Nonlinear Feedback Control for General Acyclic Traffic Networks

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Abstract: This work is devoted to the construction of explicit feedback control laws for the robust global exponential stabilization of general uncertain discrete-time acyclic traffic networks. We consider discrete-time uncertain network models which satisfy very weak assumptions. The construction of the controllers is based on recently proposed vector-Lyapunov function criteria, as well as the fact that the network is acyclic. The latter requirement is necessary for the existence of a robust, global, exponential stabilizer of the desired uncongested equilibrium point of the network. An illustrative example demonstrates the applicability of the obtained results to realistic traffic flow networks.

Keywords: nonlinear systems, discrete-time systems, acyclic networks, traffic control.

1. INTRODUCTION

Networks are large-scale entities representing different types of physical or cyber-physical systems such as fluid flow networks, communication networks, smart grids and other. Particular emphasis is given in this study to traffic networks for which a plethora of diverse infrastructures can be addressed on the basis of a unifying modeling approach (see, for example, Coogan and Arcak (2014); Fermo and Tosin (2013)). More specifically, traffic networks can be modeled as urban road networks consisting of interconnected links which are modeled as store-and-forward components (Aboudolas et al. (2009)) or cell-transmission links (Buisson et al. (1996)); large urban networks consisting of smaller homogeneous sub-networks (Aboudolas and Geroliminis (2013)); freeway networks consisting of series of links, which are modeled, e.g., via general discretized LWR models (Lebacque (1996); Karafyllis et al. (2016)) or its simplified CTM (Cell Transmission Model) version (Daganzo (1995)); large mixed (corridor) networks consisting of urban and freeway links (Papageorgiou (1995)).

Recently, many researchers have addressed the stabilization of equilibrium points of large-scale discrete-time systems. However, the verification of stability for large-scale systems still remains a challenging problem on its own. To this purpose, many tools have been proposed in the literature such as vector-Lyapunov functions that are very useful for large-scale discrete-time systems. Sufficient stability conditions by means of vector-Lyapunov functions have been proposed by Haddad and Chellaboina (2008) (pages 792-798). In addition, small-gain conditions have been proposed by Liu et al. (2012), which can be expressed by means of a vector-Lyapunov function formulation (as shown by Karafyllis and Jiang (2011), Chapter 5). Recently, sufficient conditions have been provided by Karafyllis and Papageorgiou (2015) for the robust, global, exponential stability of nonlinear, large-scale, uncertain networks by means of vector-Lyapunov functions; results that can be easily applied to traffic networks.

Following the work by Kontorinaki et al. (2016), in this work, a general model for acyclic networks consisting of an arbitrary number of elementary components with constant turning and exit rates is presented. The components of the network can be interconnected to form any two-dimensional structure with no cycles for the overall network. Specific instances of the proposed general model result in traffic network structures and problems that can be considered as special cases of the proposed network model and include all the traffic network structures mentioned above. Based on this modeling framework, the results provided by Karafyllis and Papageorgiou (2015) are utilized for the developed uncertain models of acyclic networks. More specifically, this study provides a parameterized family of explicit feedback control laws which can robustly, globally, exponentially stabilize the desired Uncongested Equilibrium Point (UEP) of a given acyclic traffic network. The achieved stabilization is robust with respect to: i) any uncertainty related to the fundamental diagram of traffic flow; as well as ii) the overall uncertain nature of the developed model when congestion phenomena are present. In fact, in the latter case, the model which describes the time evolution of the network variables is almost completely uncertain (besides the requirement of known and constant turning and exit rates). The assumptions that surround the proposed methodology are weak enough to render the methodology applicable to other kinds of acyclic networks instead of traffic networks. We emphasize here that, as it is proved by Kontorinaki et al. (2016), the requirement regarding the absence of cycles inside the network is utterly necessary for the existence of a robust global exponential stabilizer of the UEP of the network. Note that, all the proofs of the obtained results described above are omitted (due to space limitations) and can be found in the paper by Kontorinaki et al. (2016). However, in this paper an additional result which provides sufficient conditions for the robust global exponential stability of the UEP for the open-loop system is also presented (see, Section 3, Corollary 1).

The structure of the present work is as follows. Section 2 includes the model derivation as well as the discussion on the properties and the consequences of the considered modeling framework, while the main results of this work are presented in Section 3. Section 4 presents an illustrative example for a traffic control problem in a freeway-to-freeway network, where the performance of the proposed methodology for the closed-loop system is also evaluated in case modeling and measurement errors are present. Finally the concluding remarks are given in Section 5.

Definitions and Notation: In this paper, we adopt the following notation and terminology:

- $\Re_+ := [0, +\infty)$. $\Re_+^n := (\Re_+)^n$. For every set S , $S^n = \underbrace{S \times ... \times S}_{n \text{ times}}$ for every positive integer n. For a set $S \subseteq \Re^n$, $int(S)$ denotes the interior of S (which may be empty).
- By $C^0(A;\Omega)$, we denote the class of continuous functions on $A \subseteq \Re^n$, which take values in $\Omega \subseteq \Re^m$. By $C^k(A;\Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \Re^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \Re^m$.
- Let $x, y \in \Re^n$. We say that $x \leq y$ if $(y - x) \in \Re_+^n$ and we say that $x < y$ if $(y - x) \in int(\Re_+^n)$. The transpose of $x \in \Re^n$ is denoted by x'. By $|x|$ we denote the Euclidean norm of $x \in \Re^n$. For every $x \in \Re$, $[x]$ denotes the integer part of $x \in \Re$.
- We denote by I the identity matrix and we denote by $1_{n\times n} \in \Re^{n\times n}$ the matrix for which every entry is equal to one. Moreover, $1_n = (1, ..., 1)' \in \Re^n$.
- The spectral radius of $\Delta \in \Re^{n\times n}$ is denoted by $\rho(\Delta)$. When all the entries of Δ are non-negative, then we say that Δ is non-negative and we write $\Delta \in \Re_+^{n\times n}$.
- We say that the matrix $\Delta \in \Re_+^{n\times n}$ is upper (lower) triangular if all the entries below (above) the main diagonal are zero. We say that the upper (lower) triangular matrix $\Delta \in \Re_+^{n\times n}$ is strictly upper (lower) triangular if all the entries of the main diagonal are zero. The diagonal entries of an upper (lower) triangular matrix $\Delta \in \Re_+^{n\times n}$ are the eigenvalues of $\Delta \in \Re_+^{n\times n}$.

Let $X \subseteq \Re^n$, $D \subseteq \Re^l$ be non-empty sets and consider the uncertain, discrete-time, dynamical system

$$z^+ = Z(d, z), z \in X, d \in D, \quad (1)$$

where $Z : D \times X \to X$ is a mapping. The variable $z \in X$ denotes the state of (1) while here (and throughout the paper) z^+ denotes the value of the state at the next time instant, i.e., (1) expresses the recursive relation $z(t+1) = Z(d(t), z(t))$. Let $z^* \in X$ be an equilibrium point of (1), i.e., $z^* \in X$ satisfies $z^* = Z(d, z^*)$ for all $d \in D$. Notice that the requirement $z^* = Z(d, z^*)$ for all $d \in D$ implies that $d \in D$ is a vanishing perturbation, i.e., a disturbance that does not change the position of the equilibrium point of the system. Next, we use the following definitions throughout the paper.

Definition 1: *A Trapping Region (TR) for system (1) is a set $\Omega \subseteq X$ for which there exists an integer $m \geq 0$ such that for every $z_0 \in X$, $\{d(t) \in D\}_{t=0}^{\infty}$, the solution $z(t)$ of (1) with initial condition $z(0) = z_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ satisfies $z(t) \in \Omega$ for all $t \geq m$.*

A nonlinear system with a TR is a system for which all solutions enter a specific set after an initial transient period. A direct consequence of Definition 1 is that every TR for (1) must contain all equilibrium points. We next define the robust, global exponential stability notions for (1).

Definition 2: *We say that $z^* \in X$ is Robustly Globally Exponentially Stable (RGES) for system (1) if there exist constants $M, \sigma > 0$ such that for every $z_0 \in X$ and for every sequence $\{d(t) \in D\}_{t=0}^{\infty}$ the solution $z(t)$ of (1) with initial condition $z(0) = z_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ (i.e., the solution that satisfies $z(t+1) = Z(d(t), z(t))$ for all $t \geq 0$ and $z(0) = z_0$) satisfies the inequality $|z(t) - z^*| \leq M exp(-\sigma t)|z_0 - z^*|$ for all $t \geq 0$.*

2. ACYCLIC NETWORKS WITH CONSTANT TURNING AND EXIT RATES

We consider a generic network which consists of n components (cells). This network may represent a traffic flow network, a fluid flow network or another kind of network. The density of the quantity characterizing each component of the network (e.g. density of vehicles, fluid mass etc.) at time $t \geq 0$ in component $i \in \{1, ..., n\}$ is denoted by $x_i(t)$. The outflow and the inflow of the component $i \in \{1, ..., n\}$ at time $t \geq 0$ are denoted by $F_{out,i} \geq 0$ and $F_{in,i} \geq 0$, respectively. Consequently, the conservation equation for each component $i \in \{1, ..., n\}$ is given by

$$x_i^+ = x_i - F_{out,i} + F_{in,i}, i \in \{1, ..., n\}, t \geq 0. \quad (2)$$

Each component of the network has storage capacity $a_i > 0$ ($i = 1, ..., n$). Let $S = [0, a_1] \times \cdots \times [0, a_n]$ be the state space, i.e., $x \in S$. Let $v_i \geq 0$ ($i = 1, ..., n$) denote the attempted inflow to component $i \in \{1, ..., n\}$ from the region out of the network and set $v = (v_1, ..., v_n)' \in \Re_+^n$. Our first assumption is dealing with the outflows. We assume that there exist functions $f_i : D \times [0, a_i] \to \Re_+$, $s_i : D \times S \times \Re_+^n \to [0, 1]$ with $f_i(d, x_i) \leq x_i$ for all $(d, x_i) \in D\times[0, a_i]$, where $D \subseteq \Re^l$ is a non-empty, compact set, so that:

$$F_{out,i} = s_i(d, x, v) f_i(d, x_i), \text{for } i = 1, ..., n. \quad (3)$$

In fact, the functions $f_i : D \times [0, a_i] \to \Re^n$ ($i = 1, ..., n$) denote the attempted outflow from the i-th cell, i.e., the outflow that will exit the cell if there is sufficient space in the downstream cells. Particularly, the functions $f_i : D \times [0, a_i] \to \Re^n$ ($i = 1, ..., n$) remind what in the specialized literature of Traffic Engineering is called the demand-part of the fundamental diagram of the i-th cell. The functions $s_i : D \times S \times \Re_+^n \to [0, 1]$ ($i = 1, ..., n$) are introduced in order to accommodate congestion phenomena. Next, we make the following assumption for the functions $f_i : D \times [0, a_i] \to \Re^n$ $(i = 1, ..., n)$:

(H1) *For each $d \in D$, the function $f_i(d, \cdot) : [0, a_i] \to \Re_+$ satisfies $0 < f_i(d, z) < z$ for all $z \in (0, a_i]$. There exists $\delta_i \in (0, a_i]$ such that for each $d \in D$, the function $f_i(d, \cdot)$ is continuous and increasing on $[0, \delta_i]$. Moreover, there exist constants $L_i \in (0, 1)$, $G_i \in (0, 1]$, $\tilde{\delta}_i \in (0, \delta_i]$ such that $|f_i(d, z) - f_i(d, y)| \geq L_i |z - y|$ for each $d \in D$ and $y, z \in [0, \tilde{\delta}_i]$ and $|f_i(d, z) - f_i(d, y)| \leq G_i |z - y|$ for each $d \in D$ and $y, z \in [0, \delta_i]$. Finally, there exists a constant $f_i^{min} > 0$ such that for each $d \in D$ it holds that $f_i(d, z) \geq f_i^{min}$ for all $z \in [\delta_i, a_i]$.*

Remark 1: Assumption (H1) is a technical assumption that allows a very general class of functions $f_i(d, \cdot) : [0, a_i] \to \Re_+$ to be taken into account. The implications of assumption (H1) are illustrated in Fig. 1. Assumption (H1) includes the basic properties of the so-called demand function (Lebacque (1996)) in the Godunov discretization; δ_i is the critical density, where $f_i(d, \cdot)$ achieves a maximum value (capacity flow). Notice that assumption (H1) includes the possibility to consider arbitrary functions $f_i(d, \cdot)$ for overcritical densities, i.e., when $x_i > \delta_i$ (discontinuous or decreasing or, even, increasing functions, see grey area in Fig. 1).

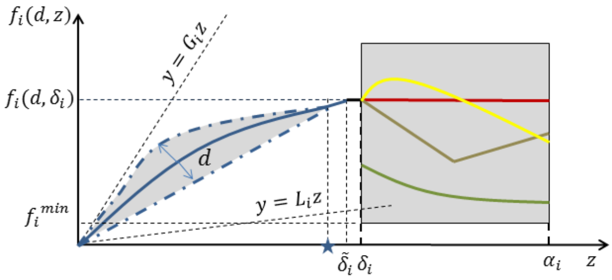

Fig. 1. Implications of Assumption (H1).

Our second assumption is dealing with the inflows. We assume that there exist functions $g_i \in C^0(D \times S; \Re_+)$, $w_i : D \times S \times \Re_+^n \to [0, 1]$ with $0 < g_i(d, x) \leq a_i - x_i$ for all $(d, x) \in D \times S$ with $x_i < a_i$ $(i = 1, ..., n)$, constants $p_{i,j} \geq 0$ with $p_{i,i} = 0$ $(i, j = 1, ..., n)$ and constants $Q_i \geq 0$ $(i = 1, ..., n)$, so that:

$$\sum_{j=1}^{n} p_{i,j} + Q_i = 1, \tag{4}$$

$$\begin{aligned} F_{in,i} = w_i(d, x, v) v_i + \sum_{j=1}^{n} p_{j,i} s_j(d, x, v) f_j(d, x_j) \\ \leq g_i(d, x), \text{ for all } i = 1, ..., n. \end{aligned} \tag{5}$$

$$\begin{aligned} &\text{if } v_i + \sum_{j=1}^{n} p_{j,i} f_j(d, x_j) \leq g_i(d, x), \text{for all } i = 1, ..., n \\ &\quad \text{then } w_i(d, x, v) = s_i(d, x, v) = 1, \text{for } i = 1, ..., n \end{aligned} \tag{6}$$

For the case of traffic networks, the functions $g_i : D \times S \to \Re_+$ remind what in the specialized literature of Traffic Engineering is called the supply function of the i-th cell. In addition, $p_{i,j}$ are turning rates and Q_i are exit rates. When $w_i(d, x, v) + \sum_{j=1}^{n} p_{j,i} s_j(d, x, v) \leq 1 + \sum_{j=1}^{n} p_{j,i}$ then we say that the i-th cell is *congested*. The functions $w_i : D \times S \times \Re_+^n \to [0, 1]$ and $s_i : D \times S \times \Re_+^n \to [0, 1]$ $(i = 1, ..., n)$ are introduced so that for each cell: (i) the actual inflow is always less than the supply (this is inequality (5)), and (ii) when the maximum value of all inflows can be accommodated then no congestion phenomena are present (this is implication (6)). Priority rules for each junction can be expressed by means of the functions $w_i : D \times S \times \Re_+^n \to [0, 1]$ and $s_i : D \times S \times \Re_+^n \to [0, 1]$ $(i = 1, ..., n)$.

For traffic flow networks, the supply function is usually given by the function $g_i(d, x) = \min(q_i, c_i(a_i - x_i))$, where $q_i > 0$ represents the maximum admissible inflow of the i-th cell and $c_i \in (0, 1]$ represents the normalized congestion wave speed. Then, the fundamental diagram of cell i is composed by the increasing function $f_i(d, x_i)$ for $x_i \in [0, \delta_i]$ and by the non-increasing function $g_i(d, x)$ for $x_i \in [\delta_i, a_i]$. Notice here that the uncertainty $d \in D$ has been introduced in order to accommodate the uncertain nature of the fundamental diagram.

Combining equations (2), (3) and (5) we obtain the following nonlinear uncertain discrete-time system:

$$\begin{aligned} x_i^+ = x_i + w_i(d, x, v) v_i - s_i(d, x, v) f_i(d, x_i) + \\ \sum_{j=1}^{n} p_{j,i} s_j(d, x, v) f_j(d, x_j), \end{aligned} \tag{7}$$

for $i = 1, ..., n$. Fig. 2 illustrates schematically the network described by the model (7). For physical reasons, we would expect a network of the form (7) under Assumption (H1) to satisfy the following three properties:

1) If the attempted external inflows $v_i \geq 0$ $(i = 1, ..., n)$ are small for a sufficiently large time period then the network densities will eventually be small.

2) If $x_i \neq 0$ for some $i = 1, ..., n$, then there is at least one non-zero outflow.

3) If the attempted external inflows $v_i \geq 0$ $(i = 1, ..., n)$ and the densities $x_i \geq 0$ $(i = 1, ..., n)$ are small, then no congestion phenomena are present in the network.

Indeed, consider a network with zero external inflows. If the network does not satisfy property 1 above then it is possible that the network retains a certain amount of density (i.e., the vehicles do not exit). The same situation would occur in the case where property 2 above does not hold. Of course, there are special cases (e.g. a gridlock around a cycle) where vehicles are trapped in the network and do not exit, but it is clear that in such situations one cannot deal with congestion phenomena via inflow control, i.e. by making the external inflows sufficiently small. Property 3 is another empirical fact that should be verified to enable inflow control: congestion phenomena are present only when the attempted external inflows $v_i \geq 0$ $(i = 1, ..., n)$ and the network densities $x_i \geq 0$ $(i = 1, ..., n)$ are sufficiently large. In the aim of guaranteeing

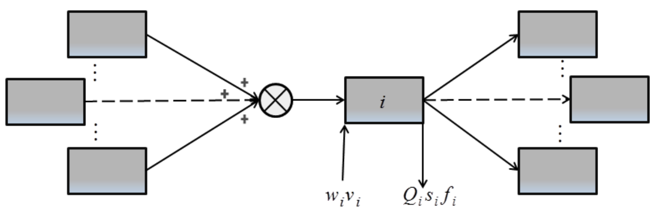

Fig. 2. Scheme of the network model.

that the considered network models actually possess the above properties, we consider only acyclic networks via the following assumption.

(H2) *The matrix $P = \{p_{i,j} : i, j = 1, ..., n\} \in [0,1]^{n\times n}$ which contains the turning rates of the acyclic network (7) is strictly upper triangular.*

Remark 2: From a graph-theoretic point of view, the vertices of a directed acyclic graph can admit a topological sorting (the starting endpoint of every edge occurs earlier in the ordering than the ending endpoint of the edge). Then, assigning the vertices of the graph to the cells of the network, for any given acyclic network, we are in a position to reorder the cells of the network into a topological sorting. The main consequence of this sorting is that the matrix $P = \{p_{i,j} : i, j = 1, ..., n\} \in [0,1]^{n\times n}$ containing the turning rates of the network becomes strictly upper triangular (Godsil and Royle (2013)).

The following technical lemmas are useful for the analysis of the acyclic networks.

Lemma 1: *For every non-negative, strictly upper triangular matrix P with $\sum_{j=1}^{n} p_{i,j} \le 1$ for all $i = 1, ..., n$, there exist positive constants $r_i > 0$ $(i = 1, ..., n)$, such that*

$$r_i > \sum_{j=1}^{n} r_j p_{i,j}, \text{for every } i = 1, ..., n. \tag{8}$$

Lemma 2: *Let $L_i \in (0,1)$ and $G_i \in (0,1]$ with $L_i \le G_i$, for $i = 1, ..., n$, be constants and let P be a non-negative, strictly upper triangular matrix with $\sum_{j=1}^{n} p_{i,j} \le 1$ for $i = 1, ..., n$. Then the matrix $I + P' diag(G) - diag(L)$ is a lower triangular matrix with $\rho(I + P' diag(G) - diag(L)) < 1$, where $G = (G_1, ..., G_n)$ and $L = (L_1, ..., L_n)$.*

The following assumption is a technical assumption, which is related to Property 2 above.

(H3) *There exist functions $\tilde{s}_i \in C^0(D \times S \times \Re_+^n; [0,1])$ with $s_i(d, x, v) \ge \tilde{s}_i(d, x, v)$ for all $(d, x, v) \in D \times S \times \Re_+^n$, and constants $v_i^{max} > 0$ $(i = 1, ..., n)$ such that the following implication holds:*

$$\begin{gathered}\text{if } x_i \tilde{s}_i(d, x, v) = 0 \text{ and } v_i < v_i^{max}, i = 1, ..., n \\ \text{then } x = 0.\end{gathered} \tag{9}$$

Remark 3: Assumption (H3) guarantees that the functions $s_i : D \times S \times \Re_+^n \to [0,1]$, which have been introduced in model (7) in order to accommodate congestion phenomena, should admit a continuous and positive definite lower bound for some $i = 1, ..., n$. Implication (9) guarantees that if the outflow of every cell of the network is zero then the density of every cell should be zero (Property 2).

We next show that assumption (H3) in conjunction with assumption (H1) and (H2) guarantees that the network (7) satisfies Properties 1, 2 above.

Proposition 1: *Consider the network (7) under assumptions (H1), (H2), (H3). Then for every constants $r_i > 0$ $(i = 1, ..., n)$ satisfying (8) and for every family of constants $\tilde{\epsilon}_i \in (0, \min(v_i^{max}, \min(g_i(d, 0) : d \in D)))$ $(i = 1, ..., n)$, there exists a constant $C > 0$ such that*

$$\left(\sum_{i=1}^{n} r_i x_i\right)^+ \le (1 - C)\sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{n} r_i v_i, \tag{10}$$

for all $(d, x) \in D \times S$ and for all $v_i \ge 0$ with $v_i \le \min(v_i^{max}, \min(g_i(d, 0) : d \in D)) - \tilde{\epsilon}_i$ $(i = 1, ..., n)$.

Inequality (10) and induction allows us to show that for every $\omega > 0$ and for sufficiently small external inflows ($v_i(t) \ge 0$ with $v_i(t) \le \min(v_i^{max}, \min(g_i(d, 0) : d \in D)) - \tilde{\epsilon}_i$ for all $t \ge 0$) there exists $T > 0$ sufficiently large such that the following estimate holds for all $t \ge T$ for the solution of (7), for every initial condition $x(0) \in S$ and for every input $\{d(t) \in D\}_{t=0}^{\infty}$:

$$\sum_{i=1}^{n} r_i x_i(t) \le \omega + C^{-1} \max_{i=1,...,n} (sup\{v_i(t) : t \ge 0\}) \sum_{i=1}^{n} r_i.$$

The above inequality shows that if the attempted external inflows $v_i \ge 0$ $(i = 1, ..., n)$ are small for a sufficiently large time period then the network densities will eventually be small. This is Property 1 stated above. Property 2 above is a direct consequence of (3), (9) and the fact that $f_i(d, x_i) = 0 \Leftrightarrow x_i = 0$ (a consequence of Assumption (H1)). Property 3 is a direct consequence of the following assumption and (6).

(H4) *There exist constants $\mu_i \in (0, \tilde{\delta}_i)$, $v_i^{max} > 0$ $(i = 1, ..., n)$, such that*

$$v_i^{max} + \sum_{j=1}^{n} p_{j,i} f_j(d, x_j) \le g_i(d, x), \tag{11}$$

for all $i = 1, ..., n$, $(d, x) \in D \times S$ with $x \le \mu$, where $\mu = (\mu_1, ..., .\mu_n)'$.

Remark 4: Assumption (H4) is a reasonable assumption: if the network densities are small (below a critical value, here denoted by μ_i) and the attempted external inflows are small (below a given v_i^{max}), then the total attempted inflow should be accommodated by the i-th cell.

Assumptions (H1), (H2), (H3) and (H4) have important consequences; some of them have been already discussed while the rest are presented in the next section. Those assumptions may fit to many kinds of networks of the form (7). In particular, for freeway traffic flow networks the aforementioned assumptions are relatively mild. It is shown by Kontorinaki et al. (2016) that the freeway models considered by Karafyllis et al. (2016) and the corresponding related assumptions are indeed special cases of the model (7) and the assumptions (H1), (H2), (H3) and (H4) respectively.

3. MAIN RESULT

Consider a network of the form (7) under assumptions (H1), (H2), (H3), (H4). We next assume the existence of a point $x^* = (x_1^*, ..., x_n^*)' \in S$ and a vector $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$ with $x_i^* \in (0, \mu_i)$ and $v_i^* < v_i^{max}$, for $i = 1, ..., n$, that satisfy the following equations:

$$f_i(d, x_i^*) = v_i^* + \sum_{j=1}^{n} p_{j,i} f_j(d, x_j^*), \tag{12}$$

for all $i = 1, ..., n$ and $d \in D$. Since $x_i^*(0, \mu_i)$, $v_i^* < v_i^{max}$ it follows from (11) that the following inequalities hold:

$$v_i^* + \sum_{j=1}^{n} p_{j,i} f_j(d, x_j^*) < g_i(d, x^*), \tag{13}$$

for all $i = 1, ..., n$ and $d \in D$. The point $x^* = (x_1^*, ..., x_n^*)' \in S$ is called the UEP of the network corresponding to the vector of external inflows $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$. Notice that the input $d \in D$ is a vanishing perturbation for system (7) with $v(t) \equiv v^*$. This is also illustrated in Fig. 1, which shows that the input $d \in D$ does not change the position of the equilibrium point (denoted by a star).

We next assume that some of the external inflows may be controlled. Let $b \in \Re_+^n$ be a vector with $b \le v^*$, let $K \in \Re_+^{n\times n}$ be a non-negative, constant matrix and let $\tau > 0$ be a constant. We set:

$$v = v^* - diag(v^* - b)\Big(1_n - h\big(1_n - \tau^{-1}Kh(x - x^*)\big)\Big), \quad (14)$$

where $h : \Re^n \to \Re_+^n$ is the mapping defined by

$$h(x) = \big(\max(0, x_1), ..., \max(0, x_n)\big)' \in \Re_+^n, \quad (15)$$

for all $x \in \Re^n$. Notice that if $b_i = v_i^*$ for some $i \in \{1, ..., n\}$ then it follows from (14) that $v_i = v_i^*$, i.e., the external inflow v_i is uncontrolled. Therefore, by assuming (14), we have taken into account all possible cases for the control of external inflows. The following theorem shows that the UEP can be robustly, globally, exponentially stabilized by the continuous feedback law (14), which regulates certain or all the external inflows.

Theorem 1: *Consider the network (7) under assumptions (H1), (H2), (H3), (H4). Assume the existence of a point* $x^* = (x_1^*, ..., x_n^*)' \in S$ *with* $x_i^* \in (0, \mu_i)$ *and a vector* $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$ *with* $v_i^* \le \min(v_i^{max}, \min(g_i(d, 0) : d \in D))$*, for* $i = 1, ..., n$*, that satisfy equations (12). Then there exists an index set* $R \subseteq \big\{i \in \{1, ..., n\} : v_i^* > 0\big\}$*, a matrix* $K \in \Re_+^{n\times n}$ *and a vector* $b \in \Re_+^n$ *with* $0 < b_i < v_i^*$ *for* $i \in R$*,* $b_i = v_i^*$ *for* $i \notin R$ *such that for every* $\tau \in (0, 1)$*,* $x^* = (x_1^*, ..., x_n^*)' \in S$ *is RGES for the closed-loop system (7) with (14).*

Theorem 1 is an existence result. However, its proof is constructive and provides formulae (or sufficient conditions) for all constants and for the index set R (see Kontorinaki et al. (2016)). Notice that the index set R is the set of all inflows that must be controlled in order to be able to guarantee that the UEP is RGES. The importance of Theorem 1 lies on the following facts:

a) It provides a family of robust, global, exponential stabilizers (parameterized by $\tau \in (0, 1)$) and an explicit feedback law (formula (14)).

b) The achieved stabilization is robust with respect to:

(i) The uncertain nature (introduced by $d \in D$) of the fundamental diagram of traffic flow (by considering uncertain demand and supply functions, $f_i(d, x_i)$ and $g_i(d, x)$ respectively).

(ii) The overall uncertain nature of the model (7) when congestion phenomena are present (by considering uncertain functions $s_i(d, \cdot, \cdot)$ and $w_i(d, \cdot, \cdot)$, with respect to $d \in D$).

The main idea behind the proof of Theorem 1 is the construction of a vector Lyapunov function for the closed-loop system. The construction of the vector Lyapunov function is based on the existence of a TR, Ω, for the system (7) in which no congestion phenomena are present. The appropriate selection of the gain matrix $K \in \Re_+^{n\times n}$ in (14) forces the selected control action to lead the state in the set Ω. In other words, the control action will first eliminate all congestion phenomena and then will drive the state to the desired equilibrium.

The following proposition shows the existence of a positively invariant region for the network (7).

Proposition 2: *Consider the network (7) under assumptions (H1), (H2), (H3), (H4). Assume the existence of a point* $x^* = (x_1^*, ..., x_n^*)' \in S$ *with* $x_i^* \in (0, \mu_i)$*, and a vector* $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$ *with* $v_i^* < v_i^{max}$*, for* $i = 1, ..., n$*, that satisfy equations (12). Then there exist constants* $\beta_i \in (x_i^*, \mu_i]$ $(i = 1, ..., n)$ *such that for every* $b \in \Re_+^n$ *with* $b \le v^*$*,* $K \in \Re_+^{n\times n}$ *and* $\tau > 0$*, it holds that*

$$x \in \Omega, d \in D \Rightarrow x^+ \in \Omega \quad (16)$$

where $\Omega = [0, \beta_1,] \times \cdots \times [0, \beta_n]$ *and* x^+ *is given by (7) with (14).*

Implication (16) shows that $\Omega \subset S$ is a positively invariant region for inputs that satisfy $d(t) \in D$ and $0 \le v(t) \le v^* - diag(v^* - b)\Big(1_n - h\big(1_n - \tau^{-1}Kh(x(t) - x^*)\big)\Big)$ for all $t \ge 0$. It should be noticed that $x^* \in int(\Omega)$, i.e., the UEP is in the interior of the positively invariant region. In order to study the stability properties of the UEP of the network (7), we need the following technical lemmas.

Lemma 3: *Consider the network (7) under assumptions (H1), (H2), (H3), (H4). Assume the existence of a point* $x^* = (x_1^*, ..., x_n^*)' \in S$ *with* $x_i^* \in (0, \mu_i)$ *and a vector* $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$ *with* $v_i^* < v_i^{max}$*, for* $i = 1, ..., n$*, that satisfy equations (12). Then there exist constants* $\beta_i \in (x_i^*, \mu_i]$ $(i = 1, ..., n)$ *such that for every* $b \in \Re_+^n$ *with* $b \le v^*$*,* $K \in \Re_+^{n\times n}$ *and* $\tau > 0$*, implication (16) holds and such that*

$$\begin{gathered} x \in \Omega, d \in D \Rightarrow \\ h(x^+ - x^*) \le \big(I + P'diag(G) - diag(L)\big)h(x - x^*) \end{gathered} \quad (17)$$

$$\begin{gathered} x \in \Omega, d \in D \Rightarrow \\ h(x^* - x^+) \le \big(I + P'diag(G) - diag(L)\big)h(x^* - x) \\ + diag(v^* - b)\tau^{-1}Kh(x - x^*), \end{gathered} \quad (18)$$

where $\Omega = [0, \beta_1,] \times \cdots \times [0, \beta_n]$*,* $h : \Re^n \to \Re_+^n$ *is the mapping defined by (15),* $L = (L_1, ..., L_n)' \in \Re^n$*,* $G = (G_1, ..., G_n)' \in \Re^n$*,* $P = \{p_{i,j} : i, j = 1, ..., n\}$ *and* x^+ *is given by (7) with (14).*

Lemma 4: *Consider the network (7) under assumptions (H1), (H2), (H3), (H4). Assume the existence of a point* $x^* = (x_1^*, ..., x_n^*)' \in S$ *with* $x_i^* \in (0, \mu_i)$ *and a vector* $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$ *with* $v_i^* < v_i^{max}$*, for* $i = 1, ..., n$*, that satisfy equations (12). Then there exist constants* $\beta_i \in (x_i^*, \mu_i]$ $(i = 1, ..., n)$ *such that for every* $b \in \Re_+^n$ *with* $b \le v^*$*,* $K \in \Re_+^{n\times n}$ *and* $\tau > 0$ *implications (16), (17), (18) hold and there exists a constant* $M > 0$ *(depending on* $b \in \Re_+^n$*,* $K \in \Re_+^{n\times n}$ *and* $\tau > 0$*), which satisfies the following property*

$$x \in S, d \in D \Rightarrow |x^+ - x^*| \le M|x - x^*|, \quad (19)$$

where x^+ *is given by (7) with (14).*

We are now ready to state the following corollary which provides sufficient conditions for the robust, global exponential stability of the UEP for the open-loop system (7) with $v = v^*$. The sufficient conditions are given by means of the selection of UEP. The proof of the corollary below is given in the Appendix.

Corollary 1: *Consider the network (7) under assumptions (H1), (H2), (H3), (H4). Assume the existence of a point* $x^* = (x_1^*, ..., x_n^*)' \in S$ *with* $x_i^* \in (0, \mu_i)$ *and a vector* $v^* = (v_1^*, ..., v_n^*)' \in \Re_+^n$ *with* $v_i^* \leq \min(v_i^{max}, \min(g_i(d, 0) : d \in D))$, *for* $i = 1, ..., n$, *that satisfy equations (12). Let* $r = (r_1, ..., r_n)' \in int(\Re_+^n)$ *be a vector of constants satisfying (8) and let* $C > 0$ *be the corresponding constant for which inequality (10) holds for all* $(d, x) \in D \times S$ *and for* $v_i = v_i^*$ $(i = 1, ..., n)$. *Assume that*

$$r'v^* \leq C \min_{i=1,...,n} (r_i x_i^*). \tag{20}$$

Then the equilibrium point $x^* = (x_1^*, ..., x_n^*)' \in S$ *is RGES for the open-loop system (7) with* $v = v^*$.

4. ILLUSTRATIVE EXAMPLE

Consider a 3-lane freeway-to-freeway traffic network of the form (7) with $n = 8$ cells. The traffic network consists of two smaller freeways, 2 km each; the first is composed by the cells $i = 1, 2, 3, 4$, and the second is composed by the cells $i = 5, 6, 7, 8$ (Fig. 3). The cells are homogeneous, each cell being 0.5 km in length. The whole network admits two external inflows; one external inflow at the upstream boundary of the first cell and one external inflow at the upstream boundary of the fifth cell, while there are no intermediate external inflows ($v_i = 0, i \neq 1, 5$ and $v_1, v_5 \neq 0$). At the end of the first freeway (4^{th} cell) there is an off-ramp which becomes an on-ramp for the second freeway at the upstream boundary of the 7^{th} cell (Fig. 3). According to this configuration, the exit and turning rates of the freeway are $Q_i = 0$ for $i \neq 4, 8$, $Q_4 = 0.5$, $Q_8 = 1$, $p_{i,j} = 1$ for $j = i + 1$ and $i \neq 4$, $p_{4,7} = 0.5$ and $p_{i,j} = 0$ for all other cases $(i, j = 1, ..., 8)$. Consequently, the only control possibilities are the inflows v_1, v_5. It should be noted here that the 7^{th} cell is a bottleneck for the overall network due to the ramp that joins both freeways. Congestion may be created in the 7^{th} cell, due to high on-ramp demand from the 1^{st} and the 5^{th} cells, and spill back to both freeways depending on the priority rules.

All the following simulation tests have been conducted using constant priority rules for the junctions. More specifically, the functions $s_i(d, x, v)$ $(i = 1, ..., 8)$ have been defined so as to incorporate into the model (7) a full priority rate for the external inflows and the mainstream flow coming from the 6^{th} cell against the mainstream flow coming from the 4^{th} cell. Moreover, the demand and the supply functions have been defined so as to reflect the uncertainty, d, derived from the fundamental diagram of traffic flow. More specifically, we assume that the demand functions are given as a convex combination of several functions ϕ_i (e.g., linear or quadratic) (for $i = 1, ..., m$, where $m = 1, 2, ...$), satisfying assumption (H1) and guaranteeing that the uncertainty d is a vanishing perturbation

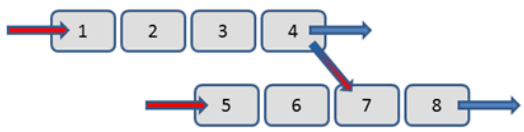

Fig. 3. The scheme of the freeway-to-freeway network.

for the system (7). Fig. 4 illustrates the representation of the above specifications for the demand and the supply functions. The grey area in Fig. 4 represents any possible demand and supply functions. For the exact specification of the functions $s_i(d, x, v)$, the demand and the supply functions see Kontorinaki et al. (2016).

Assuming that the simulation time step is $T = 15$ s, we can apply appropriate transformations to common traffic units. All flows and densities are measured in [veh], however, transformations in common traffic units are given for the most critical variables wherever it is needed.

Each cell has the same critical density $\delta_i = 55 + 2\epsilon$ [veh] with $\epsilon = 10^{-5}$ (corresponding to 36.7 [veh/km/lane]) and the same jam density $a_i = 170$ [veh] (corresponding to 113.3 [veh/km/lane]), for $i = 1, ..., 8$. Furthermore, the considered supply functions yield to a congestion wave speed within approximately 26 to 36 [km/h] and a maximum inflow approximately between 2000 to 2750 [veh/h/lane]. For the overall system (7), the uncertainty $d(t) = (d_1(t), ..., d_4(t)) \in D$ is a time-varying parameter taking values from a uniform distribution within $D = [0, 1]^3 \times [0.22, 0.3]$ (d_i, for $i = 1, 2, 3$, involve within the uncertainty derived by the demand functions and d_4 involves within the uncertainty derived by the supply functions). It can be verified (see Kontorinaki et al. (2016)) that Assumptions (H1), (H2), (H3) and (H4) are satisfied for the selected modeling framework.

Thus, in this example, we have that $R = \{1, 5\}$. Our goal is to globally exponentially stabilize the system at an UEP which is as close as possible to the critical density (due to the fact that the flow value at the critical density is the largest). Equation (12) and inequality (13) are satisfied by selecting $v^* = (25, 0, 0, 0, 12.5, 0, 0, 0)$ and $x^* = (55, 55, 55, 55, 27.5, 27.5, 55, 55)$. The above UEP is not open-loop globally exponentially stable due to the existence of additional (congested) equilibria. This is shown in Fig. 5(a), where the solution of the open-loop system, with constant inflows $v = v^* = (25, 0, 0, 0, 12.5, 0, 0, 0)$, constant $d(t) \equiv (1, 0, 0, 0.5)$ and $x_0 = (a_1, ..., a_8)$, is attracted by a congested equilibrium leading to outflow, which is 7.4 [veh] lower than the capacity flow of the 4^{th} cell and 4.9 [veh] lower than the capacity flow of the 8^{th} cell. Therefore, if the objective is the operation of

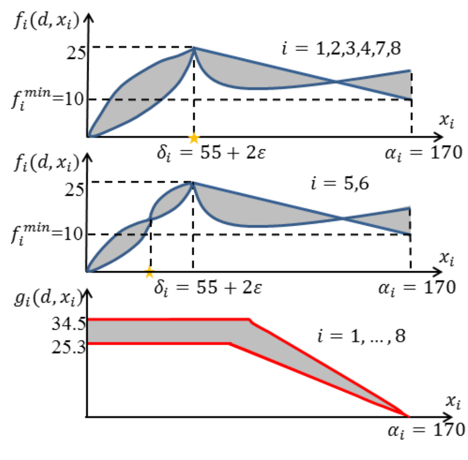

Fig. 4. Specification of the parameters of the demand and the supply functions of every cell.

the freeway with largest possible outflow, then a control strategy will be needed. However, the present simulation study (by considering Corollary 1) indicated that there exists v^*, appropriately small, so that the UEP is RGES for the open-loop system (7) with $v = v^*$. For this example there are 2 controllable inflows, thus, the UEP is RGES for $0 \le v \le v^* = (15, 0, 0, 0, 12.5, 0, 0, 0)$ but also for $0 \le v \le v^* = (20, 0, 0, 0, 10, 0, 0, 0)$.

We constructed the matrix K and the constants τ and b using the sufficient conditions provided from the proofs of the technical lemmas and propositions (Kontorinaki et al. (2016)). Thus, we selected $K = 0.004 \cdot 1_{n\times n}$, $\tau = 1/4$ and $b_1 = b_5 = 0.5$ ($b_i = 0$ for $i \ne 1, 5$) which satisfy those conditions and allow for a good control performance with respect to overshooting effects. We performed a simulation study with respect to various initial conditions for the closed-loop system. The indicative Fig. 5(b) shows the response of the density of every cell for the closed-loop system (7), (14) with constant $d(t) \equiv (1, 0, 0, 0.5)$ and $x_0 = (a_1, ..., a_n)$. Furthermore, Fig. 5(c) illustrates the evolution of the Euclidean norm of the deviation of the state from the UEP for the open-loop system (6) with $v = v^*$ (red color) and for the closed-loop system (6) with (13) (blue color) for $x_0 = (a_1, ..., a_n)$. Both Fig. 5(b) and Fig. 5(c) indicate that the feedback regulator respond very satisfactorily in this test exhibiting a fast convergence to the UEP.

We test also the performance of the proposed control scheme with respect to measurement errors. For this test, the measurements that feed the feedback law (14) are given by $\hat{x}(t) = \hat{P}\big((1 - \hat{d}(t)A)x(t)\big)$, where $\hat{P}$ is a projection operator within the set S, $\hat{d}(t)$ is a uniformly distributed function within $[0, 1]$ and $A = 0.2$. The selected criterion, which reflects the performance of the controller, is the total amount of Vehicles Exiting the Network during the simulation horizon (VEN_h), i.e.,

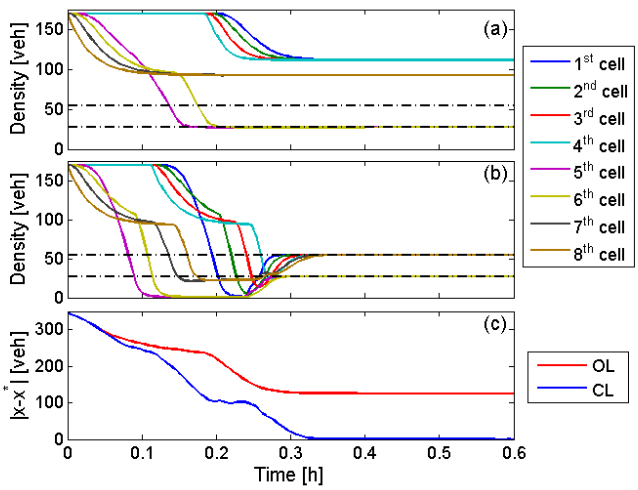

Fig. 5. The response of the densities (a) for the open-loop system (7) with $v = v^*$, (b) for the closed-loop system (7) with (14) and (c) the evolution of the Euclidean norm of the deviation of the solution $x(t)$ from the UEP, i.e. $|x(t) - x^*|$.

$$VEN_h = \sum_{k=0}^{h} \big(Q_4 s_4(d(k), x(k), v(k)) f_4(d(k), x_4(k)) + f_8(d(k), x_8(k))\big), \tag{21}$$

where $h = KT$, with $K = 1, 2, ...$, corresponds to the simulation time horizon (in [hours]). The goal of any control strategy is to maximize this criterion which is also equivalent to the minimization of the total time that vehicles spent within the traffic network.

Table 1 shows the values of $VEN_{0.6}$ (in [veh]) for the open-loop system (7) with $v = v^*$ (OL), for the closed-loop system (7) with (14) (CL) and for the closed-loop system (7) with (14) with measurement errors (CL-ME). The results have been derived using different initial conditions for two cases: i) constant $d \in D$ and ii) time-varying $d \in D$. The utilized initial conditions are $x_{0_1} = (a_1, ..., a_8)$, $x_{0_2} = (150, 140, 60, 120, 120, 100, 160, 130)$, $x_{0_3} = (50, 50, 50, 50, 27, 27, 80, 60)$ and $x_{0_4} = (60, 65, 60, 65, 20, 25, 60, 65)$. Table 1 indicates that in all cases the selected criterion is significantly higher for the closed-loop system (6) with and without measurement errors comparing with the no-control case. For the case where $d \in D$ is constant, the average amelioration of the criterion values (CL and CL-ME) for most of the cases is 25% comparing with the no-control case; while, for time-varying $d \in D$, the average amelioration becomes even higher, i.e., 42%.

Table 1. $VEN_{0.6}$ (in [veh]).

	OL	CL	CL-ME
Constant d			
x_{0_1}	3555	4165	4220
x_{0_2}	3640	4570	4630
x_{0_3}	4035	5255	5260
x_{0_4}	4190	5290	5275
Time Varying d			
x_{0_1}	3075	4095	4200
x_{0_2}	3140	4540	4560
x_{0_3}	3475	5220	5180
x_{0_4}	3710	5265	5215

5. CONCLUDING REMARKS

This work provided a rigorous methodology for the construction of a parameterized family of explicit feedback laws that guarantee the RGES of the UEP for general nonlinear uncertain discrete-time acyclic traffic networks.The construction of the global exponential feedback stabilizer is based on a vector Lyapunov function approach as well as certain important properties of acyclic traffic networks. Moreover, this work provided sufficient conditions for the RGES of the UEP for the corresponding open-loop system. The applicability and the efficacy of the obtained results to real control problems is demonstrated by conducting a simulation study, using a freeway-to-freeway network, with respect to various initial conditions as well as the presence of measurement and modeling errors.

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Appendix A. PROOF OF COROLLARY 1

In order to show that the UEP $x^* = (x_1^*, ..., x_n^*)' \in S$ is RGES for the open-loop system (7) with $v = v^*$ so that inequality (20) holds, we will use Theorem 2.3 in (Karafyllis and Papageorgiou (2015)). To do so, first we will show that the set $\Omega = [0, \beta_1] \times \cdots \times [0, \beta_n]$ is a TR for the open-loop system (7) with $v = v^*$. For the above implication, it suffices to show that for every $x_0 \in S$ and $\{d(t) \in D\}_{t=0}^{\infty}$ the solution $x(t)$ of the open-loop system (7) with $v = v^*$ and initial condition $x(0) = x_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ satisfies $x(t) \in \Omega$ for all $t \geq m$. We select $m \in \{1, 2, ..., \}$ so that

$$m := \left\lceil \frac{ln\big(C \min_{i=1,...,n}(r_i\beta_i) - r'v^*\big) - ln(Cr'\alpha)}{ln(1-C)} \right\rceil + 1 \tag{A.1}$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in int(\Re_+^n)$ and we proceed by contradiction. Suppose that there exists $x_0 \in S$, $\{d(t) \in D\}_{t=0}^{\infty}$ such that the solution $x(t)$ of the open-loop system (7) with $v = v^*$ and initial condition $x(0) = x_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ satisfies $x(t) \notin \Omega$ for certain $t \geq m$. Since the set $\Omega = [0, \beta_1] \times \cdots \times [0, \beta_n]$ is positively invariant (a direct consequence of (16) and the fact that $b = v^*$), it follows that $x(q) \notin \Omega$ for all $q = 0, 1, ..., m$. Define

$$N(q) := r'x(q) \tag{A.2}$$

and notice that (10) with $v = v^*$ implies the following estimate for all $q = 0, 1, ..., m$:

$$N(q+1) \leq (1-C)N(q) + r'v^*. \tag{A.3}$$

Inequality (A.3) implies the following estimate for all $q = 0, 1, ..., m+1$:

$$N(q) \leq (1-C)^q N(0) + C^{-1} r'v^* \big(1 - (1-C)^q\big). \tag{A.4}$$

Since $N(0) = r'x(0) = r'x_0 \leq r'a$ for all $x_0 \in S$, we obtain from (A.4) for all $q = 0, 1, ..., m+1$:

$$N(q) \leq (1-C)^q r'a + C^{-1} r'v^*. \tag{A.5}$$

Inequality (A.5) in conjunction with definition (A.1) implies that $N(m) \leq \min_{i=1,...,n}(r_i\beta_i)$, which combined with definition (A.2) shows that $x(m) \in \Omega$, a contradiction. Thus, Ω is a TR for the open-loop system (7) with $v = v^*$.

We next define

$$\begin{aligned} V_i(x) &:= \max(0, x_i - x_i^*), \text{ for } i = 1, ..., n, \\ V_i(x) &:= \max(0, x_i^* - x_i), \text{ for } i = n+1, ..., 2n. \end{aligned} \tag{A.6}$$

Notice that inequalities

$$\frac{1}{\sqrt{n}}|x - x^*| \leq \max_{i=1,...,2n} V_i(x) = \max_{i=1,...,n} |x_i - x_i^*| \leq |x - x^*| \tag{A.7}$$

hold for all $x \in S$. Applying, Lemma 3 with $b = v^*$ in conjuction with definition (A.6) we get the vector inequality

$$V(x^+) \leq \Gamma V(x) \tag{A.8}$$

for all $(d, x) \in D \times \Omega$, where $V(x) = \big(V_1(x), ..., V_{2n}\big)' \in \Re^{2n}$ and

$$\Gamma := \begin{bmatrix} I + P'diag(G) - diag(L) & 0 \\ 0 & I + P'diag(G) - diag(L) \end{bmatrix} \tag{A.9}$$

Lemma 2 guarantees that the spectral radius of the matrix $I + P'diag(G) - diag(L)$ is less than one. It follows that the spectral radius of Γ as defined by (A.9), is less than one since $I + P'diag(G) - diag(L)$ and therefore Γ are lower triangular matrices. Applying Theorem 2.3 in (Karafyllis and Papageorgiou (2015)) in conjuction with (A.8) and Lemma 4, we conclude that the UEP is RGES for the open-loop system (7) with $v = v^*$. The proof is complete.