left(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_{\rm r}}\right)\boldsymbol{\mu}\right)\right\}. \tag{12}$$

[1]Operator $\mathrm{vec}(\cdot)$ accounts for column-by-column vectorization of a matrix.

A natural approach to (12) would be an exhaustive search among all 2^{NP} data sequences $\mathbf{s} \in \{\pm 1\}^{NP}$, but such a receiver is impractical even for moderate values of P, since its complexity grows exponentially with P. In the sequel, we present an efficient algorithm that performs the maximization in (12) with $\mathcal{O}(P^{2D})$ calculations if $\boldsymbol{\mu}$ belongs to the span of $\mathbf{C}_h$.

Using identities for the determinant and inverse of a rank-deficient update [38], we compute

$$
\begin{aligned}
|\mathbf{C}_y(\mathbf{s})| &= \left|\sigma_v^2 \mathbf{I}_{M_\mathrm{r}TP}\right| \left|\mathbf{I}_D + \frac{1}{\sigma_v^2}\mathbf{Q}^H(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}})\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right)\mathbf{Q}\right| \\
&= \sigma_v^{2M_\mathrm{r}TP}\left|\mathbf{I}_D + \frac{1}{\sigma_v^2}\mathbf{Q}^H\left(\mathbf{D}^*(\mathbf{s})\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right)\mathbf{Q}\right| \overset{(8)}{=} \sigma_v^{2M_\mathrm{r}TP}\left|\mathbf{I}_D + \frac{T}{\sigma_v^2}\mathbf{Q}^H(\mathbf{I}_{M_\mathrm{t}P} \otimes \mathbf{I}_{M_\mathrm{r}})\,\mathbf{Q}\right| \\
&= \sigma_v^{2M_\mathrm{r}TP}\left|\mathbf{I}_D + \frac{T}{\sigma_v^2}\mathbf{Q}^H\mathbf{Q}\right| = \sigma_v^{2M_\mathrm{r}TP}\left|\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right|
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\mathbf{C}_y^{-1}(\mathbf{s}) &= \frac{1}{\sigma_v^2}\mathbf{I}_{M_\mathrm{r}TP} - \frac{1}{\sigma_v^2}\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right)\mathbf{Q}\left(\mathbf{I}_D + \frac{1}{\sigma_v^2}\mathbf{Q}^H(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}})\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right)\mathbf{Q}\right)^{-1}\mathbf{Q}^H(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}})\,\frac{1}{\sigma_v^2} \\
&\overset{(8)}{=} \frac{1}{\sigma_v^2}\mathbf{I}_{M_\mathrm{r}TP} - \frac{1}{\sigma_v^4}\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right)\mathbf{Q}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\mathbf{Q}^H(\mathbf{I}_{M_\mathrm{t}P} \otimes \mathbf{I}_{M_\mathrm{r}})\,\mathbf{Q}\right)^{-1}\mathbf{Q}^H(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}) \\
&= \frac{1}{\sigma_v^2}\mathbf{I}_{M_\mathrm{r}TP} - \frac{1}{\sigma_v^4}\left(\mathbf{D}^T(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right)\mathbf{Q}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-1}\mathbf{Q}^H\left(\mathbf{D}^*(\mathbf{s}) \otimes \mathbf{I}_{M_\mathrm{r}}\right),
\end{aligned} \tag{14}
$$

where $\boldsymbol{\Sigma} \triangleq \mathbf{Q}^H\mathbf{Q}$ is a $D \times D$ diagonal matrix with the D positive eigenvalues of $\mathbf{C}_h$ on its diagonal. For notation simplicity, we define

$$
\mathbf{U} \triangleq \mathbf{Q}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}} \tag{15}
$$

and observe that

$$
\mathbf{U}^H\mathbf{U} = \left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{Q}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}} = \left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}}\boldsymbol{\Sigma}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}} = \left(\boldsymbol{\Sigma}^{-1} + \frac{T}{\sigma_v^2}\mathbf{I}_D\right)^{-1}. \tag{16}
$$

We observe that $|\mathbf{C}_y(\mathbf{s})|$ is independent of the transmitted sequence $\mathbf{s}$, drop it from the maximization in (12), and substitute (15) in (14) and then back in (12) to obtain

$$\begin{aligned}
\hat{\mathbf{s}}_{\text{opt}}=&\underset{\mathbf{s}\in\{\pm1\}^{NP}}{\operatorname{argmax}}\Bigg\{-\big(\mathbf{y}-(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}})\boldsymbol{\mu}\big)^H\Big(\frac{1}{\sigma_v^2}\mathbf{I}_{M_\text{r}TP}-\frac{1}{\sigma_v^4}(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}})\,\mathbf{U}\,\mathbf{U}^H(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}})\Big)\big(\mathbf{y}-(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}})\boldsymbol{\mu}\big)\Bigg\}\\
=&\underset{\mathbf{s}\in\{\pm1\}^{NP}}{\operatorname{argmax}}\Bigg\{-\tfrac{1}{\sigma_v^2}\mathbf{y}^H\mathbf{y}+\tfrac{1}{\sigma_v^2}\mathbf{y}^H\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\boldsymbol{\mu}+\tfrac{1}{\sigma_v^2}\boldsymbol{\mu}^H\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{y}-\tfrac{1}{\sigma_v^2}\boldsymbol{\mu}^H\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\boldsymbol{\mu}\\
&+\tfrac{1}{\sigma_v^4}\mathbf{y}^H\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{U}\,\mathbf{U}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{y}\\
&-\tfrac{1}{\sigma_v^4}\mathbf{y}^H\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{U}\,\mathbf{U}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\boldsymbol{\mu}\\
&-\tfrac{1}{\sigma_v^4}\boldsymbol{\mu}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{U}\,\mathbf{U}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{y}\\
&+\tfrac{1}{\sigma_v^4}\boldsymbol{\mu}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{U}\,\mathbf{U}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\boldsymbol{\mu}\Bigg\}\\
\overset{(8)}{=}&\underset{\mathbf{s}\in\{\pm1\}^{NP}}{\operatorname{argmax}}\Bigg\{\tfrac{1}{\sigma_v^2}\mathbf{y}^H\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{U}\,\mathbf{U}^H\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{y}\\
&+\mathbf{y}^H\,\big(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\Big(\mathbf{I}_{M_\text{t}M_\text{r}P}-\tfrac{T}{\sigma_v^2}\mathbf{U}\,\mathbf{U}^H\Big)\,\boldsymbol{\mu}+\boldsymbol{\mu}^H\,\Big(\mathbf{I}_{M_\text{t}M_\text{r}P}-\tfrac{T}{\sigma_v^2}\mathbf{U}\,\mathbf{U}^H\Big)\,\big(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\big)\,\mathbf{y}\Bigg\}.
\end{aligned} \tag{17}$$

We continue our algorithmic developments by defining the matrices $\mathbf{X}\overset{\triangle}{=}[\mathbf{X}_1\ldots\mathbf{X}_N]\in\mathbb{C}^{M_\text{t}\times TN}$, $\mathbf{S}\overset{\triangle}{=}\big[\mathbf{s}^{(1)}\ldots\mathbf{s}^{(P)}\big]\in\{\pm1\}^{N\times P}$, and $\mathbf{G}(\mathbf{s})\overset{\triangle}{=}\big[\mathbf{C}\big(\mathbf{s}^{(1)}\big)\ \ldots\ \mathbf{C}\big(\mathbf{s}^{(P)}\big)\big]\in\mathbb{C}^{M_\text{t}\times TP}$. We observe that $\mathbf{s}=\text{vec}(\mathbf{S})$, $\mathbf{G}(\mathbf{s})\mathbf{G}^H(\mathbf{s})=TP\mathbf{I}_{M_\text{t}}$ due to (2), and

$$\begin{aligned}
\mathbf{G}(\mathbf{s})&=\left[\sum_{n=1}^{N}\mathbf{X}_n s_n^{(1)}\ \ldots\ \sum_{n=1}^{N}\mathbf{X}_n s_n^{(P)}\right]=\big[[\mathbf{X}_1\ldots\mathbf{X}_N]\big(\mathbf{s}^{(1)}\otimes\mathbf{I}_T\big)\ \ldots\ [\mathbf{X}_1\ldots\mathbf{X}_N]\big(\mathbf{s}^{(P)}\otimes\mathbf{I}_T\big)\big]\\
&=\mathbf{X}\big[\big(\mathbf{s}^{(1)}\otimes\mathbf{I}_T\big)\ \ldots\ \big(\mathbf{s}^{(P)}\otimes\mathbf{I}_T\big)\big]=\mathbf{X}\big(\big[\mathbf{s}^{(1)}\ldots\mathbf{s}^{(P)}\big]\otimes\mathbf{I}_T\big)=\mathbf{X}\left(\mathbf{S}\otimes\mathbf{I}_T\right).
\end{aligned} \tag{18}$$

Moreover, we denote by $\mathbf{e}_p$ the pth column of $\mathbf{I}_P$, $p=1,2,\ldots,P$, and rewrite $\mathbf{D}(\mathbf{s})$ as

$$\begin{aligned}
\mathbf{D}(\mathbf{s})&=\text{diag}\left(\left[\mathbf{C}\big(\mathbf{s}^{(1)}\big),\ldots,\mathbf{C}\big(\mathbf{s}^{(P)}\big)\right]\right)=\begin{bmatrix}\mathbf{G}(\mathbf{s})\left(\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\right)\\ \vdots\\ \mathbf{G}(\mathbf{s})\left(\mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\right)\end{bmatrix}=(\mathbf{I}_P\otimes\mathbf{G}(\mathbf{s}))\begin{bmatrix}\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\\ \vdots\\ \mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\end{bmatrix}\\
&=(\mathbf{I}_P\otimes\mathbf{G}(\mathbf{s}))\left(\tilde{\mathbf{I}}_P\otimes\mathbf{I}_T\right)\overset{(18)}{=}(\mathbf{I}_P\otimes\mathbf{X}\left(\mathbf{S}\otimes\mathbf{I}_T\right))\left(\tilde{\mathbf{I}}_P\otimes\mathbf{I}_T\right)
\end{aligned} \tag{19}$$

where $\tilde{\mathbf{I}}_P\overset{\triangle}{=}\left[\mathbf{e}_1\mathbf{e}_1^T\ \ \cdots\ \ \mathbf{e}_P\mathbf{e}_P^T\right]^T\in\{0,1\}^{P^2\times P}$. Then, the vector $\left(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\right)\mathbf{y}$ that appears in the maximization problem in (17) is reexpressed as

$$\begin{aligned}
&\left(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\text{r}}\right)\mathbf{y}\overset{(19)}{=}\left(\left(\mathbf{I}_P\otimes\mathbf{X}^*\left(\mathbf{S}\otimes\mathbf{I}_T\right)\right)\left(\tilde{\mathbf{I}}_P\otimes\mathbf{I}_T\right)\otimes\mathbf{I}_{M_\text{r}}\right)\text{vec}(\mathbf{Y})\\
\overset{(9)}{=}&\ \text{vec}\left(\mathbf{Y}\left(\tilde{\mathbf{I}}_P^T\otimes\mathbf{I}_T\right)\left(\mathbf{I}_P\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)\mathbf{I}_{M_\text{t}P}\right)\overset{(9)}{=}\left(\mathbf{I}_{M_\text{t}P}\otimes\mathbf{Y}\left(\tilde{\mathbf{I}}_P^T\otimes\mathbf{I}_T\right)\right)\text{vec}\left(\mathbf{I}_P\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right).
\end{aligned} \tag{20}$$

We observe that

$$\begin{aligned}
&\text{vec}\left(\mathbf{I}_P\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)=\text{vec}\left(\left[\mathbf{e}_1\ \ldots\ \mathbf{e}_P\right]\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)\\
=&\ \text{vec}\left(\left[\mathbf{e}_1\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\ \ldots\ \mathbf{e}_P\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right]\right)=\begin{bmatrix}\text{vec}\left(\mathbf{e}_1\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)\\ \vdots\\ \text{vec}\left(\mathbf{e}_P\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)\end{bmatrix}
\end{aligned} \tag{21}$$

where, for any $p=1,2,\ldots,P$,

$$\mathrm{vec}\big(\mathbf{e}_p\otimes\big(\mathbf{S}^T\otimes\mathbf{I}_T\big)\mathbf{X}^H\big)=\mathrm{vec}\big(\mathbf{e}_p\cdot 1\otimes\mathbf{I}_{PT}\big(\mathbf{S}^T\otimes\mathbf{I}_T\big)\mathbf{X}^H\big)=\mathrm{vec}\big(\left(\mathbf{e}_p\otimes\mathbf{I}_{PT}\right)\big(1\otimes\big(\mathbf{S}^T\otimes\mathbf{I}_T\big)\mathbf{X}^H\big)\big)$$
$$=\mathrm{vec}\big(\left(\mathbf{e}_p\otimes\mathbf{I}_{PT}\right)\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\mathbf{I}_{M_\mathrm{t}}\big)\overset{(9)}{=}\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_p\otimes\mathbf{I}_{PT}\right)\mathrm{vec}\left(\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right). \tag{22}$$

We also denote by $\tilde{\mathbf{X}}_m$ the matrix that contains the mth rows of all N space-time matrices, that is

$$\tilde{\mathbf{X}}_m\triangleq\begin{bmatrix}[\mathbf{X}_1]_{m,:}\\ \vdots\\ [\mathbf{X}_N]_{m,:}\end{bmatrix}\in\mathbb{C}^{N\times T},\ m=1,\ldots,M_\mathrm{t}, \tag{23}$$

and observe that

$$\mathbf{X}^H=[\mathbf{X}_1\ldots\mathbf{X}_N]^H=\left[\mathrm{vec}\left(\tilde{\mathbf{X}}_1^H\right)\ \ldots\ \mathrm{vec}\left(\tilde{\mathbf{X}}_{M_\mathrm{t}}^H\right)\right]. \tag{24}$$

Then,

$$\mathrm{vec}\left(\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)\overset{(24)}{=}\mathrm{vec}\left(\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\left[\mathrm{vec}\left(\tilde{\mathbf{X}}_1^H\right)\ \ldots\ \mathrm{vec}\left(\tilde{\mathbf{X}}_{M_\mathrm{t}}^H\right)\right]\right) \tag{25}$$
$$=\mathrm{vec}\Big(\Big[\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathrm{vec}\Big(\tilde{\mathbf{X}}_1^H\Big)\ \ldots\ \left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathrm{vec}\Big(\tilde{\mathbf{X}}_{M_\mathrm{t}}^H\Big)\Big]\Big)\overset{(9)}{=}\mathrm{vec}\Big(\Big[\mathrm{vec}\Big(\tilde{\mathbf{X}}_1^H\mathbf{S}\Big)\ \ldots\ \mathrm{vec}\Big(\tilde{\mathbf{X}}_{M_\mathrm{t}}^H\mathbf{S}\Big)\Big]\Big)$$
$$\overset{(9)}{=}\mathrm{vec}\Big(\Big[\Big(\mathbf{I}_P\otimes\tilde{\mathbf{X}}_1^H\Big)\mathrm{vec}(\mathbf{S})\ \ldots\ \Big(\mathbf{I}_P\otimes\tilde{\mathbf{X}}_{M_\mathrm{t}}^H\Big)\mathrm{vec}(\mathbf{S})\Big]\Big)=\mathrm{vec}\big(\big[\mathbf{Z}_1^H\mathbf{s}\ \ldots\ \mathbf{Z}_{M_\mathrm{t}}^H\mathbf{s}\big]\big)=\begin{bmatrix}\mathbf{Z}_1^H\mathbf{s}\\ \vdots\\ \mathbf{Z}_{M_\mathrm{t}}^H\mathbf{s}\end{bmatrix}=\mathbf{Z}^H\mathbf{s}$$

where $\mathbf{Z}_m\triangleq\mathbf{I}_P\otimes\tilde{\mathbf{X}}_m\in\mathbb{C}^{NP\times TP}$, $m=1,\ldots,M_\mathrm{t}$, and $\mathbf{Z}\triangleq[\mathbf{Z}_1\ \ldots\ \mathbf{Z}_{M_\mathrm{t}}]\in\mathbb{C}^{NP\times M_\mathrm{t}TP}$. Substituting (25) in (22) and then back in (21), we obtain

$$\mathrm{vec}\left(\mathbf{I}_P\otimes\left(\mathbf{S}^T\otimes\mathbf{I}_T\right)\mathbf{X}^H\right)=\begin{bmatrix}(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\otimes\mathbf{I}_{PT})\mathbf{Z}^H\mathbf{s}\\ \vdots\\ (\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\otimes\mathbf{I}_{PT})\mathbf{Z}^H\mathbf{s}\end{bmatrix}=\left(\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\otimes\mathbf{I}_P\\ \vdots\\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\otimes\mathbf{I}_P\end{bmatrix}\otimes\mathbf{I}_T\right)\mathbf{Z}^H\mathbf{s}. \tag{26}$$

Using (26), eq. (20) becomes

$$\left(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\mathbf{y}=\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}\right)\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\tilde{\mathbf{I}}_P^T\otimes\mathbf{I}_T\right)\left(\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\otimes\mathbf{I}_P\\ \vdots\\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\otimes\mathbf{I}_P\end{bmatrix}\otimes\mathbf{I}_T\right)\mathbf{Z}^H\mathbf{s}$$
$$=\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}\right)\left(\begin{bmatrix}\left(\mathbf{I}_{M_\mathrm{t}}\otimes\tilde{\mathbf{I}}_P^T\right)\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\otimes\mathbf{I}_P\right)\\ \vdots\\ \left(\mathbf{I}_{M_\mathrm{t}}\otimes\tilde{\mathbf{I}}_P^T\right)\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\otimes\mathbf{I}_P\right)\end{bmatrix}\otimes\mathbf{I}_T\right)\mathbf{Z}^H\mathbf{s}=\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}\right)\left(\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\tilde{\mathbf{I}}_P^T\left(\mathbf{e}_1\otimes\mathbf{I}_P\right)\\ \vdots\\ \mathbf{I}_{M_\mathrm{t}}\otimes\tilde{\mathbf{I}}_P^T\left(\mathbf{e}_P\otimes\mathbf{I}_P\right)\end{bmatrix}\otimes\mathbf{I}_T\right)\mathbf{Z}^H\mathbf{s}$$
$$=\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}\right)\left(\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\mathbf{e}_1^T\\ \vdots\\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\mathbf{e}_P^T\end{bmatrix}\otimes\mathbf{I}_T\right)\mathbf{Z}^H\mathbf{s}=\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}\right)\mathbf{E}\mathbf{Z}^H\mathbf{s} \tag{27}$$

where $\mathbf{E}\triangleq\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\mathbf{e}_1^T\\ \vdots\\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\mathbf{e}_P^T\end{bmatrix}\otimes\mathbf{I}_T\in\{0,1\}^{M_\mathrm{t}P^2T\times M_\mathrm{t}PT}$. Due to (27), the first, second, and third part of the maximization argument in (17) become

$$\frac{1}{\sigma_v^2}\mathbf{s}^T\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}^H\right)\mathbf{U}\,\mathbf{U}^H\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}\right)\mathbf{E}\mathbf{Z}^H\mathbf{s}, \tag{28}$$

$$\mathbf{s}^T\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}^H\right)\left(\mathbf{I}_{M_{\mathrm{t}}M_{\mathrm{r}}P}-\frac{T}{\sigma_v^2}\mathbf{U}\,\mathbf{U}^H\right)\boldsymbol{\mu},\text{ and }\ \boldsymbol{\mu}^H\left(\mathbf{I}_{M_{\mathrm{t}}M_{\mathrm{r}}P}-\frac{T}{\sigma_v^2}\mathbf{U}\,\mathbf{U}^H\right)\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}\right)\mathbf{E}\mathbf{Z}^H\mathbf{s}, \tag{29}$$

respectively. We append a 1 to the end of the data vector $\mathbf{s}$, define $\tilde{\mathbf{s}}\triangleq\left[\mathbf{s}^T\ 1\right]^T$, and obtain

$$\hat{\mathbf{s}}_{\mathrm{opt}}=\left[\hat{\tilde{\mathbf{s}}}_{\mathrm{opt}}\right]_{1:NP,1} \tag{30}$$

where, using (28) and (29) in (17),

$$\hat{\tilde{\mathbf{s}}}_{\mathrm{opt}}=\underset{\substack{\tilde{\mathbf{s}}\in\{\pm1\}^{NP+1}\\ \tilde{s}_{NP+1}=1}}{\arg\max}\ \tilde{\mathbf{s}}^T\begin{bmatrix}\frac{1}{\sigma_v^2}\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}^H\right)\mathbf{U}\,\mathbf{U}^H\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}\right)\mathbf{E}\mathbf{Z}^H & \mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}^H\right)\left(\mathbf{I}_{M_{\mathrm{t}}M_{\mathrm{r}}P}-\frac{T}{\sigma_v^2}\mathbf{U}\,\mathbf{U}^H\right)\boldsymbol{\mu}\\ \boldsymbol{\mu}^H\left(\mathbf{I}_{M_{\mathrm{t}}M_{\mathrm{r}}P}-\frac{T}{\sigma_v^2}\mathbf{U}\,\mathbf{U}^H\right)\left(\mathbf{I}_{M_{\mathrm{t}}P}\otimes\mathbf{Y}\right)\mathbf{E}\mathbf{Z}^H & 0\end{bmatrix}\tilde{\mathbf{s}}. \tag{31}$$

In the sequel, we show that (31) is order-$2D$ polynomially solvable when the matrix of interest -up to diagonal manipulations- has at most D nonzero eigenvalues that are also positive. For this purpose, we present the following two lemmas.

Lemma 1 *Every matrix* $\mathbf{A}\in\mathbb{C}^{N\times N}$ *is binary-quadratic-form-optimization-equivalent (BQFO-equivalent) to* $\dot{\mathbf{A}}=\mathbf{A}+\mathit{diag}(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^{N\times1}$, *i.e.* $\mathbf{s}^T\mathbf{A}\mathbf{s}$ *and* $\mathbf{s}^T\dot{\mathbf{A}}\mathbf{s}$ *are maximized (minimized) by the same binary vector* $\mathbf{s}\in\{\pm1\}^N$.

Proof: For any $\mathbf{s}\in\{\pm1\}^N$, $\mathbf{s}^T\dot{\mathbf{A}}\mathbf{s}=\mathbf{s}^T(\mathbf{A}+\mathrm{diag}(\mathbf{x}))\mathbf{s}=\mathbf{s}^T\mathbf{A}\mathbf{s}+\sum_{n=1}^N x_ns_n^2=\mathbf{s}^T\mathbf{A}\mathbf{s}+\sum_{n=1}^N x_n$ where $\sum_{n=1}^N x_n$ is a real constant that does not affect the maximization (minimization) of the quadratic form. □

Lemma 2 *Let* $\mathbf{B}\in\mathbb{C}^{(N-1)\times M}$, $\mathbf{C}\in\mathbb{C}^{M\times D}$, $\mathbf{x}\in\mathbb{C}^{M\times1}$, *and*

$$\mathbf{A}\triangleq\begin{bmatrix}\mathbf{B}\mathbf{C}\mathbf{C}^H\mathbf{B}^H & \mathbf{B}\left(\mathbf{I}_M-\mathbf{C}\mathbf{C}^H\right)\mathbf{x}\\ \mathbf{x}^H\left(\mathbf{I}_M-\mathbf{C}\mathbf{C}^H\right)\mathbf{B}^H & 0\end{bmatrix}\in\mathbb{C}^{N\times N}. \tag{32}$$

If $\mathbf{x}$ *belongs to the range of* $\mathbf{C}$, *i.e.* $\mathbf{x}=\mathbf{C}\mathbf{a}$, $\mathbf{a}\in\mathbb{C}^{D\times1}$, *then* $\mathbf{A}$ *is BQFO-equivalent to the positive (semi)definite matrix*

$$\dot{\mathbf{A}}\triangleq\begin{bmatrix}\mathbf{B}\mathbf{C}\\ \mathbf{a}^H\left(\mathbf{I}_D-\mathbf{C}^H\mathbf{C}\right)\end{bmatrix}\left[\mathbf{C}^H\mathbf{B}^H\ \ (\mathbf{I}_D-\mathbf{C}^H\mathbf{C})\mathbf{a}\right]. \tag{33}$$

Proof: If $\mathbf{x}=\mathbf{C}\mathbf{a}$, $\mathbf{a}\in\mathbb{C}^{D\times1}$, then

$$\begin{aligned}\mathbf{A}&=\begin{bmatrix}\mathbf{B}\mathbf{C}\mathbf{C}^H\mathbf{B}^H & \mathbf{B}\left(\mathbf{I}_M-\mathbf{C}\mathbf{C}^H\right)\mathbf{C}\mathbf{a}\\ \mathbf{a}^H\mathbf{C}^H\left(\mathbf{I}_M-\mathbf{C}\mathbf{C}^H\right)\mathbf{B}^H & 0\end{bmatrix}=\begin{bmatrix}\mathbf{B}\mathbf{C}\mathbf{C}^H\mathbf{B}^H & \mathbf{B}\left(\mathbf{C}-\mathbf{C}\mathbf{C}^H\mathbf{C}\right)\mathbf{a}\\ \mathbf{a}^H\left(\mathbf{C}^H-\mathbf{C}^H\mathbf{C}\mathbf{C}^H\right)\mathbf{B}^H & 0\end{bmatrix}\\ &=\begin{bmatrix}\mathbf{B}\mathbf{C}\mathbf{C}^H\mathbf{B}^H & \mathbf{B}\mathbf{C}\left(\mathbf{I}_D-\mathbf{C}^H\mathbf{C}\right)\mathbf{a}\\ \mathbf{a}^H\left(\mathbf{I}_D-\mathbf{C}^H\mathbf{C}\right)\mathbf{C}^H\mathbf{B}^H & 0\end{bmatrix}=\dot{\mathbf{A}}+\mathrm{diag}\left(\begin{bmatrix}\mathbf{0}_{(N-1)\times1}\\ -\left\|\left(\mathbf{I}_D-\mathbf{C}^H\mathbf{C}\right)\mathbf{a}\right\|^2\end{bmatrix}\right).\end{aligned} \tag{34}$$

Due to Lemma 1, $\mathbf{A}$ is BQFO-equivalent to $\dot{\mathbf{A}}$. $\square$

From (15), we observe that $\mathbf{Q}$ and $\mathbf{U}$ have identical ranges. If $\boldsymbol{\mu}$ belongs to the range of $\mathbf{C}_h = \mathbf{Q}\,\mathbf{Q}^H$ (equivalently, the range of $\frac{1}{\sigma_v}\mathbf{U}$, i.e. $\boldsymbol{\mu} = \frac{1}{\sigma_v}\mathbf{U}\mathbf{a}$ for some $\mathbf{a} \in \mathbb{C}^D$), then we obtain $\mathbf{a} = \sigma_v \left(\mathbf{U}^H\mathbf{U}\right)^{-1}\mathbf{U}^H\boldsymbol{\mu}$, set $\mathbf{B} \triangleq \frac{1}{\sqrt{T}}\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_tP} \otimes \mathbf{Y}^H\right)$, $\mathbf{C} \triangleq \frac{\sqrt{T}}{\sigma_v}\mathbf{U}$, and $\mathbf{x} \triangleq \sqrt{T}\boldsymbol{\mu}$ in Lemma 2, and rewrite (31) as

$$\begin{aligned}
\hat{\tilde{\mathbf{s}}}_{\mathrm{opt}} &= \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max}\ \tilde{\mathbf{s}}^T \begin{bmatrix} \frac{1}{\sigma_v}\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_tP} \otimes \mathbf{Y}^H\right)\mathbf{U} \\ \sigma_v\boldsymbol{\mu}^H\mathbf{U}\left(\mathbf{U}^H\mathbf{U}\right)^{-1}\left(\mathbf{I}_D - \frac{T}{\sigma_v^2}\mathbf{U}^H\mathbf{U}\right) \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_v}\mathbf{U}^H(\mathbf{I}_{M_tP} \otimes \mathbf{Y})\mathbf{E}\mathbf{Z}^H & \sigma_v\left(\mathbf{I}_D - \frac{T}{\sigma_v^2}\mathbf{U}^H\mathbf{U}\right)\left(\mathbf{U}^H\mathbf{U}\right)^{-1}\mathbf{U}^H\boldsymbol{\mu} \end{bmatrix} \tilde{\mathbf{s}} \\
&= \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max} \left\| \begin{bmatrix} \frac{1}{\sigma_v}\mathbf{U}^H(\mathbf{I}_{M_tP} \otimes \mathbf{Y})\mathbf{E}\mathbf{Z}^H & \sigma_v\left(\left(\mathbf{U}^H\mathbf{U}\right)^{-1} - \frac{T}{\sigma_v^2}\mathbf{I}_D\right)\mathbf{U}^H\boldsymbol{\mu} \end{bmatrix} \tilde{\mathbf{s}} \right\|^2 \\
&\overset{(15)}{\underset{(16)}{=}} \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max} \left\| \underbrace{\begin{bmatrix} \frac{1}{\sigma_v}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}}\mathbf{Q}^H(\mathbf{I}_{M_tP} \otimes \mathbf{Y})\mathbf{E}\mathbf{Z}^H & \sigma_v\boldsymbol{\Sigma}^{-1}\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}}\mathbf{Q}^H\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{A}^H} \tilde{\mathbf{s}} \right\|^2 \\
&= \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max}\ \tilde{\mathbf{s}}^T\mathbf{A}\mathbf{A}^H\tilde{\mathbf{s}} = \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max}\ \Re\left\{\tilde{\mathbf{s}}^T\mathbf{A}\mathbf{A}^H\tilde{\mathbf{s}}\right\} = \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max}\ \tilde{\mathbf{s}}^T\left(\Re\{\mathbf{A}\}\Re\{\mathbf{A}\}^T + \Im\{\mathbf{A}\}\Im\{\mathbf{A}\}^T\right)\tilde{\mathbf{s}} \\
&= \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max}\ \tilde{\mathbf{s}}^T\left[\Re\{\mathbf{A}\}\ \Im\{\mathbf{A}\}\right]\left[\Re\{\mathbf{A}\}\ \Im\{\mathbf{A}\}\right]^T\tilde{\mathbf{s}} = \underset{\substack{\tilde{\mathbf{s}} \in \{\pm 1\}^{NP+1} \\ \tilde{s}_{NP+1}=1}}{\arg\max} \left\|\mathbf{V}^T\tilde{\mathbf{s}}\right\| \qquad (35)
\end{aligned}$$

where

$$\mathbf{A} \triangleq \begin{bmatrix} \frac{1}{\sigma_v}\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_tP} \otimes \mathbf{Y}^H\right)\mathbf{Q} \\ \sigma_v\boldsymbol{\mu}^H\mathbf{Q}\boldsymbol{\Sigma}^{-1} \end{bmatrix} \left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}} \in \mathbb{C}^{(NP+1)\times D} \qquad (36)$$

and $\mathbf{V} \triangleq [\Re\{\mathbf{A}\}\ \ \Im\{\mathbf{A}\}] \in \mathbb{R}^{(NP+1)\times 2D}$. The computation of $\hat{\tilde{\mathbf{s}}}_{\mathrm{opt}}$ in (35) (hence, $\hat{\mathbf{s}}_{\mathrm{opt}}$ in (30)) can be implemented with complexity $\mathcal{O}\left(P^{2D}\right)$ if we follow the multiple-auxiliary-angle methodology that has been introduced in [33]-[36] for the problem of rank-deficient quadratic form maximization and is presented below.

We introduce the spherical coordinates $\phi_1 \in (-\pi, \pi]$, $\phi_2, \ldots, \phi_{2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ and define the spherical coordinate vector $\boldsymbol{\phi} \triangleq [\phi_1, \phi_2, \ldots, \phi_{2D-1}]^T$ and the $2D \times 1$ hyperpolar vector

$$\mathbf{c}(\boldsymbol{\phi}) \triangleq \begin{bmatrix} \sin\phi_1 \\ \cos\phi_1 \sin\phi_2 \\ \vdots \\ \cos\phi_1 \ldots \cos\phi_{2D-2}\sin\phi_{2D-1} \\ \cos\phi_1 \ldots \cos\phi_{2D-2}\cos\phi_{2D-1} \end{bmatrix}. \qquad (37)$$

Due to Cauchy-Schwartz Inequality which states that, for any $\mathbf{v} \in \mathbb{R}^{2D}$, $\mathbf{v}^T\mathbf{c}(\boldsymbol{\phi}) \leq \|\mathbf{v}\|\|\mathbf{c}(\boldsymbol{\phi})\|$ with equality if and only if $\phi_1, \ldots, \phi_{2D-1}$ are the spherical coordinates of $\mathbf{v}$, for the optimization problem in (35) we obtain

$$\begin{aligned}\max_{\tilde{\mathbf{s}}\in\{\pm 1\}^{NP+1}} \left\|\mathbf{V}^T\tilde{\mathbf{s}}\right\| &= \max_{\tilde{\mathbf{s}}\in\{\pm 1\}^{NP+1}} \max_{\phi_1\in[-\pi,\pi]} \max_{\phi_2,\ldots,\phi_{2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]} \left\{\tilde{\mathbf{s}}^T\mathbf{V}\mathbf{c}(\boldsymbol{\phi})\right\} \\ &= \max_{\phi_1\in(-\pi,\pi]} \max_{\phi_2,\ldots,\phi_{2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]} \sum_{n=1}^{NP+1} \max_{\tilde{s}_n=\pm 1} \left\{\tilde{s}_n \mathbf{V}_{n,:}\mathbf{c}(\boldsymbol{\phi})\right\}\end{aligned} \tag{38}$$

by interchanging the maximizations in (38). Eq. (38) shows that $\hat{\tilde{\mathbf{s}}}_{\text{opt}}$ can be obtained by scanning the hypersphere defined by the auxiliary angles $\phi_1, \phi_2, \ldots, \phi_{2D-1}$ and determining the optimal vector $\hat{\tilde{\mathbf{s}}}(\boldsymbol{\phi})$ for every given point $\boldsymbol{\phi} \in (-\pi, \pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]^{2D-2}$. Interestingly, for any point $\boldsymbol{\phi}$, the maximizing argument of each term of the sum in (38) depends only on the corresponding row of $\mathbf{V}$ and is determined by $\tilde{s}_n = \operatorname{sgn}\left(\mathbf{V}_{n,:}\,\mathbf{c}(\boldsymbol{\phi})\right),\ n = 1, \ldots, NP+1$.

$$\tilde{\mathbf{s}}(\boldsymbol{\phi}) = \operatorname{sgn}(\mathbf{V}\mathbf{c}(\boldsymbol{\phi})) \tag{39}$$

$\tilde{\mathbf{s}}(\boldsymbol{\phi}) = \operatorname{sgn}(\mathbf{V}\mathbf{c}(\boldsymbol{\phi}))$ and the optimal vector $\hat{\tilde{\mathbf{s}}}_{\text{opt}}$ in (35) is met if we scan the entire set $(-\pi, \pi] \times (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2}$. Moreover, opposite binary vectors $\tilde{\mathbf{s}}$ and $-\tilde{\mathbf{s}}$ result in the same metric in (35), thus, we can ignore the values of ϕ_1 in $\left(-\pi, -\frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \pi\right]$ and rewrite the optimization problem in (38) as

$$\max_{\boldsymbol{\phi}\in\varPhi} \sum_{n=1}^{NP+1} \max_{\tilde{s}_n=\pm 1} \left\{\tilde{s}_n \mathbf{V}_{n,:}\mathbf{c}(\boldsymbol{\phi})\right\} \tag{40}$$

$\max_{\boldsymbol{\phi}\in\varPhi} \sum_{n=1}^{NP+1} \max_{\tilde{s}_n=\pm 1} \left\{\tilde{s}_n \mathbf{V}_{n,:}\mathbf{c}(\boldsymbol{\phi})\right\}$ where $\varPhi \stackrel{\triangle}{=} (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-1}$. Finally, we collect all candidate binary vectors into set

$$\mathcal{S} \stackrel{\triangle}{=} \bigcup_{\boldsymbol{\phi}\in\varPhi} \{\tilde{\mathbf{s}}(\boldsymbol{\phi})\} \subseteq \{\pm 1\}^{NP+1} \tag{41}$$

and observe that $\arg\max_{\tilde{\mathbf{s}}\in\{\pm 1\}^{NP+1}} \left\|\mathbf{V}^T\tilde{\mathbf{s}}\right\| \in \mathcal{S}$, hence $\hat{\tilde{\mathbf{s}}}_{\text{opt}} = \hat{\tilde{\mathbf{s}}}'_{\text{opt}} \cdot \left[\hat{\tilde{\mathbf{s}}}'_{\text{opt}}\right]_{NP+1,1}$ where

$$\hat{\tilde{\mathbf{s}}}'_{\text{opt}} = \arg\max_{\tilde{\mathbf{s}}\in\mathcal{S}} \left\|\mathbf{V}^T\tilde{\mathbf{s}}\right\|. \tag{42}$$

In [35], it was shown that the decision $\tilde{s}_n = \operatorname{sgn}\left(\mathbf{V}_{n,:}\,\mathbf{c}(\boldsymbol{\phi})\right)$ is equivalent to

$$\tilde{s}_n = \begin{cases} -\operatorname{sgn}(V_{n,1}), & \phi_1 \in \left(-\frac{\pi}{2}, \tan^{-1}\left(-\frac{\mathbf{V}_{n,2:2D}\mathbf{c}(\boldsymbol{\phi}_{2:2D-1})}{V_{n,1}}\right)\right] \\ \operatorname{sgn}(V_{n,1}), & \phi_1 \in \left(\tan^{-1}\left(-\frac{\mathbf{V}_{n,2:2D}\mathbf{c}(\boldsymbol{\phi}_{2:2D-1})}{V_{n,1}}\right), \frac{\pi}{2}\right]. \end{cases} \tag{43}$$

The function $\phi = \tan^{-1}\left(-\frac{\mathbf{V}_{n,2:2D}\mathbf{c}(\boldsymbol{\phi}_{2:2D-1})}{V_{n,1}}\right)$ determines a hypersurface which partitions $\varPhi$ into two regions. One region corresponds to $\tilde{s}_n = -1$ and the other one corresponds to $\tilde{s}_n = +1$. Therefore, the $NP+1$ rows of $\mathbf{V}$ are associated with $NP+1$ corresponding hypersurfaces that partition the hypercube $\varPhi$ into K cells $C_1, C_2, \ldots, C_K$ such that $\bigcup_{k=1}^{K} C_k = \varPhi$, $C_k \cap C_j \neq 0\ \forall\ k \neq j$, and each cell C_k corresponds to a distinct vector $\tilde{\mathbf{s}}_k \in \{\pm 1\}^{NP+1}$. See, for example, Fig. 1 where we present the cells that are formed by an eight-row three-column matrix $\mathbf{V}$ and a spherical vector $\boldsymbol{\phi} = [\phi_1\ \phi_2]^T$.[2] In [35], it was shown that $K = \sum_{d=0}^{2D-1}\binom{NP}{d}$, therefore all candidate vectors form set $\mathcal{S}$ with cardinality $|\mathcal{S}| = \sum_{d=0}^{2D-1}\binom{NP}{d} = \mathcal{O}\left(P^{2D-1}\right)$. The construction of $\mathcal{S}$ is of special interest since it determines the overall performance of the proposed method. An algorithm for the construction of $\mathcal{S}$ was developed in [35] and is available at http://www.telecom.tuc.gr/∼karystinos. Interestingly, the algorithm's complexity for the construction of $\mathcal{S}$ is $\mathcal{O}(P^{2D})$ for any given matrix $\mathbf{V}$.

In our developments in this section, $\mathbf{V}$ (which is a function of $\mathbf{A}$) has to be computed by the receiver and subsequently fed to the algorithm of [35]. In (36), it is seen that $\mathbf{A}$ is a function of the received data matrix $\mathbf{Y}$, matrices $\mathbf{Z}$, $\mathbf{E}$, $\mathbf{Q}$, and $\boldsymbol{\Sigma}$, vector $\boldsymbol{\mu}$, and scalar σ_v. We note that

$$\begin{aligned}
\mathbf{Z}\mathbf{E}^T\!\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}^H\right)\mathbf{Q} &= \mathbf{Z}\left[\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\ \ldots\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\right]\left(\mathbf{I}_P\otimes\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}^H\right)\mathbf{Q}\\
&= \mathbf{Z}\left[\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\right)\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}^H\right)\ \ldots\ \left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\right)\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}^H\right)\right]\mathbf{Q}\\
&= \mathbf{Z}\left[\mathbf{I}_{M_\mathrm{t}}\otimes\left(\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\ \ldots\ \mathbf{I}_{M_\mathrm{t}}\otimes\left(\mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\right]\begin{bmatrix}[\,\mathbf{Q}\,]_{1:M_\mathrm{t}M_\mathrm{r},:}\\ \vdots\\ [\,\mathbf{Q}\,]_{(P-1)M_\mathrm{t}M_\mathrm{r}+1:PM_\mathrm{t}M_\mathrm{r},:}\end{bmatrix}\\
&= \sum_{p=1}^{P}\left[\mathbf{Z}_1\ \ldots\ \mathbf{Z}_{M_\mathrm{t}}\right]\left(\mathbf{I}_{M_\mathrm{t}}\otimes\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\right)[\,\mathbf{Q}\,]_{(p-1)M_\mathrm{t}M_\mathrm{r}+1:pM_\mathrm{t}M_\mathrm{r},:}\,. \qquad (44)
\end{aligned}$$

For convenience, we define $\mathbf{Q}_p \stackrel{\triangle}{=} [\,\mathbf{Q}\,]_{(p-1)M_\mathrm{t}M_\mathrm{r}+1:pM_\mathrm{t}M_\mathrm{r},:}$ and observe that, for any $p = 1, 2, \ldots, P$,

$$\begin{aligned}
&\left[\mathbf{Z}_1\ \ldots\ \mathbf{Z}_{M_\mathrm{t}}\right]\left(\mathbf{I}_{M_\mathrm{t}}\otimes\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\right)\mathbf{Q}_p = \left[\mathbf{I}_P\otimes\tilde{\mathbf{X}}_1\ \ldots\ \mathbf{I}_P\otimes\tilde{\mathbf{X}}_{M_\mathrm{t}}\right]\left(\mathbf{I}_{M_\mathrm{t}}\otimes\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\right)\mathbf{Q}_p\\
&= \left[\left(\mathbf{I}_P\otimes\tilde{\mathbf{X}}_1\right)\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\ \ldots\ \left(\mathbf{I}_P\otimes\tilde{\mathbf{X}}_{M_\mathrm{t}}\right)\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\mathbf{I}_T\right)\mathbf{Y}^H\right]\mathbf{Q}_p \qquad (45)\\
&= \left[\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\tilde{\mathbf{X}}_1\right)\mathbf{Y}^H\ \ldots\ \left(\mathbf{e}_p\mathbf{e}_p^T\otimes\tilde{\mathbf{X}}_{M_\mathrm{t}}\right)\mathbf{Y}^H\right]\mathbf{Q}_p = \sum_{m=1}^{M_\mathrm{t}}\left(\mathbf{e}_p\mathbf{e}_p^T\otimes\tilde{\mathbf{X}}_m\right)\mathbf{Y}^H[\mathbf{Q}_p]_{(m-1)M_\mathrm{r}+1:mM_\mathrm{r},:}\\
&= \sum_{m=1}^{M_\mathrm{t}}\begin{bmatrix}\mathbf{0}_{(p-1)N\times M_\mathrm{r}}\\ \tilde{\mathbf{X}}_m\left[\mathbf{Y}^H\right]_{(p-1)T+1:pT,:}\\ \mathbf{0}_{(P-p)N\times M_\mathrm{r}}\end{bmatrix}[\mathbf{Q}_p]_{(m-1)M_\mathrm{r}+1:mM_\mathrm{r},:} = \begin{bmatrix}\mathbf{0}_{(p-1)N\times D}\\ \sum_{m=1}^{M_\mathrm{t}}\tilde{\mathbf{X}}_m\left[\mathbf{Y}^H\right]_{(p-1)T+1:pT,:}[\mathbf{Q}_p]_{(m-1)M_\mathrm{r}+1:mM_\mathrm{r},:}\\ \mathbf{0}_{(P-p)N\times D}\end{bmatrix}.
\end{aligned}$$

Computation of $\tilde{\mathbf{X}}_m\left[\mathbf{Y}^H\right]_{(p-1)T+1:pT,:}[\mathbf{Q}_p]_{(m-1)M_\mathrm{r}+1:mM_\mathrm{r},:}$, $m = 1, \ldots, M_\mathrm{t}$, requires $\mathcal{O}(NTM_\mathrm{r} + NM_\mathrm{r}D)$ calculations and the sum in (45) consists of M_t such products resulting in a total of $\mathcal{O}(M_\mathrm{t}(NTM_\mathrm{r} + NM_\mathrm{r}D))$ calculations. In addition, (44) contains P such sums, hence the computational complexity of $\mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y}^H\right)\mathbf{Q}$ is $\mathcal{O}(PM_\mathrm{t}(NTM_\mathrm{r} + NM_\mathrm{r}D))$. Computation of the row vector $\sigma_v\boldsymbol{\mu}^H\mathbf{Q}\boldsymbol{\Sigma}^{-1}$ that appears in the bottom row of $\mathbf{A}$ requires $\mathcal{O}\left(M_\mathrm{t}M_\mathrm{r}PD + D\right)$ calculations. Finally, the multiplication of the leftmost matrix in (36) with $\left(\mathbf{I}_D + \frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}}$ costs $\mathcal{O}\left((NP+1)D\right)$. Therefore, the overall complexity overhead for the computation of $\mathbf{V}$ becomes

[2] We selected an odd number of columns for matrix $\mathbf{V}$ and even number of elements for $\boldsymbol{\phi}$ just for illustration purposes. It should be mentioned that $\mathbf{V}$ and $\boldsymbol{\phi}$ always consist of an even number of columns and odd number of elements, respectively.

$\mathcal{O}(PM_{\rm t}(NTM_{\rm r}+NM_{\rm r}D)+M_{\rm t}M_{\rm r}PD+D+(NP+1)D)$ which is linear in the sequence length P for constant values of $M_{\rm t}$, $M_{\rm r}$, N, T, and D (that is, fixed number of antennas, space-time coding rate, and channel covariance rank). We conclude that the overall complexity of the proposed receiver is $\mathcal{O}\left(P^{2D}\right)$.

As a brief summary, the sequence of calculations of the proposed ML noncoherent receiver is as follows. The whole data record of TP received vectors is utilized to form the received matrix $\mathbf{Y}$. Then, matrix $\mathbf{V}$ is computed as a function of $\mathbf{A}$ in (36) with complexity $\mathcal{O}(P)$. Finally, the quadratic-form-maximization algorithm of [35] is operated on $\mathbf{V}$ to produce the optimal data bit sequence $\hat{\hat{\mathbf{s}}}_{\rm opt}$ with complexity $\mathcal{O}\left(P^{2D}\right)$. Therefore, the overall complexity of the proposed receiver is $\mathcal{O}\left(P^{2D}\right)$. We conclude that, for the general case of time-varying Ricean fading, ML noncoherent OSTBC detection is attained with polynomial complexity if the mean channel vector belongs to the range of the channel covariance matrix whose rank is not a function of the sequence length; provided the latter condition, the polynomial complexity of our proposed optimal receiver depends strictly on the rank of the channel covariance matrix.

4 Special Channel Model Cases

In the previous section we considered a time-varying Ricean fading MIMO channel model and showed that ML noncoherent OSTBC detection is attained with polynomial complexity if the mean channel vector belongs to the range of the channel covariance matrix whose rank is not a function of the sequence length. Since the time-invariant Ricean, time-varying Rayleigh, and time-invariant Rayleigh channel models are special cases of the general model that we considered, we immediately conclude that polynomial ML noncoherent detection complexity is also attained provided the same conditions with the time-varying Ricean channel model case hold. Especially for (time-varying or time-invariant) Rayleigh fading, the channel mean is zero which always belongs to the range of the channel covariance matrix, hence polynomial-complexity ML noncoherent detection is always attained provided that the channel covariance rank is not a function of the sequence length. In this section, we examine separately the three special cases of the general model of the previous section, identify conditions for polynomial solvability of the ML noncoherent detection problem, and report exact complexity requirements of the proposed polynomial-complexity ML noncoherent detector. An interesting outcome of our analysis for the time-invariant Rayleigh fading channel is that *polynomial-complexity ML noncoherent detection is always feasible through the proposed algorithm* and -in contrast to Ricean fading- the complexity of the optimal receiver is reduced if the channel covariance rank is lower.

Case I: Time-varying Rayleigh fading

Due to Rayleigh fading, we have $\boldsymbol{\mu}=\mathbf{0}$, hence the bottom row of $\mathbf{A}$ in (36) becomes zero and the ML detector of (35) simplifies to $\hat{\mathbf{s}}_{\rm opt} = \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}} \left\|\mathbf{V}_1^T\mathbf{s}\right\|$ where $\mathbf{V}_1 \stackrel{\triangle}{=} [\Re\{\mathbf{A}_1\}\ \Im\{\mathbf{A}_1\}]$ and $\mathbf{A}_1 \stackrel{\triangle}{=} \mathbf{Z}\mathbf{E}^T\left(\mathbf{I}_{M_{\rm t}P}\otimes\mathbf{Y}^H\right)\mathbf{Q}\left(\mathbf{I}_D+\frac{T}{\sigma_v^2}\boldsymbol{\Sigma}\right)^{-\frac{1}{2}}$.

Case II: Time-invariant Ricean fading

We have $\mathbf{h} = \mathbf{1}_P \otimes \underline{\mathbf{h}}$ where $\underline{\mathbf{h}} = \mathrm{vec}\left(\mathbf{H}^{(1)}\right)$ is a circular complex Gaussian vector with mean $\underline{\boldsymbol{\mu}}$ and covariance matrix $\mathbf{C}_{\underline{h}} = \underline{\mathbf{Q}}\,\underline{\mathbf{Q}}^H$. Then, the covariance matrix of the channel vector $\mathbf{h}$ equals $\mathbf{C}_h = \mathbf{1}_P\mathbf{1}_P^T \otimes \mathbf{C}_{\underline{h}} = \mathbf{1}_P\mathbf{1}_P^T \otimes \underline{\mathbf{Q}}\,\underline{\mathbf{Q}}^H = (\mathbf{1}_P \otimes \underline{\mathbf{Q}})(\mathbf{1}_P \otimes \underline{\mathbf{Q}})^H$, therefore $\mathbf{Q} = \mathbf{1}_P \otimes \underline{\mathbf{Q}}$ and

$$
\begin{aligned}
\mathbf{Q}^H(\mathbf{I}_{M_\mathrm{t}P}\otimes\mathbf{Y})\mathbf{E} &= \left(\mathbf{1}_P^T\otimes\underline{\mathbf{Q}}^H\right)(\mathbf{I}_P\otimes\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y})\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\\ \vdots \\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\end{bmatrix} = \left(\mathbf{1}_P^T\otimes\underline{\mathbf{Q}}^H(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y})\right)\begin{bmatrix}\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_1\mathbf{e}_1^T\otimes\mathbf{I}_T\\ \vdots \\ \mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_P\mathbf{e}_P^T\otimes\mathbf{I}_T\end{bmatrix}\\
&= \sum_{p=1}^{P}\underline{\mathbf{Q}}^H(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y})\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{e}_p\mathbf{e}_p^T\otimes\mathbf{I}_T\right) = \underline{\mathbf{Q}}^H(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y})\left(\mathbf{I}_{M_\mathrm{t}}\otimes\left(\sum_{p=1}^{P}\mathbf{e}_p\mathbf{e}_p^T\right)\otimes\mathbf{I}_T\right)\\
&= \underline{\mathbf{Q}}^H\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}\right)\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{I}_P\otimes\mathbf{I}_T\right) = \underline{\mathbf{Q}}^H\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}\right). \qquad (46)
\end{aligned}
$$

In addition, $\boldsymbol{\Sigma} = \mathbf{Q}^H\mathbf{Q} = (\mathbf{1}_P^T\otimes\underline{\mathbf{Q}}^H)(\mathbf{1}_P\otimes\underline{\mathbf{Q}}) = \mathbf{1}_P^T\mathbf{1}_P\otimes\underline{\mathbf{Q}}^H\underline{\mathbf{Q}} = P\otimes\underline{\boldsymbol{\Sigma}} = P\underline{\boldsymbol{\Sigma}}$, where $\underline{\boldsymbol{\Sigma}} \triangleq \underline{\mathbf{Q}}^H\underline{\mathbf{Q}}$, and $\boldsymbol{\mu}^H\mathbf{Q}\boldsymbol{\Sigma}^{-1} = \left(\mathbf{1}_P^T\otimes\underline{\boldsymbol{\mu}}^H\right)\left(\mathbf{1}_P\otimes\underline{\mathbf{Q}}\right)\frac{1}{P}\underline{\boldsymbol{\Sigma}}^{-1} = \frac{1}{P}\left(P\otimes\underline{\boldsymbol{\mu}}^H\underline{\mathbf{Q}}\right)\underline{\boldsymbol{\Sigma}}^{-1} = \underline{\boldsymbol{\mu}}^H\underline{\mathbf{Q}}\underline{\boldsymbol{\Sigma}}^{-1}$. Then, the ML detector in (35) simplifies to $\hat{\tilde{\mathbf{s}}}_{\mathrm{opt}} = \arg\max_{\substack{\tilde{\mathbf{s}}\in\{\pm1\}^{NP+1}\\ \tilde{s}_{NP+1}=1}}\left\|\mathbf{V}_2^T\tilde{\mathbf{s}}\right\|$ where $\mathbf{V}_2 \triangleq [\Re\{\mathbf{A}_2\}\ \Im\{\mathbf{A}_2\}]$ and $\mathbf{A}_2 \triangleq \begin{bmatrix}\frac{1}{\sigma_v}\mathbf{Z}\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}^H\right)\underline{\mathbf{Q}}\\ \sigma_v\underline{\boldsymbol{\mu}}^H\underline{\mathbf{Q}}\underline{\boldsymbol{\Sigma}}^{-1}\end{bmatrix}\left(\mathbf{I}_D + \frac{TP}{\sigma_v^2}\underline{\boldsymbol{\Sigma}}\right)^{-\frac{1}{2}}$. Of course, such a simplification is possible when $\underline{\boldsymbol{\mu}}$ belongs to the range of $\mathbf{C}_{\underline{h}}$.

Case III: Time-invariant Rayleigh fading

We have $\underline{\boldsymbol{\mu}} = \mathbf{0}$, hence the ML detector becomes $\hat{\mathbf{s}}_{\mathrm{opt}} = \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\|\mathbf{V}_3^T\mathbf{s}\right\|$ where $\mathbf{V}_3 \triangleq [\Re\{\mathbf{A}_3\}\ \Im\{\mathbf{A}_3\}]$ and $\mathbf{A}_3 \triangleq \mathbf{Z}\left(\mathbf{I}_{M_\mathrm{t}}\otimes\mathbf{Y}^H\right)\underline{\mathbf{Q}}\left(\mathbf{I}_D + \frac{TP}{\sigma_v^2}\underline{\boldsymbol{\Sigma}}\right)^{-\frac{1}{2}}$. Note that the latter is always feasible, since $\boldsymbol{\mu} = \mathbf{0}$ always belongs to the range of $\mathbf{C}_h$.

It is interesting to mention that the ML noncoherent detector in the case of time-invariant Rayleigh fading (Case III) simplifies to the popular trace detector [5], [10] in the special cases of i.i.d. channel coefficients or joint channel estimation and data detection due to channel statistics uncertainty at the receiver. Indeed, if the channel coefficients are i.i.d., then $\mathbf{C}_{\underline{h}}$, $\underline{\mathbf{Q}}$, and $\underline{\boldsymbol{\Sigma}}$ are scaled versions of $\mathbf{I}_{M_\mathrm{t}M_\mathrm{r}}$, $D = M_\mathrm{t}M_\mathrm{r}$, and the ML detector in (17) becomes

$$
\begin{aligned}
\hat{\mathbf{s}}_{\mathrm{opt}} &= \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\{\mathbf{y}^H\left(\mathbf{D}^T(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\left(\mathbf{1}_P\otimes\mathbf{I}_{M_\mathrm{t}M_\mathrm{r}}\right)\left(\mathbf{1}_P^T\otimes\mathbf{I}_{M_\mathrm{t}M_\mathrm{r}}\right)\left(\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\mathbf{y}\right\}\\
&= \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\{\mathbf{y}^H\left(\mathbf{D}^T(\mathbf{s})\left(\mathbf{1}_P\otimes\mathbf{I}_{M_\mathrm{t}}\right)\otimes\mathbf{I}_{M_\mathrm{r}}\right)\left(\left(\mathbf{1}_P^T\otimes\mathbf{I}_{M_\mathrm{t}}\right)\mathbf{D}^*(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\mathbf{y}\right\} \qquad (47)\\
&= \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\{\mathbf{y}^H\left(\mathbf{G}^T(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\left(\mathbf{G}^*(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\mathbf{y}\right\} = \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\|\left(\mathbf{G}^*(\mathbf{s})\otimes\mathbf{I}_{M_\mathrm{r}}\right)\mathrm{vec}(\mathbf{Y})\right\|^2\\
&\overset{(9)}{=} \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\|\mathrm{vec}\left(\mathbf{Y}\mathbf{G}^H(\mathbf{s})\right)\right\|^2 = \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\left\|\mathbf{Y}\mathbf{G}^H(\mathbf{s})\right\|_\mathrm{F}^2 = \arg\max_{\mathbf{s}\in\{\pm1\}^{NP}}\mathrm{tr}\left\{\mathbf{Y}\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\mathbf{Y}^H\right\}.
\end{aligned}
$$

In the second special case, the receiver does not have knowledge of the channel statistics, hence joint ML estimation of $\underline{\mathbf{H}}$ and detection of $\mathbf{s}$ is performed according to $\widehat{\{\underline{\mathbf{H}}, \mathbf{s}\}} = \arg\min_{\substack{\underline{\mathbf{H}}\in\mathbb{C}^{M_\mathrm{r}\times M_\mathrm{t}}\\ \mathbf{s}\in\{\pm1\}^{NP}}}\|\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s})\|_\mathrm{F}^2$. Then,

$$\hat{\mathbf{s}}_{\mathrm{GLRT}} = \arg\min_{\mathbf{s}\in\{\pm 1\}^{NP}} \left\{ \min_{\underline{\mathbf{H}}\in\mathbb{C}^{M_{\mathrm{r}}\times M_{\mathrm{t}}}} \|\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s})\|_{\mathrm{F}}^2 \right\} \tag{48}$$

is the generalized-likelihood ratio test (GLRT) detection of $\mathbf{s}$ [17], [39]. For any $\mathbf{s} \in \{\pm 1\}^{NP}$, the inner minimization in (48) results in

$$\hat{\underline{\mathbf{H}}}(\mathbf{s}) = \arg \min_{\underline{\mathbf{H}}\in\mathbb{C}^{M_{\mathrm{r}}\times M_{\mathrm{t}}}} \|\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s})\|_{\mathrm{F}}^2 = \frac{1}{TP}\mathbf{Y}\mathbf{G}^H(\mathbf{s}) \tag{49}$$

which is obtained by setting $\frac{\partial}{\partial \underline{\mathbf{H}}} \|\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s})\|_{\mathrm{F}}^2 = 0$ and solving for $\underline{\mathbf{H}}$ using matrix differentiation [40] that gives

$$\begin{aligned}
&\frac{\partial}{\partial \underline{\mathbf{H}}} \|\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s})\|_{\mathrm{F}}^2 = \frac{\partial}{\partial \underline{\mathbf{H}}}\mathrm{tr}\left\{(\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s}))^H(\mathbf{Y} - \underline{\mathbf{H}}\mathbf{G}(\mathbf{s}))\right\} \\
&= \frac{\partial}{\partial \underline{\mathbf{H}}}\mathrm{tr}\left\{\mathbf{Y}^H\mathbf{Y}\right\} - \frac{\partial}{\partial \underline{\mathbf{H}}}\mathrm{tr}\left\{\mathbf{G}(\mathbf{s})\mathbf{Y}^H\underline{\mathbf{H}}\right\} - \frac{\partial}{\partial \underline{\mathbf{H}}}\mathrm{tr}\left\{\mathbf{Y}\mathbf{G}^H(\mathbf{s})\underline{\mathbf{H}}^H\right\} + \frac{\partial}{\partial \underline{\mathbf{H}}}\mathrm{tr}\left\{\mathbf{G}(\mathbf{s})\mathbf{G}^H(\mathbf{s})\underline{\mathbf{H}}^H\underline{\mathbf{H}}\right\} \\
&= \mathbf{0} - \mathbf{Y}^*\mathbf{G}^T(\mathbf{s}) + \mathbf{0} + TP\underline{\mathbf{H}}^*.
\end{aligned} \tag{50}$$

Using (49), the GLRT decision in (48) becomes

$$\begin{aligned}
\hat{\mathbf{s}}_{\mathrm{GLRT}} &= \arg\min_{\mathbf{s}\in\{\pm 1\}^{NP}} \mathrm{tr}\left\{\left(\mathbf{Y} - \frac{1}{TP}\mathbf{Y}\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\right)\left(\mathbf{Y} - \frac{1}{TP}\mathbf{Y}\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\right)^H\right\} \\
&= \arg\min_{\mathbf{s}\in\{\pm 1\}^{NP}} \left\{-\frac{2}{TP}\mathrm{tr}\left\{\mathbf{Y}\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\mathbf{Y}^H\right\} + \frac{1}{(TP)^2}\mathrm{tr}\left\{\mathbf{Y}\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\mathbf{Y}^H\right\}\right\} \\
&= \arg\max_{\mathbf{s}\in\{\pm 1\}^{NP}} \mathrm{tr}\left\{\mathbf{Y}\mathbf{G}^H(\mathbf{s})\mathbf{G}(\mathbf{s})\mathbf{Y}^H\right\}.
\end{aligned} \tag{51}$$

Apparently, (47) and (51) are identical problems. Both constitute special cases of Case III and can be solved in polynomial time $\mathcal{O}(P^{2M_{\mathrm{t}}M_{\mathrm{r}}})$ if we follow the proposed approach.

We conclude that for time-invariant Rayleigh fading the ML noncoherent detector can always operate with polynomial complexity in the sequence length P, the order of which is determined strictly by the rank D of the channel covariance matrix. That is, the lower the Rayleigh channel covariance rank the lower the receiver complexity. Therefore, the worst-case complexity is $\mathcal{O}\left(P^{2M_{\mathrm{t}}M_{\mathrm{r}}}\right)$ and is met, for example, when the channel coefficients are i.i.d. Similar properties hold for time-varying Rayleigh fading, as long as the rank D of the correlation matrix is not a function of the sequence length P. Instead, for Ricean channel distribution, polynomial complexity is attained through the proposed receiver if the mean channel vector belongs to the range of the channel covariance matrix. Hence, in contrast to Rayleigh fading where low-rank correlation is desired, full-rank correlation is sufficient to guarantee polynomial detection complexity in the case of time-invariant Ricean fading. In the following section, we illustrate our theoretic findings.

5 Simulation Studies

We consider a 2×2 MIMO system that employs Alamouti space-time coding (with rate $R = \frac{N}{T} = 1$, since $N = T = 2$) to transmit binary data in an unknown Rayleigh or Ricean fading channel environment. Space-time ambiguity induced by the rotatability of the Alamouti code [41] is resolved by employing differential space-time modulation [24] due to which the pth transmitted space-time block is $\mathbf{C}^{(p)} = \mathbf{C}^{(p-1)}\mathbf{X}^{(p)}$ where $\mathbf{X}^{(p)} = \begin{bmatrix} s_1^{(p)} & 0 \\ 0 & s_2^{(p)} \end{bmatrix}$ if $s_1^{(p)}s_2^{(p)} > 0$, $\mathbf{X}^{(p)} = \begin{bmatrix} 0 & s_2^{(p)} \\ -s_1^{(p)} & 0 \end{bmatrix}$ if $s_1^{(p)}s_2^{(p)} < 0$, and $\mathbf{C}^{(0)} = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$ so that $\mathbf{C}^{(p)}$ follows the Alamouti code structure, for any $p = 0, 1, 2, \ldots$. For the covariance matrix of the vectorized channel matrix we adopt the model of [42], according to which $\mathbf{C}_{\underline{h}} = \begin{bmatrix} 1 & r & t & w_1 \\ r^* & 1 & w_2 & t \\ t^* & w_2^* & 1 & r \\ w_1^* & t^* & r^* & 1 \end{bmatrix}$. In our studies we set $t = r = 0$ and $w_1 = w_2 = 1$, a setup that exhibits higher ergodic capacity in comparison to the one with independent channel coefficients [42]. Observe that the rank of such a matrix is 2, therefore the overall complexity of the proposed ML receiver becomes $\mathcal{O}(P^4)$. We present results averaged over $1,000$ channel realizations.

In Fig. 2, we study the Rayleigh fading channel case and present the bit error rate (BER) of the one-shot coherent MRC receiver and the ML noncoherent receiver implemented with complexity $\mathcal{O}(P^4)$ by the proposed algorithm as a function of the transmitted signal-to-noise ratio (SNR) for sequence lengths $P = 2$ and 53. For comparison purposes, we also present the BER of the pilot-assisted noncoherent receiver [29] implemented with complexity $\mathcal{O}(P^4)$ and the pilot-assisted coherent receiver [30] implemented with linear complexity. We observe that -as expected- the conventional one-lag ($P = 2$) noncoherent receiver exhibits a $2 - 3$dB loss in comparison with the coherent MRC receiver. The SNR loss is reduced to $1 - 1.5$dB by ML sequence detection for $P = 53$, i.e. block detection of 106 bits, which is implemented by the proposed algorithm with polynomial complexity while the conventional sequence detection implementation would require an exhaustive search among 2^{104} vectors of length 106. The pilot-assisted noncoherent receiver [29] operates with rate $\frac{P-1}{P+1}$ since it consumes one additional block for the initial pilot transmission and one additional block for differential encoding and exhibits similar performance with the proposed ML receiver for $P = 2$. However, for $P = 53$ the proposed ML noncoherent receiver outperforms the pilot-assisted noncoherent one when they operate with the same complexity $\mathcal{O}(P^4)$. The proposed receiver is also superior to the pilot-assisted coherent receiver [30]. For the latter, we used one pilot OSTBC matrix and $P-1$ information OSTBC matrices to maintain the information rate $\frac{P-1}{P}$ of the differential encoding scheme associated with the proposed ML noncoherent receiver.

In Fig. 3, we set the SNR to 6dB and present BER and computational complexity curves of the proposed ML and pilot-assisted [29] noncoherent receivers versus sequence length P. We observe that, if the pilot-assisted receiver operates with the same complexity with the proposed ML receiver, then its performance deteriorates as the sequence length P increases. In Fig. 3(a), the BER of the coherent MRC receiver is also presented as a performance lower bound. Fig. 3(b) demonstrates the significant complexity gain offered by the proposed algorithm. We recall that exhaustive search fails for a sequence length $P > 15$ while the proposed algorithm maintains ML performance with polynomial computational complexity. For example, if $P = 53$ ($NP = 106$), then the conventional receiver implementation would require an exhaustive search among $2^{104} \approx 2 \cdot 10^{31}$ binary vectors of length 106 while the proposed implementation performs a search among $\binom{105}{0} + \binom{105}{1} + \binom{105}{2} + \binom{105}{3} \approx 2 \cdot 10^5$ binary vectors of length 106. Finally, to demonstrate the efficiency of our proposed algorithm for the Ricean fading channel case, relevant performance and complexity results are presented in Figs. 4 and 5 where the mean channel vector is selected to belong to the range of the channel covariance matrix.

6 Conclusions

We proved that ML noncoherent sequence detection is always polynomially solvable with respect to the sequence length for OSTBC and correlated (in general) Rayleigh channel coefficients and developed a novel algorithm that performs ML noncoherent OSTBC detection with polynomial complexity whose order is completely determined by the channel covariance matrix rank. Our proposed detector operates in polynomial time for Ricean channels as well, if the mean channel vector belongs to the range of the channel covariance matrix. Hence, low-rank channel correlation is preferred for Rayleigh channels while full-rank channel correlation is desired for Ricean channels to guarantee polynomial ML detection complexity. For the the cases of time-varying Rayleigh or Ricean channel matrices, similar conclusions were drawn.

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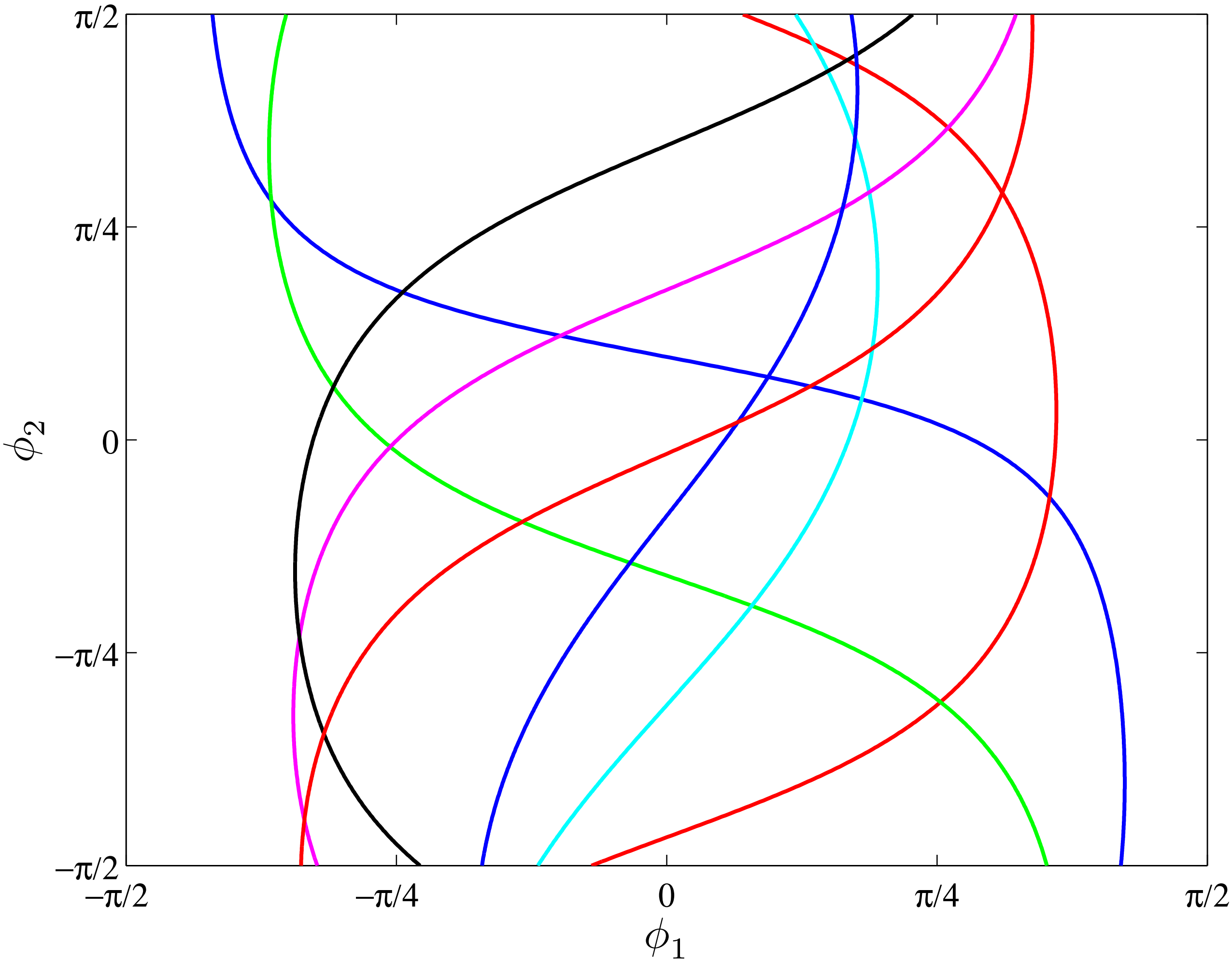

Fig. 1. Partition of Φ into cells.

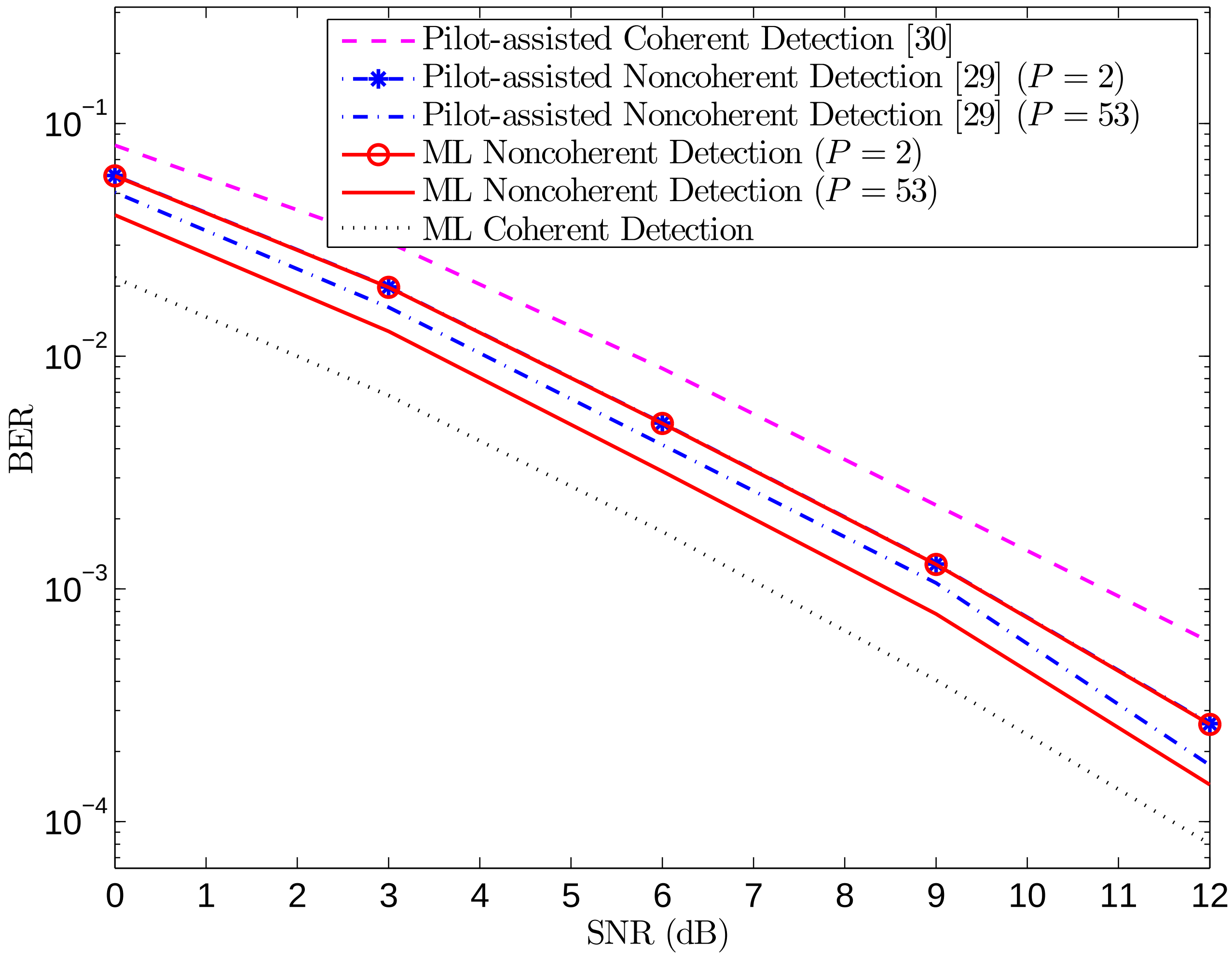

Fig. 2. BER versus SNR for ML and pilot-assisted [30] coherent OSTBC receivers and proposed ML and pilot-assisted [29] noncoherent OSTBC receivers with sequence length $P = 2$ (conventional receiver) and $P = 53$ (106 bits) upon Rayleigh fading.

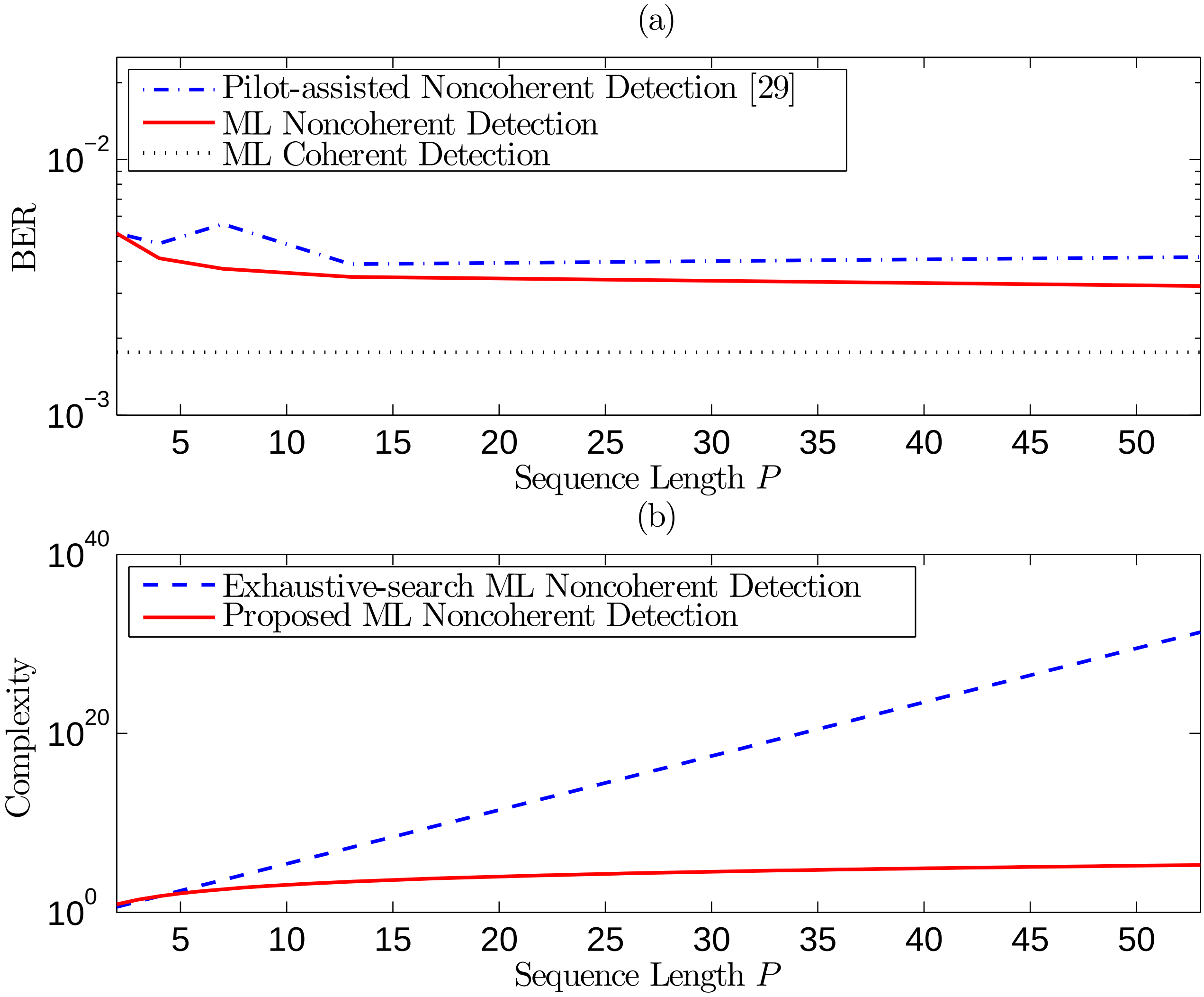

Fig. 3. (a) BER and (b) complexity versus sequence length P for SNR = 6dB and Rayleigh fading.

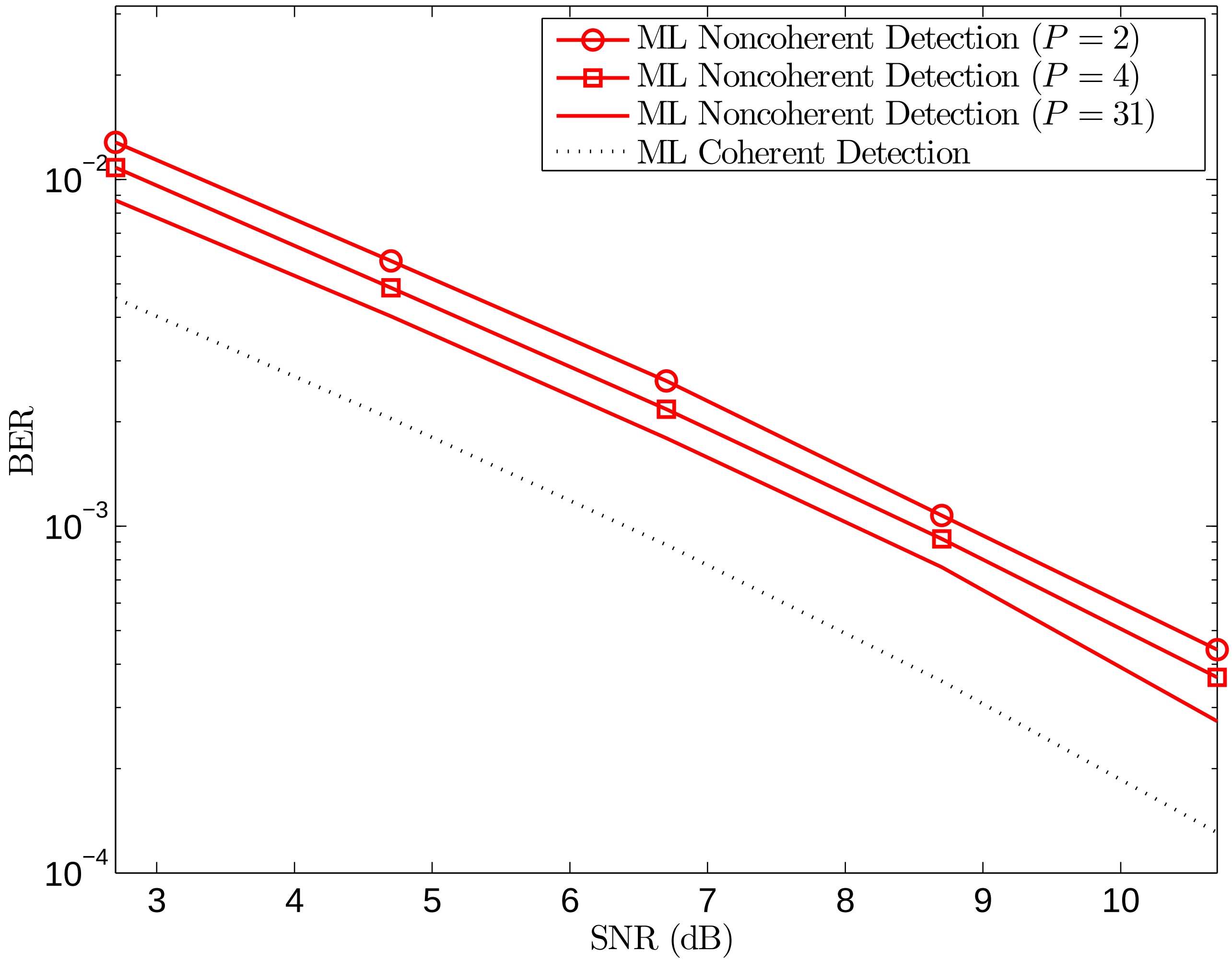

Fig. 4. BER versus SNR for ML coherent OSTBC receiver and ML noncoherent OSTBC receivers with sequence length $P = 2$ (conventional receiver), $P = 4$ (8 bits), and $P = 31$ (62 bits) upon Ricean fading.

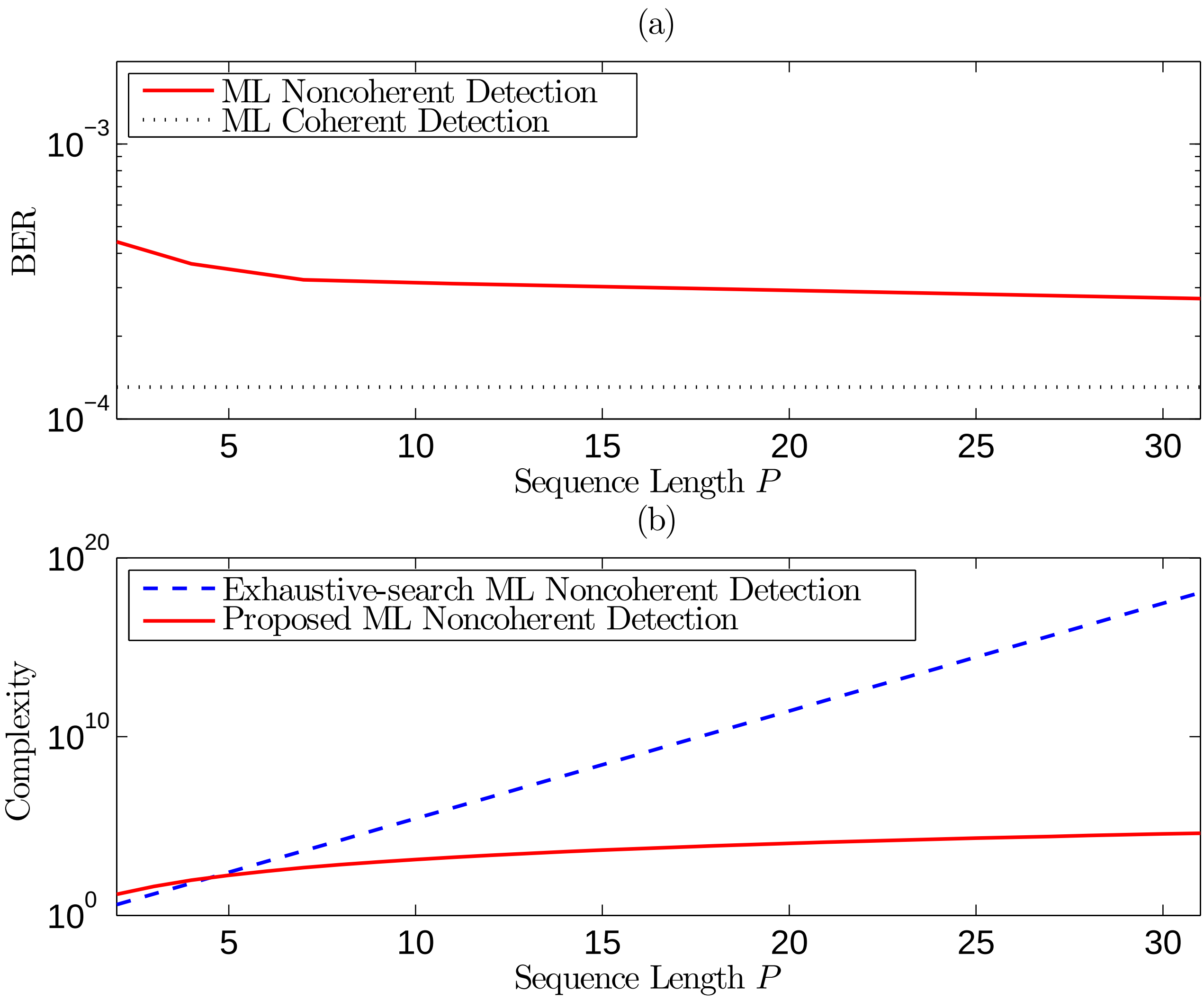

Fig. 5. (a) BER and (b) complexity versus sequence length P for SNR = 10.7dB and Ricean fading.