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Polynomial-Complexity Computation of the M-phase Vector that Maximizes a Rank-Deficient Quadratic Form

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Abstract

The maximization of a positive (semi)definite complex quadratic form over a finite alphabet is $\mathcal{NP}$-hard and achieved through exhaustive search when the form has full rank. However, if the form is rank-deficient, the optimal solution can be computed with only polynomial complexity in the length N of the maximizing vector. In this work, we consider the general case of a rank-D positive (semi)definite complex quadratic form and develop a method that maximizes the form with respect to a M-phase vector with polynomial complexity. Our method utilizes auxiliary hyperspherical coordinates and partitions the multidimensional space into a polynomial-size set of regions, where each region corresponds to a distinct M-phase vector. The vector that maximizes the rank-D quadratic form is shown to belong to the polynomial-size set of vectors. Therefore, the proposed method efficiently reduces the size of the feasible set from exponential to polynomial. We also develop an algorithm that constructs the polynomial-size candidate set in polynomial time and observe that it is fully parallelizable and rank-scalable.

I. INTRODUCTION

The problem of unconstrained complex quadratic maximization over a finite alphabet captures many problems that are of interest to the communications and signal processing community. Many recent developments on the semidefinite relaxation (SDR) technique have indicated that SDR is capable of providing near optimal (and sometimes accurate) approximations in polynomial time [1]. Although SDR algorithm has a theoretical guarantee that the solution of the quadratic problem has, at worst, moderate approximation accuracy[1] [2], in most of the times it remains an approximation algorithm that does not guarantee the computation of the optimal solution.

Recently, the authors in [7], based on the seminal work of Goemans and Williamson [8], [9], studied some approximation algorithms for the class of complex quadratic optimization problems with discrete decision variables: maximize $\mathbf{z}^{\mathcal{H}}\mathbf{Q}\mathbf{z}$ s.t. $z_k \in \{1, \omega, \dots, \omega^{M-1}\},\ k = 1, \dots, n$ where $M \geq 2$ and $\omega = e^{j\frac{2\pi}{M}}$. In this work, they prove that the model presented in [9] for MAX-3-CUT ($M = 3$) turns out to be a special case of the general model proposed in [7]. Interestingly, they prove that for $\forall M \geq 2$ the problem remains $\mathcal{NP}$-hard.

Interestingly, it has been recently proven that the maximization of a quadratic form with a binary vector argument[2] is no longer $\mathcal{NP}$-hard if the rank of the form is not a function of the problem size [3], [4].[3] Specifically, based

[1]Practically, the performance of this algorithm is substantially better than that of the worst case.

[2]In this work, a vector is called binary if and only if each element of it equals $+1$ or -1. Contrarily, if each element of it equals 0 or 1, then the vector is said to belong to the $0/1$ field.

[3]An obvious example is the maximization of the rank-1 quadratic form where the optimal argument vector is provided by the hard-limiter output when applied to the maximum-eigenvalue eigenvector of the matrix parameter.

on the equivalence of the maximization of any rank-deficient quadratic form over the binary field with the rank-deficient maximization over the $0/1$ field [3], it is proved that the latter can be computed optimally and efficiently in polynomial time through a variety of computational geometry (CG) algorithms, such as the incremental algorithm for cell enumeration in arrangements [5] and the reverse search [6].[4] However, the aforementioned CG algorithms are studied only for binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) modulations.

From a different perspective, the authors in [10] present an algorithm for the efficient computation of the binary vector of length N that maximizes a rank-D quadratic form with polynomial complexity where $D \leq N$. The algorithm uses $D-1$ auxiliary hyperspherical coordinates that partition the multidimensional hypercube into distinct regions of polynomial size, each of which corresponds to a different candidate vector. Therefore, the method in [10] reduces the size of the candidate vector set from exponential to polynomial.

In the present work, we modify the algorithm in [10] to serve complex-domain optimization problems such as, for example, maximum-likelihood sequence detection (MLSD) of uncoded input sequence in the presence of frequency non-selective/time selective fading with channel state information (CSI) [12], maximum-likelihood (ML) noncoherent single-input multiple-output (SIMO) detection of arbitrary-order M-PSK [11], multiuser detection in M-PSK code-division multiple-access (CDMA) [13]-[16], hard-ML M-PSK detection for multiple-input, multiple-output (MIMO) channels [17]-[19], blind ML detection of orthogonal space-time block codes (OSTBC) [20] and quadrature-amplitude modulation (QAM) and PSK codebooks for limited multiple-input multiple-output (MIMO) beamforming [21]. Specifically, we introduce as many auxiliary hyperspherical coordinates as twice the rank of the problem reduced by one and partition the multidimensional space into a polynomial-size set of distinct regions, each of which is associated with a different M-phase vector. The proposed algorithm turns out to be time and memory efficient, fully parallelizable and rank-scalable.

The paper is organized as follows. In the following section, we present the original problem and its characteristics. Section III is devoted to the theoretical developments of the proposed algorithm for the maximization of a rank-deficient quadratic form with a M-phase vector argument. The implementation of the proposed algorithm is discussed in more detail in Section IV. Concluding remarks are found in Section V.

Notation: Upper and lower case bold symbols denote matrices and column vectors, respectively; x_i denotes the i-th element of vector $\mathbf{x}$ and $A_{i,j}$ the (i,j)-th entry of matrix $\mathbf{A}$; $\mathbf{A}_{i:j,k:l}$ follows a MATLAB-like notation that denotes the submatrix of $\mathbf{A}$ that consists of the i-th up to j-th rows and k-th up to l-th columns of it; $(\cdot)^*$ denotes conjugation; $(\cdot)^T$ transpose; $(\cdot)^{\mathcal{H}}$ Hermitian transpose; $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix with $\mathbf{x}$ on its diagonal; $\mathbf{I}_{N\times N}$ the $N \times N$ identity matrix; $\mathbf{0}_{N\times 1}$ the $N \times 1$ vector of all zeros; $\|\cdot\|$ the Frobenius norm; $|\cdot|$ the cardinality of a set and $j \triangleq \sqrt{-1}$.

[4] It should be noted that although the incremental algorithm is optimal in terms of the rank-deficient quadratic maximization, it lacks of parallelizability and memory management efficiency. On the other hand, the reverse search is highly parallelizable, speed and memory efficient and, as a result, has been utilized for the maximization of a rank-deficient quadratic form over the $0/1$ field.

II. Problem Statement

We consider the quadratic expression

$$\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s} \tag{1}$$

where $\mathbf{Q} \in \mathbb{C}^{N\times N}$ is a positive (semi)definite matrix and $\mathbf{s} \in \mathcal{A}_M^N$ is a M-PSK N-tuple vector argument. We assume $\mathcal{A}_M = \left\{ e^{\frac{j2\pi m}{M}} \mid m = 0,1,\ldots,M-1 \right\}$ as the M-phase alphabet and $M \in \{2^k \mid k = 1,2\ldots\}$.

In our problem, we focus on the computation of the M-PSK vector that maximizes the quadratic form

$$\mathbf{s}_{\text{opt}} \triangleq \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}. \tag{2}$$

Since $\mathbf{Q}$ is symmetric, the matrix can be represented in terms of its eigenvalues and eigenvectors using spectral factorization

$$\mathbf{Q} = \sum_{n=1}^{N} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}},\ \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N,\ \ \mathbf{q}_n \in \mathbb{C}^N,\ \ \|\mathbf{q}_n\| = 1,\ \ \mathbf{q}_n^{\mathcal{H}}\mathbf{q}_k = 0,\ \ n \neq k,\ \ n,k = 1,2,\ldots,N, \tag{3}$$

where λ_n and $\mathbf{q}_n$ are the n-th eigenvalue and eigenvector of the matrix $\mathbf{Q}$, respectively. If $\lambda_N > 0$, then $\mathbf{Q}$ is full rank and our problem in (2) becomes $\mathcal{NP}$-hard where the computation of $\mathbf{s}_{\text{opt}}$ can be implemented using exhaustive search over the set $\mathcal{A}_M^N$ with complexity[5] $\mathcal{O}(M^N)$ since $|\mathcal{A}_M^N| = M^N$.

On the other hand, if $\lambda_n = \lambda_{n+1} = \cdots = \lambda_N = 0, n \in \{2,3,\ldots,N\}$, then $\mathbf{Q}$ is rank-deficient. Therefore, in the following, without loss of generality (w.l.o.g.), $\mathbf{Q}$ is assumed a positive (semi)definite complex matrix with rank $D \leq N$, i.e

$$\mathbf{Q} = \sum_{n=1}^{D} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}},\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D > 0. \tag{4}$$

Furthermore, since $\lambda_n > 0, n = 1,2,\ldots,D$, we define the weighted principal component

$$\mathbf{v}_n \triangleq \sqrt{\lambda_n}\mathbf{q}_n,\ \ n = 1,2,\ldots,D, \tag{5}$$

and the corresponding $\mathbf{V} \in \mathbb{C}^{N\times D}$ complex matrix

$$\mathbf{V} \triangleq \begin{bmatrix}\mathbf{v}_1\ \mathbf{v}_2\ \ldots \mathbf{v}_D\end{bmatrix} \tag{6}$$

such that $\mathbf{V}\mathbf{V}^{\mathcal{H}} = \sum_{n=1}^{D} \mathbf{v}_n\mathbf{v}_n^{\mathcal{H}} = \sum_{n=1}^{D} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}} = \mathbf{Q}$. Thus, our initial problem statement in (2) can be transformed into the following optimization problem

$$\mathbf{s}_{\text{opt}} \triangleq \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{V}^{\mathcal{H}}\mathbf{s}\right\}. \tag{7}$$

We underline that $\mathbf{V}$ is a full rank complex matrix and matrices $\mathbf{Q}$ and $\mathbf{V}$ have the same rank $D \leq N$.

Special emphasis for the binary real case was given recently in [10] where an efficient algorithm for the computation of $\mathbf{s}_{\text{opt}} \in \{\pm 1\}$ was developed. In this work, the authors present an algorithm that utilizes auxiliary hypershperical coordinates and partitions the multidimensional space into a polynomial-size set of regions

[5]Since rotated candidate vectors $\hat{\mathbf{s}} = \mathbf{s}e^{j\frac{2\pi m}{M}}, m = 0,1,\ldots,M-1$, give the same result in our maximization problem, i.e. $\hat{\mathbf{s}}^{\mathcal{H}}\mathbf{Q}\hat{\mathbf{s}} = (\mathbf{s}e^{j\frac{2\pi m}{M}})^{\mathcal{H}}\mathbf{Q}(\mathbf{s}e^{j\frac{2\pi m}{M}}) = \mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$, we can focus only on the $\frac{1}{M}$-th of the elements of $\mathcal{A}_M^N$. In this case, the complexity of the resulting maximization quadratic form reduces to $\mathcal{O}(M^{N-1})$, which is still intractable for moderate values of N.

$|\mathcal{S}(\mathbf{V}_{N\times D})| = \sum_{d=0}^{D-1}\binom{N-1}{d}$. This set is computed with complexity $\mathcal{O}(N^D)$ and it is proved that the procedure is fully parallelizable and rank-scalable.

In parallel, the authors in [22] propose an algorithm using combinatorial geometry where the maximizing argument of the quadratic form $\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$ is computed with polynomial complexity in the length N of the parameter vector $\mathbf{s}$ if the rank of $\mathbf{Q}$ is fixed, but in fact exponential in N if observation matrix $\mathbf{Q}$ has rank that depends on N. They do so by applying CG methods to construct a subset of $\mathcal{A}_M^N$ where $M \in \{2,4\}$ that contains $\sum_{i=0}^{2D-1}\binom{N\log_2(M)-1}{i}$ vectors among which one vector is the maximizer of $\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$. The complexity of the resulting algorithm is of the order $\mathcal{O}(N^{2D})$ and $\mathcal{O}((2N)^{2D})$ for BPSK and QPSK, respectively. On the other hand, the proposed method in [11] relies on the same principles as the CG algorithm devised in [22] but is more general in that it is also applicable to modulation with $M > 4$. In that work, the authors do not explicitly recommend an efficient algorithm to generate the sufficient set $\mathcal{T}$ over which the search is performed and identify the incremental algorithm for cell enumeration in arrangements [5] as a tool for solution.

In the next section, we use the framework presented in [10] and propose a more generalized algorithm for the maximization of a rank-deficient quadratic form over any M-ary PSK alphabet where $M \in \{2^k \mid k = 1, 2 \ldots\}$. Specifically, we introduce $2D-1$ auxiliary hyperspherical coordinates and show that there exists a set $\mathcal{S}(\mathbf{V}_{N\times D}) \subset \mathcal{A}_M^N$ which has cardinality $|\mathcal{S}(\mathbf{V}_{N\times D})| = \sum_{d=1}^{D}\sum_{i=0}^{d-1}\binom{N}{i}\binom{N-i}{2(d-i)-1}\left(\frac{M}{2}\right)^{2(d-i)-2}\left(\frac{M}{2}-1\right)^{i}$, can be computed in polynomial time and contains the optimal vector $\mathbf{s}_{\mathrm{opt}}$ in (7). The proposed algorithm constructs the reduced-size candidate set $|\mathcal{S}(\mathbf{V}_{N\times D})|$ with complexity $\mathcal{O}((\frac{MN}{2})^{2D})$ and is fully parallelizable and rank-scalable.

III. Efficient Maximization of Rank-deficient Quadratic Form with a MPSK Vector Argument

A. Problem Reformulation

Since $\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{V}^{\mathcal{H}}\mathbf{s} = \|\mathbf{V}^{\mathcal{H}}\mathbf{s}\|^2$, we can rewrite our problem as

$$\mathbf{s}_{\mathrm{opt}} \triangleq \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}}\mathbf{s}\|. \tag{8}$$

We have constructed $\mathbf{V}$ in such a way that it emerges as a full rank $N \times D$ matrix with $D \leq N$. W.l.o.g., we assume that each row of $\mathbf{V}$ has at least one nonzero element, i.e. $\mathbf{V}_{n,1:D} \neq \mathbf{0}_{1\times D}, \forall n \in \{1, 2, \ldots, N\}$. In opposite case the value of the maximization argument $s_n, n \in \{1, 2, \ldots, N\}$, related with the n-th all-zero row of $\mathbf{V}_{N\times D}$ would have no effect on the maximization procedure. Therefore, assuming that we have $\mathcal{K} \in \{1, 2, \ldots N\}$ rows of $\mathbf{V}$ equal to $\mathbf{0}_{1\times D}$, we can simply ignore these rows, reduce the dimension of our problem from N to $N - \mathcal{K}$ and assign arbitrary values to the elements of the maximizing vector related to the $\mathcal{K}$ all-zero rows of $\mathbf{V}$.

Let $\boldsymbol{\phi}_{i:j} \triangleq [\phi_i, \phi_{i+1}, \ldots, \phi_j]^T$. To develop an efficient method for the maximization in (8), we introduce $2D-1$ auxiliary hyperspherical coordinates $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ and define the hyperspherical real vector with unit radial coordinate r,

$$\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \begin{bmatrix} \sin\phi_1 \\ \cos\phi_1 \sin\phi_2 \\ \cos\phi_1 \cos\phi_2 \sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \sin\phi_{2D-1} \\ \left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \cos\phi_{2D-1} \end{bmatrix}_{2D\times 1} \tag{9}$$

according to the following lemma. The proof is provided in the Appendix.

Lemma 1: Let $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$. Then a spherical coordinate system can be defined in a $(2D)$-dimensional Euclidean space where each point $\mathbf{x} \triangleq [x_1 \ x_2 \ \dots \ x_{2D}]^T \in \mathbb{R}^{2D}$ can be described by coordinates consisting of a radial coordinate r

$$r \triangleq \sqrt{x_1^2 + x_2^2 + \cdots + x_{2D}^2}, \ \ r \geq 0, \tag{10}$$

and $2D-1$ angular coordinates $\boldsymbol{\phi}_{1:2D-1}$ as follows

$$\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \begin{bmatrix} r\sin\phi_1 \\ r\cos\phi_1 \sin\phi_2 \\ r\cos\phi_1 \cos\phi_2 \sin\phi_3 \\ \vdots \\ r\left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \sin\phi_{2D-1} \\ r\left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \cos\phi_{2D-1} \end{bmatrix}_{2D\times 1} \tag{11}$$

■

Furthermore, we define the $D \times 1$ hyperspherical complex vector

$$\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \tilde{\mathbf{c}}_{2:2:2D}(\boldsymbol{\phi}_{1:2D-1}) + j\tilde{\mathbf{c}}_{1:2:2D-1}(\boldsymbol{\phi}_{1:2D-1})$$

$$= \begin{bmatrix} \cos\phi_1 \sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1 \cos\phi_2 \cos\phi_3 \sin\phi_4 + j\cos\phi_1 \cos\phi_2 \sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \cos\phi_{2D-1} + j\left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \sin\phi_{2D-1} \end{bmatrix} = \begin{bmatrix} c_1(\boldsymbol{\phi}_{1:2}) \\ c_2(\boldsymbol{\phi}_{1:4}) \\ \vdots \\ c_D(\boldsymbol{\phi}_{1:2D-1}) \end{bmatrix}_{D\times 1} \tag{12}$$

From Cauchy-Swartz inequality, we observe that for any $\mathbf{a} \in \mathbb{C}^D$,

$$\left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right| \leq \|\mathbf{a}\|\|\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\| = \|\mathbf{a}\|, \tag{13}$$

since $\|\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\| = \sqrt{\left|c_1(\boldsymbol{\phi}_{1:2D-1})\right|^2 + \cdots + \left|c_D(\boldsymbol{\phi}_{1:2D-1})\right|^2} = 1$. The equality of (13) is achieved if and only if $\phi_1, \phi_2, \ldots, \phi_{2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ are the hyperspherical coordinates of vector $\mathbf{a}$, i.e. if [6]

$$\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{14}$$

since $\left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right| = \left|\mathbf{a}^{\mathcal{H}}\frac{\mathbf{a}}{\|\mathbf{a}\|}\right| = \|\mathbf{a}\|$. Using the above, a critical equality for our subsequent developments is

$$\mathbf{s}_{\text{opt}} = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}}\mathbf{s}\| = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\boldsymbol{\phi}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \left|\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right|. \tag{15}$$

Furthermore, we observe that for any $\mathbf{a} \in \mathbb{C}^D$ and any $\hat{\theta} \in (-\pi, \pi]$,

$$\Re\left\{\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{-j\hat{\theta}}\right\} \leq \left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right| \tag{16}$$

with equality if and only if $\hat{\theta} = \theta$ where $\theta \triangleq \arg\left\{\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\}$ since in this case $\Re\Big\{\underbrace{\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})}_{\in\mathbb{C}}e^{-j\hat{\theta}}\Big\} =$ $\Re\left\{\left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right|e^{j\theta}e^{-j\hat{\theta}}\right\} \overset{\hat{\theta}=\theta}{=} \left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right|$.

It can be easily observed that expressions (13) and (16) are simultaneously satisfied with equality if and only if $\phi_1, \phi_2, \ldots, \phi_{2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ are the hyperspherical coordinates of vector $\mathbf{a}$ ***and*** $\hat{\theta} = \theta$. In this case, (15) can be further transformed into:

$$\mathbf{s}_{\text{opt}} = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\boldsymbol{\phi}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{-j\theta}\right\}, \tag{17}$$

where $\theta = \arg\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\}$.

Now, let $\boldsymbol{\phi}_{1:2D-1}, \hat{\boldsymbol{\phi}}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ and $\theta \in (-\pi, \pi]$. Then, for any $\boldsymbol{\phi}_{1:2D-1}$ and θ, there always exist $\hat{\boldsymbol{\phi}}_{1:2D-1}$ such that $\mathbf{c}(\hat{\boldsymbol{\phi}}_{1:2D-1}) = \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{-j\theta}$, i.e. $\hat{\boldsymbol{\phi}}_{1:2D-1}$ are the hyperspherical coordinates of complex vector $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{-j\theta}$. Conversely, for any $\hat{\boldsymbol{\phi}}_{1:2D-1}$ and θ, there always exist $\boldsymbol{\phi}_{1:2D-1}$ such that $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = \mathbf{c}(\hat{\boldsymbol{\phi}}_{1:2D-1})e^{j\theta}$, i.e. $\boldsymbol{\phi}_{1:2D-1}$ are the hyperspherical coordinates of complex vector $\mathbf{c}(\hat{\boldsymbol{\phi}}_{1:2D-1})e^{j\theta}$. Therefore, eq. (17) can be equivalently rewritten as

$$\begin{aligned}
\mathbf{s}_{\text{opt}} &= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\boldsymbol{\phi}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{-j\theta}\right\} && (18)\\
&= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\hat{\boldsymbol{\phi}}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\hat{\boldsymbol{\phi}}_{1:2D-1})\right\} && (19)\\
&= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\boldsymbol{\phi}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\} && (20)
\end{aligned}$$

since (20) defines a maximization that runs over the whole domain of the hyperspherical coordinates and for notational convenience, we redefine $\hat{\boldsymbol{\phi}}_{1:2D-1} \triangleq \boldsymbol{\phi}_{1:2D-1}$.

Next, we note the following: assume that for specific hyperspherical coordinates $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$, the expression in (20) gives as a result a M-PSK vector $\mathbf{s} \in \mathcal{A}_M^N$ which, in combination with $\boldsymbol{\phi}_{1:2D-1}$, constitute the tuple $\left(\mathbf{s}, \angle\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right) \in \mathcal{A}_M^N \times \left[(-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]\right]$. We observe that for $\forall\alpha \in \arg\{\mathcal{A}_M\}$

[6]We observe that the equality of (13) is also achieved for any rotated version of $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$, i.e. $e^{j\omega}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = e^{j\omega}\frac{\mathbf{a}}{\|\mathbf{a}\|}$ for any $\omega \in (-\pi, \pi]$ since $\left|\mathbf{a}^{\mathcal{H}}e^{j\omega}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right| = \left|\mathbf{a}^{\mathcal{H}}e^{j\omega}\frac{\mathbf{a}}{\|\mathbf{a}\|}\right| = \left|e^{j\omega}\|\mathbf{a}\|\right| = \|\mathbf{a}\|$. But, for clarity reasons, we present the case for $\omega = 0$ in the above statement.

where $\arg\{\mathcal{A}_M\} = \left\{\frac{2\pi m}{M} \mid m = 0, 1, \ldots, M-1\right\}$, the tuple $\left(e^{j\alpha}\mathbf{s}, \angle e^{j\alpha}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right) = \left(\hat{\mathbf{s}}, \angle \mathbf{c}(\hat{\boldsymbol{\phi}}_{1:2D-1})\right) \in \mathcal{A}_M^N \times \left[(-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]\right]$ gives the same value as $\left(\mathbf{s}, \angle \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right)$ in the maximization metric of (20) since $\Re\left\{\hat{\mathbf{s}}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\hat{\boldsymbol{\phi}}_{1:2D-1})\right\} = \Re\left\{(e^{j\alpha}\mathbf{s})^{\mathcal{H}}\mathbf{V}e^{j\alpha}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\} = \Re\left\{e^{-j\alpha}\mathbf{s}^{\mathcal{H}}\mathbf{V}e^{j\alpha}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\} = \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\}$. This means that, given any tuple $\left(\mathbf{s}, \angle \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right) \in \mathcal{A}_M^N \times \left[(-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]\right]$, we have $M = |\mathcal{A}_M|$ different rotated versions of this M-phase vector that belong in $\mathcal{A}_M^N$ and give the same result in the maximization metric of (20) for $\forall\alpha$. The value of α is chosen according to the following proposition. The proof is provided in the Appendix.

Proposition 1: Given a hyperspherical complex vector $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$ and $\boldsymbol{\phi}_{2D-1} \in (-\pi, \pi]$, there always exists an angle $\alpha \in \arg\{\mathcal{A}_M\}$ that relocates the angular coordinate ϕ_{2D-1} of the hyperspherical vector $\left\{\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha}\right\}$ in the interval $(-\frac{\pi}{M}, \frac{\pi}{M}]$. ■

Using proposition 1, w.l.o.g., we choose $\alpha \in \arg\{\mathcal{A}_M\}$ such that $\boldsymbol{\phi}_{2D-1} \in (-\frac{\pi}{M}, \frac{\pi}{M}]$. Thus, (20) becomes

$$\mathbf{s}_{\text{opt}} = \arg \max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\boldsymbol{\phi}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\}. \tag{21}$$

Dropping the $\arg$ operator and interchanging the maximizations in (21) we obtain the equivalent problem

$$\max_{\phi_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]} \sum_{n=1}^{N} \max_{s_n\in\mathcal{A}_M} \Re\left\{s_n^*\mathbf{V}_{n,1:D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right\}, \quad \Phi \triangleq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{22}$$

B. Decision Functions and Candidate Vector Set $\mathcal{S}(\mathbf{V}_{N\times D})$

We observe that the original maximization problem in (8) is decomposed in a set of symbol-by-symbol maximization rules for a given set of angles $\boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$. For such a set of angles, the maximization argument of the sum in (22), e.g. symbol s_n, depends only on the corresponding row of matrix $\mathbf{V}$. As $\boldsymbol{\phi}_{1:2D-1}$ vary, the decision in favor of s_n is maintained as long as a decision boundary is not crossed. On a M-PSK complex unit circle, a decision boundary $\mathcal{B}_k$ is defined as a complex exponential of the form

$$\mathcal{B}_k = e^{j\pi\frac{2k+1}{M}}, \;\; k = 0, 1, \ldots, \frac{M}{2} - 1, \tag{23}$$

that passes through the origin of the complex unit circle and separates the alphabet $\mathcal{A}_M$ into two disjoint sets: $\dot{\mathcal{A}}_M^{(k)} \subset \mathcal{A}_M$ and $\ddot{\mathcal{A}}_M^{(k)} \subset \mathcal{A}_M$ where

$$\dot{\mathcal{A}}_M^{(k)} = \left\{e^{\frac{j2\pi m}{M}} \;\middle|\; m \in \left\{\left(k+1, \ldots, k + \frac{M}{2}\right) \bmod M\right\}\right\}, \tag{24}$$

$$\ddot{\mathcal{A}}_M^{(k)} = \left\{e^{\frac{j2\pi m}{M}} \;\middle|\; m \in \left\{\left(k + \frac{M}{2} + 1, \ldots, k + M\right) \bmod M\right\}\right\}, \tag{25}$$

and $\dot{\mathcal{A}}_M^{(k)} \cap \ddot{\mathcal{A}}_M^{(k)} = \{\emptyset\}$.

Due to the structure of $\mathcal{A}_M$ and given the definitions above, the $\frac{M}{2}$ decision boundaries for the determination of s_n are given by

$$\mathbf{V}_{n,1:D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = Ae^{j\pi\frac{2k+1}{M}}, \;\; A \in \mathbb{R} - \{0\}, \;\; k = 0, 1, \ldots, \frac{M}{2} - 1, \tag{26}$$

or equivalently

$$\Im\Big\{e^{-j\pi\frac{2k+1}{M}}\mathbf{V}_{n,1:D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\Big\}=0,\;\; k=0,1,\ldots,\frac{M}{2}-1. \tag{27}$$

For $n=1,2,\ldots,N$ and $k=0,1,\ldots,\frac{M}{2}-1$, we can write (27) in matrix form

$$\Im\left(\underbrace{\begin{bmatrix} e^{-j\frac{\pi}{M}}\mathbf{V}_{N\times D}\\ \vdots \\ e^{-j\pi\frac{M-1}{M}}\mathbf{V}_{N\times D}\end{bmatrix}}_{\hat{\mathbf{V}}_{\frac{MN}{2}\times D}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\right)=\mathbf{0}_{\frac{MN}{2}\times 1}\Leftrightarrow \tag{28}$$

$$\Im\Big\{\big[\Re(\hat{\mathbf{V}})+j\Im(\hat{\mathbf{V}})\big]\big[\tilde{\mathbf{c}}_{2:2:2D}(\phi_{1:2D-1})+j\tilde{\mathbf{c}}_{1:2:2D-1}(\phi_{1:2D-1})\big]\Big\}=\mathbf{0}_{\frac{MN}{2}\times 1}\Leftrightarrow \tag{29}$$

$$\Re(\hat{\mathbf{V}})\tilde{\mathbf{c}}_{1:2:2D-1}(\phi_{1:2D-1})+\Im(\hat{\mathbf{V}})\tilde{\mathbf{c}}_{2:2:2D}(\phi_{1:2D-1})=\mathbf{0}_{\frac{MN}{2}\times 1}\Leftrightarrow \tag{30}$$

$$\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}\tilde{\mathbf{c}}(\phi_{1:2D-1})=\mathbf{0}_{\frac{MN}{2}\times 1}\Leftrightarrow \tag{31}$$

$$\tilde{\mathbf{V}}_{l,1:2D}\tilde{\mathbf{c}}(\phi_{1:2D-1})=0,\;\; l=1,2,\ldots,\frac{MN}{2} \tag{32}$$

where

$$\tilde{\mathbf{V}}_{:,1:2:2D-1}=\Re\big(\hat{\mathbf{V}}\big)\quad\text{and}\quad\tilde{\mathbf{V}}_{:,2:2:2\mathrm{D}}=\Im\big(\hat{\mathbf{V}}\big). \tag{33}$$

From the construction of $\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}$, it can be easily observed that each row of $\mathbf{V}$ is rotated by each of the $\frac{M}{2}$ exponentials $e^{-j\pi\frac{2k+1}{M}}$ that represent the decision boundaries $\mathcal{B}_k, k=0,1,\ldots,\frac{M}{2}-1$. Therefore, using the rows of $\tilde{\mathbf{V}}$, we can define N different groups $\boldsymbol{\mathcal{G}}^{(n)}, n\in\{1,2,\ldots,N\}$, where each group is related to the n-th row of $\mathbf{V}$ and consists of $\frac{M}{2}$ rows, each of which comes from a different rotated version of the n-th row of $\mathbf{V}$. Thus, we have

$$\boldsymbol{\mathcal{G}}^{(n)}\triangleq\begin{bmatrix}\tilde{\mathbf{V}}_{n,1:2D}\\ \tilde{\mathbf{V}}_{(n+N),1:2D}\\ \vdots\\ \tilde{\mathbf{V}}_{\left(n+(\frac{M}{2}-1)N\right),1:2D}\end{bmatrix}_{\frac{M}{2}\times 2D}\quad,\forall n\in\{1,2,\ldots,N\}. \tag{34}$$

For a given point $\boldsymbol{\phi}_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ and according to (32), each row of $\boldsymbol{\mathcal{G}}^{(n)}, n\in\{1,2,\ldots,N\}$, defines a decision expression for s_n

$$\boldsymbol{\mathcal{G}}^{(n)}_{i,1:2D}\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1})\gtrless 0,\;\; i=1,2,\ldots,\frac{M}{2}, \tag{35}$$

that separates $\mathcal{A}_M$ into two *distinct* sets, $\dot{\mathcal{A}}_M^{(i-1)}$ and $\ddot{\mathcal{A}}_M^{(i-1)}$ as defined in (24)-(25), and indicates in which set s_n belongs to. Let $\mathcal{R}_i^{(n)} \in \{\dot{\mathcal{A}}_M^{(i-1)}, \ddot{\mathcal{A}}_M^{(i-1)}\}$ be the set such that $s_n \in \mathcal{R}_i^{(n)}$ for $i \in \{1, 2, \ldots, \frac{M}{2}\}$, according to (35). Then, the final decision about s_n is taken by computing the intersection of $\mathcal{R}_i^{(n)}$ for $\forall i = 1, 2, \ldots, \frac{M}{2}$, i.e.

$$s_n = \bigcap_{i=1}^{\frac{M}{2}} \mathcal{R}_i^{(n)}. \tag{36}$$

Motivated by the statements above and the inner maximization rule in (22), for each $D \times 1$ complex vector $\mathbf{v}$ we define the decision function s that maps $\boldsymbol{\phi}_{1:2D-1}$ to $\mathcal{A}_M$ according to

$$s(\mathbf{v}^T; \boldsymbol{\phi}_{1:2D-1}) \triangleq \arg \max_{s \in \mathcal{A}_M} \Re\{s^* \mathbf{v}^T \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\}. \tag{37}$$

Furthermore, for the given $N \times D$ complex observation matrix $\mathbf{V}$, we can construct the vector decision function $\mathbf{s}$ using (37) where each point $\boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ is mapped to a candidate M-PSK vector according to

$$\mathbf{s}(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{1:2D-1}) \triangleq \begin{bmatrix} s(\mathbf{V}_{1,1:D}; \boldsymbol{\phi}_{1:2D-1}) \\ s(\mathbf{V}_{2,1:D}; \boldsymbol{\phi}_{1:2D-1}) \\ \vdots \\ s(\mathbf{V}_{N,1:D}; \boldsymbol{\phi}_{1:2D-1}) \end{bmatrix} = \begin{bmatrix} \arg\max_{s \in \mathcal{A}_M} \Re\{s^* \mathbf{V}_{1,1:D} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\} \\ \arg\max_{s \in \mathcal{A}_M} \Re\{s^* \mathbf{V}_{2,1:D} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\} \\ \vdots \\ \arg\max_{s \in \mathcal{A}_M} \Re\{s^* \mathbf{V}_{N,1:D} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\} \end{bmatrix} \tag{38}$$

$$= \arg \max_{\mathbf{s} \in \mathcal{A}_M^N} \Re\{\mathbf{s}^{\mathcal{H}} \mathbf{V} \mathbf{c}(\boldsymbol{\phi}_{1:2D-1})\}. \tag{39}$$

Computing $\mathbf{s}(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{1:2D-1})$ for $\forall \boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, we collect all M-phase candidate vectors into a set

$$\mathcal{S}(\mathbf{V}_{N\times D}) \triangleq \bigcup_{\boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]} \Big\{ \mathbf{s}(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{1:2D-1}) \Big\} \subseteq \mathcal{A}_M^N. \tag{40}$$

Since $\boldsymbol{\phi}_{1:2D-1}$ take values from the set $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, our problem in (7) becomes

$$\mathbf{s}_{\text{opt}} \triangleq \arg \max_{\mathbf{s} \in \mathcal{S}(\mathbf{V})} \Big\{ \mathbf{s}^{\mathcal{H}} \mathbf{V} \mathbf{V}^{\mathcal{H}} \mathbf{s} \Big\}, \tag{41}$$

i.e. the M-phase candidate vector $\mathbf{s}_{\text{opt}}$ that maximizes the expression above belongs into the set $\mathcal{S}(\mathbf{V}_{N\times D})$.

In the following, we (i) show that $|\mathcal{S}(\mathbf{V}_{N\times D})| = \sum_{d=1}^{D} \sum_{i=0}^{d-1} \binom{N}{i} \binom{N-i}{2(d-i)-1} \left(\frac{M}{2}\right)^{2(d-i)-2} \left(\frac{M}{2} - 1\right)^i$ and (ii) develop an algorithm for the construction of $\mathcal{S}(\mathbf{V}_{N\times D})$ with complexity $\mathcal{O}((\frac{MN}{2})^{2D})$.

C. Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

According to eq. (32), we can derive $\frac{MN}{2}$ different decision rules that separate the $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ space (and moreover the space $\Phi^{2D-2} \times (-\pi, \pi]$) into distinct regions, each of which is associated with a different M-PSK candidate vector $\mathbf{s}$. More specifically, the rows of $\tilde{\mathbf{V}}_{\frac{MN}{2} \times 2D}$ determine $\frac{MN}{2}$ hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})$ that partition the $(2D-1)$-dimensional hypercube $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ into K non-interleaving cells $C_1, C_2, \ldots, C_K$ such that the union of all cells is equal to $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ and the intersection of any two distinct cells, say C_k, C_j for $k \neq j$, is empty. Each cell C_k corresponds to a *distinct* $\mathbf{s}_k \in \mathcal{A}_M^N$ in the sense that $\mathbf{s}(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{1:2D-1}) = \mathbf{s}_k$ for any $\boldsymbol{\phi}_{1:2D-1} \in C_k$ and $\mathbf{s}_k \neq \mathbf{s}_j$ if $k \neq j, k, j \in \{1, 2, \ldots, K\}$.

Before we present some further results on the behavior of such hypersurfaces, it is illustrative to present some partitions of $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ for various values of D, M and N. As first example, we set $D = 2$, $N = 4$ and $M = 8$ and draw an arbitrary rank-2 complex matrix $\mathbf{V}_{4\times 2}$ with $\mathbf{V}_{n,1:2} \neq \mathbf{0}_{1\times 2}, n = 1, 2, ..., N$. Since we study the case $\phi_{2D-1} \in (-\frac{\pi}{M}, \frac{\pi}{M}]$, we are interested only in cells that belong into the region $\boldsymbol{\phi}_{1:3} \in \Phi^2 \times (-\frac{\pi}{8}, \frac{\pi}{8}]$. According to the decision boundary rule in (32), we plot in Fig. 1(a) the hypersurface $\mathcal{H}(\tilde{\mathbf{V}}_{1,1:4})$ described by the expression $\phi_1 = \tan^{-1}\left(-\frac{\tilde{\mathbf{V}}_{(1,2:4)}\tilde{\mathbf{c}}(\boldsymbol{\phi}_{2:3})}{\tilde{V}_{(1,1)}}\right) = \tan^{-1}\left(-\frac{\tilde{V}_{(1,2)}\sin\phi_2+\tilde{V}_{(1,3)}\cos\phi_2\sin\phi_3+\tilde{V}_{(1,4)}\cos\phi_2\cos\phi_3}{\tilde{V}_{(1,1)}}\right)$ that originates from the first row of $\mathbf{V}_{4\times 2}$. As (35) states, the hypersurface creates two non-overlapping regions in the three-dimensional space. In Figs. 1(b) and 1(c), we add two more hypersurfaces, $\mathcal{H}(\tilde{\mathbf{V}}_{2,1:4})$ and $\mathcal{H}(\tilde{\mathbf{V}}_{3,1:4})$, originating from the second and third row of $\mathbf{V}_{4\times 2}$, respectively. We observe that the hypersurfaces intersect to a single point $\boldsymbol{\phi}(\mathbf{V}_{4\times 2}; \{1, 2, 3\})$ and the three-dimensional space is partitioned into regions (cells) each of which corresponds to a distinct M-phase candidate vector $\mathbf{s} \in \mathcal{S}(\mathbf{V}_{4\times 2})$.[7]

Two basic properties of such intersections are presented in the following proposition. The proof is given in the Appendix.

Proposition 2: Let $\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}$ be a real matrix constructed from a rank-D complex matrix $\mathbf{V}_{N\times D}$ with $\mathbf{V}_{n,1:D} \neq \mathbf{0}_{1\times D}, \forall n \in \{1, 2, \dots N\}$. The following hold true.

(i) Each subset of $\left\{\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}), \dots, \mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})\right\}$ that consists of $2D-1$ hypersurfaces has either a single or uncountably many intersections in $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$.

(ii) Each combination of $2D-1$ hypersurfaces from the set $\left\{\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}), \dots, \mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})\right\}$ has a unique intersection point that constitutes a vertex of a cell if and only if no more than two hypersurfaces originate from the same row of the matrix $\mathbf{V}$. ■

Let $\mathcal{I}_{2D-1} \triangleq \{i_1, i_2, \dots, i_{2D-1}\} \subset \{1, 2, \dots, \frac{MN}{2}\}$ denote the subset of $2D-1$ indices that correspond to hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2D}), \dots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2D-1},1:2D})$. We detect the following cases:

(a) Intersections of $2D-1$ hypersurfaces where at most two surfaces originate from the same row of $\mathbf{V}$.

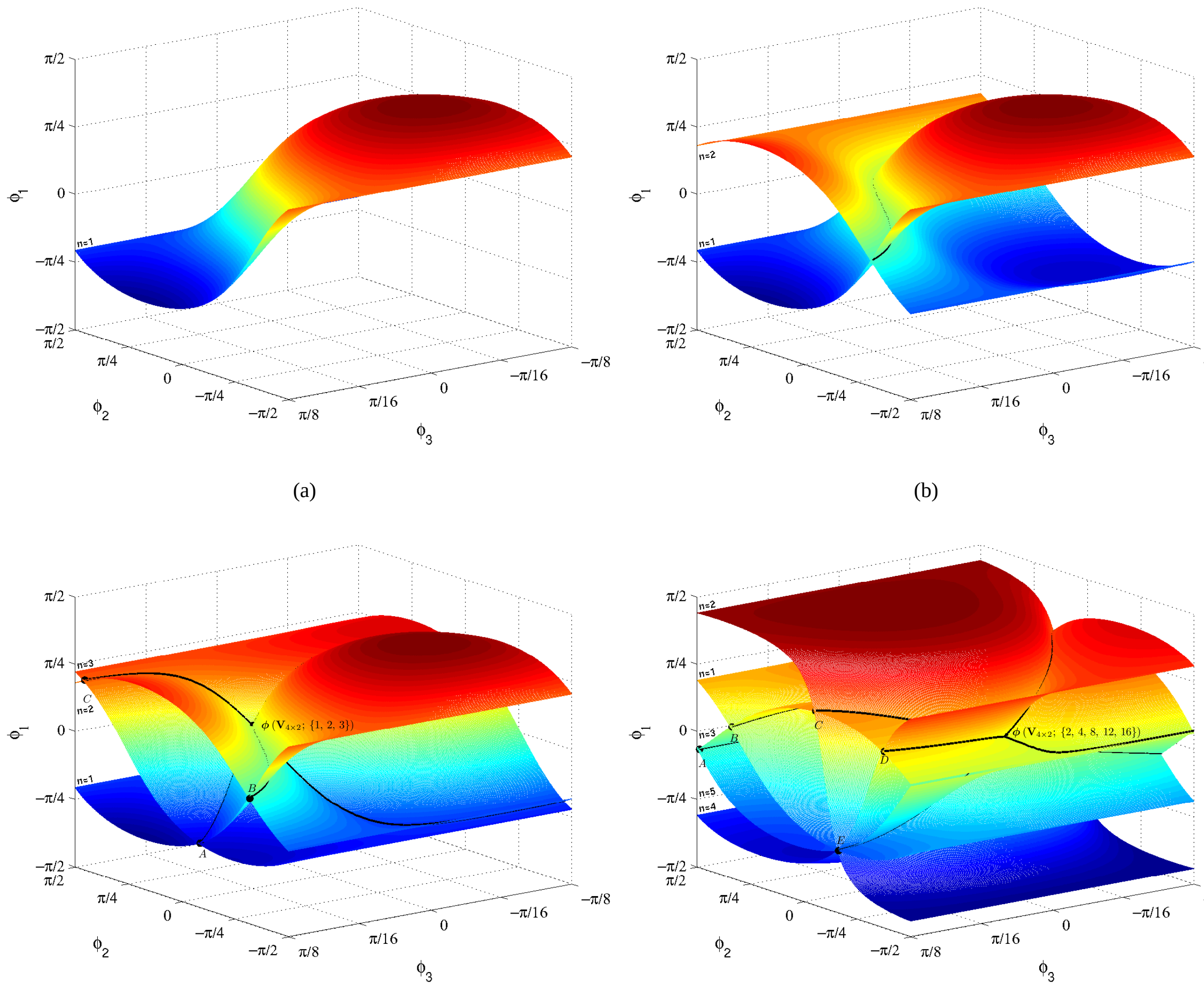
(b) Intersections of $2D-1$ hypersurfaces where at least three surfaces originate from the same row of $\mathbf{V}$.

According to Proposition 3, Part (ii), combinations of the form (b) do not have a unique intersection point but infinitely many intersection points; thus no cell is created and these combinations can be ignored. Extending the previous example, we present in Fig. 2 the intersection of $\frac{M}{2}$ hypersurfaces that originate from the first row of $\mathbf{V}_{4\times 2}$ and are related with the decision of argument s_1. Such an ensemble of hypersurfaces partitions the hypercube $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ into M regions, each of which is mapped to a unique element of the $\mathcal{A}_M$ alphabet. A very important observation for our subsequent developments is presented in the following corollary.

Corollary 1: All $\frac{M}{2}$ hypersurfaces originating from the same row of $\mathbf{V}$ intersect to a common axis. ■

For example, in Fig. 2 all hypersurfaces have uncountably many intersection points that form a common line intersection. The dimensionality of the common axis depends on the rank D of the observation matrix $\mathbf{V}$ and equals

[7] For visualization purposes, we do not plot the complete partition.



(a) (b) (c) (d)

Fig. 1.

to $2(D-1)$. Thus, hypersurfaces coming from the same row of $\mathbf{V}_{N\times D}$ intersect to a common one-dimensional line if $D = 2$, to a common four-dimensional hyperplane if $D = 3$, etc.[8]

On the other hand, combinations of the form (a) have a unique intersection point $\boldsymbol{\phi}(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1}) \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ that leads $\mathcal{Q}$ cells, say $C_1(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1}), C_2(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1}), \ldots, C_{\mathcal{Q}}(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$, $\mathcal{Q} \in \left\{(\frac{M}{2}-1)^0, (\frac{M}{2}-1)^1, \ldots, (\frac{M}{2}-1)^{D-1}\right\}$ and each cell is associated with a distinct M-phase candidate vector $\mathbf{s}_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$, $q = 1, 2, \ldots, \mathcal{Q}$, in the sense that $\mathbf{s}_q(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{2D-1}) = \mathbf{s}_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ for all $\boldsymbol{\phi}_{1:2D-1} \in C_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ and $\boldsymbol{\phi}(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ is a single point of $C_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ where ϕ_{2D-1} is minimized. We underline that not any such combination intersects into the region of interest $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$; thus there are combinations of hypersurfaces that intersect at a single point $\boldsymbol{\phi}(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ where $\phi_{2D-1} \notin (-\frac{\pi}{M}, \frac{\pi}{M}]$. As described later,

[8]For $D \geq 3$, we cannot visualize the resulting partitions and the common intersection axes.

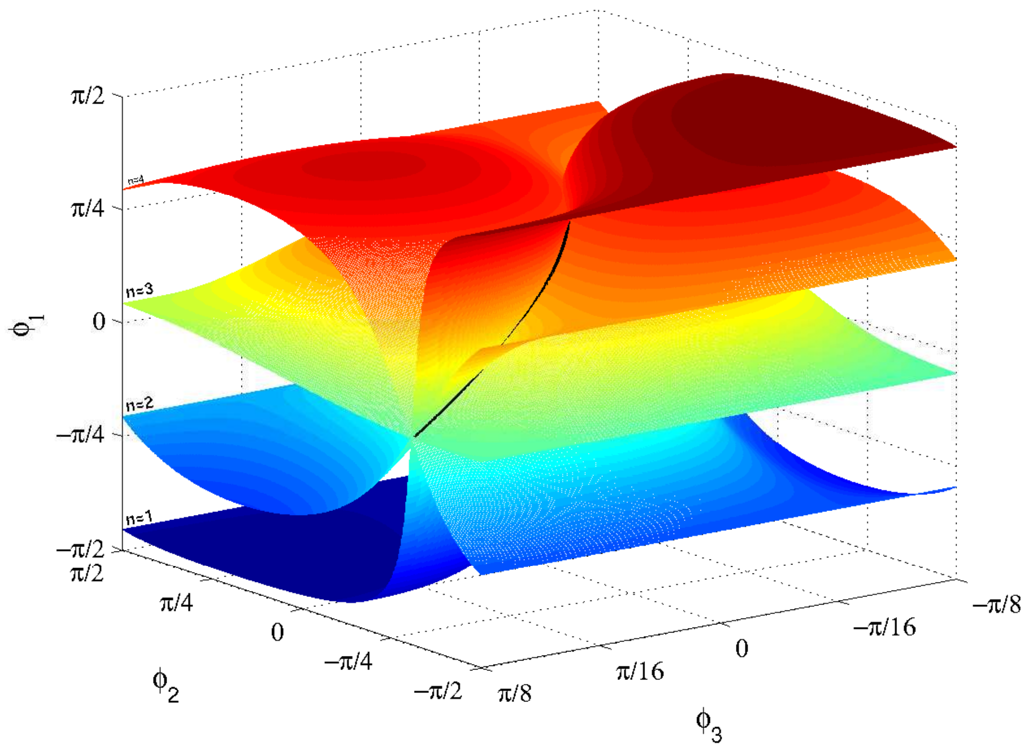


Fig. 2.

any such case can be ignored since there always exists a combination of hypersurfaces with $\phi(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1}) \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ which "leads" cells associated with equivalent candidate vectors. The number of cells $\mathcal{Q}$ "led" by an intersection point depends on the number of participating pairs of hypersurfaces that originate from the same row of matrix $\mathbf{V}$. More specifically,

- if none pairs of hypersurfaces come from the same row of $\mathbf{V}$ (i.e. all hypersurfaces originate from different rows of $\mathbf{V}$), then the intersection point "leads" only one cell,
- if just one pair of hypersurfaces comes from the same row of $\mathbf{V}$, then the intersection point "leads" $\left(\frac{M}{2}-1\right)$ cells,
- ...
- if $D-1$ pairs of hypersurfaces come from the same row of $\mathbf{V}$, then the intersection point "leads" $\left(\frac{M}{2}-1\right)^{D-1}$ cells.

To better understand the above statements, we present two different examples. For this purpose, we consider the same complex matrix $\mathbf{V} \in \mathbb{C}^{N\times D}$ as the previous example where $N=4, D=2$ and $M=8$ and assume the intersection depicted in Fig. 1(c). Since all hypersurfaces participating in the intersection come from different rows of $\mathbf{V}$ there is no other hypersurface from the set $\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})$ that passes

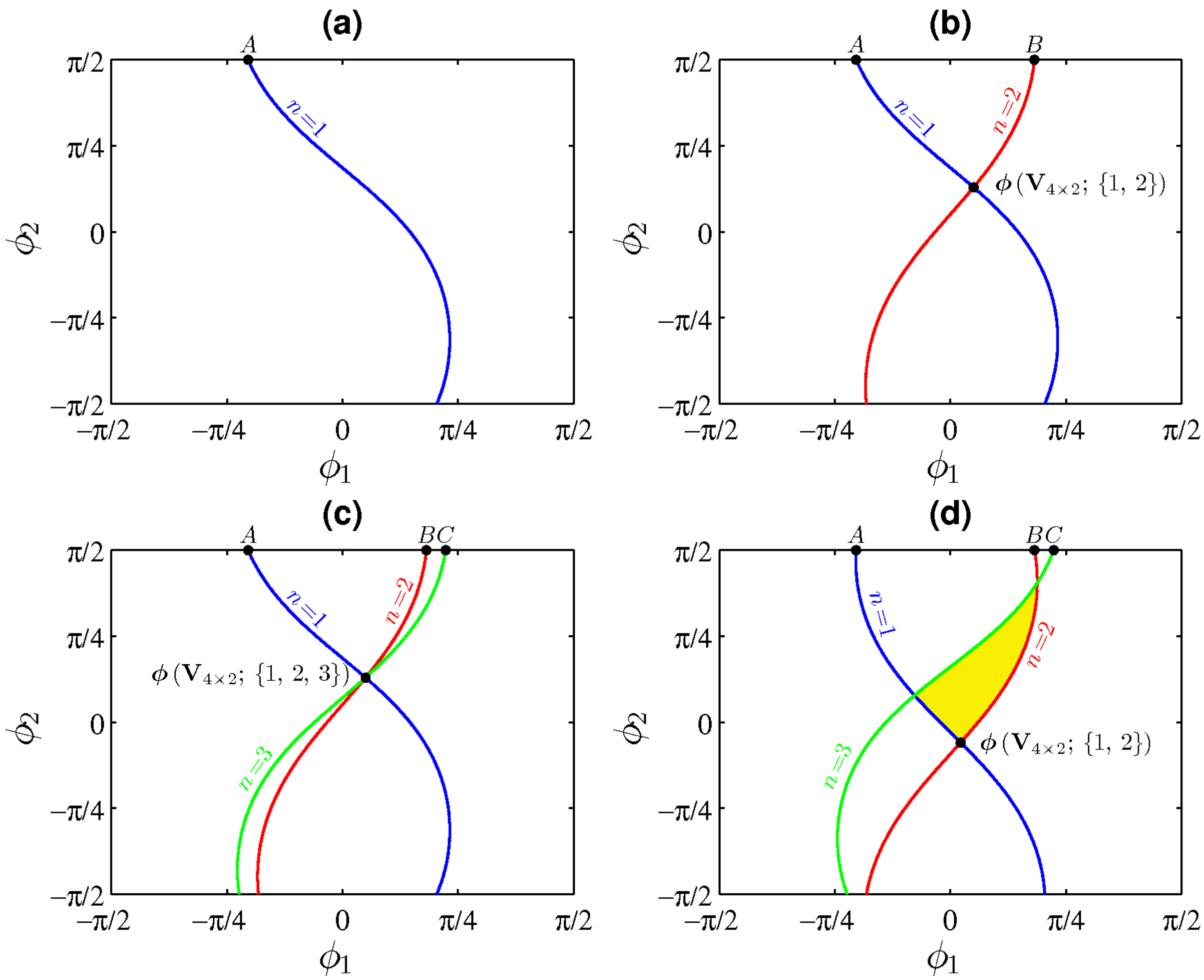


Fig. 3.

through this intersection point.[9] Let $\hat{\phi}_3 = \arg_{\phi_3}\{\boldsymbol{\phi}(\mathbf{V}_{4\times 2};\{1,2,3\})\}$. In Figs. 3([a b c]), we plot the hypersurfaces depicted in Fig. 1(c) for $\phi_3 = \hat{\phi}_3$ and in Fig. 3(d), we sketch the same hypersurfaces for $\phi_3 = \hat{\phi}_3 + d\phi$ where $d\phi$ is an arbitrary small positive quantity. We observe that as ϕ_3 increases, curve $n = 3$ moves away from intersection $\boldsymbol{\phi}(\mathbf{V}_{4\times 2};\{1,2\})$, thus creating a new cell [see the highlighted cell in Fig. 3(d)] that corresponds to a *distinct* $\mathbf{s}_k \in \mathcal{A}_M^N$.

To illustrate an example with one pair of hypersurfaces originating from the same row of the observation matrix, we assume the intersection $\boldsymbol{\phi}(\mathbf{V}_{4\times 2};\{2,6,3\})$ of two hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{2,1:4}), \mathcal{H}(\tilde{\mathbf{V}}_{6,1:4})$ coming from the second row of $\mathbf{V}_{4\times 2}$and one hypersurface $\mathcal{H}(\tilde{\mathbf{V}}_{3,1:4})$ from the third row of $\mathbf{V}_{4\times 2}$ and plot in Figs. 4([a b c]) these hypersurfaces ($n = 1$, $n = 2$ and $n = 3$, respectively) for $\phi_3 = \hat{\phi}_3$ where $\hat{\phi}_3 = \arg_{\phi_3}(\boldsymbol{\phi}(\mathbf{V}_{4\times 2};\{2,6,3\})$. But, as corollary 1 states, since all $\frac{M}{2}$ hypersurfaces originating from a specific row of $\mathbf{V}_{N\times D}$ intersect at a common

[9] In the sequel, we consider the most computationally demanding case of distinct intersections.

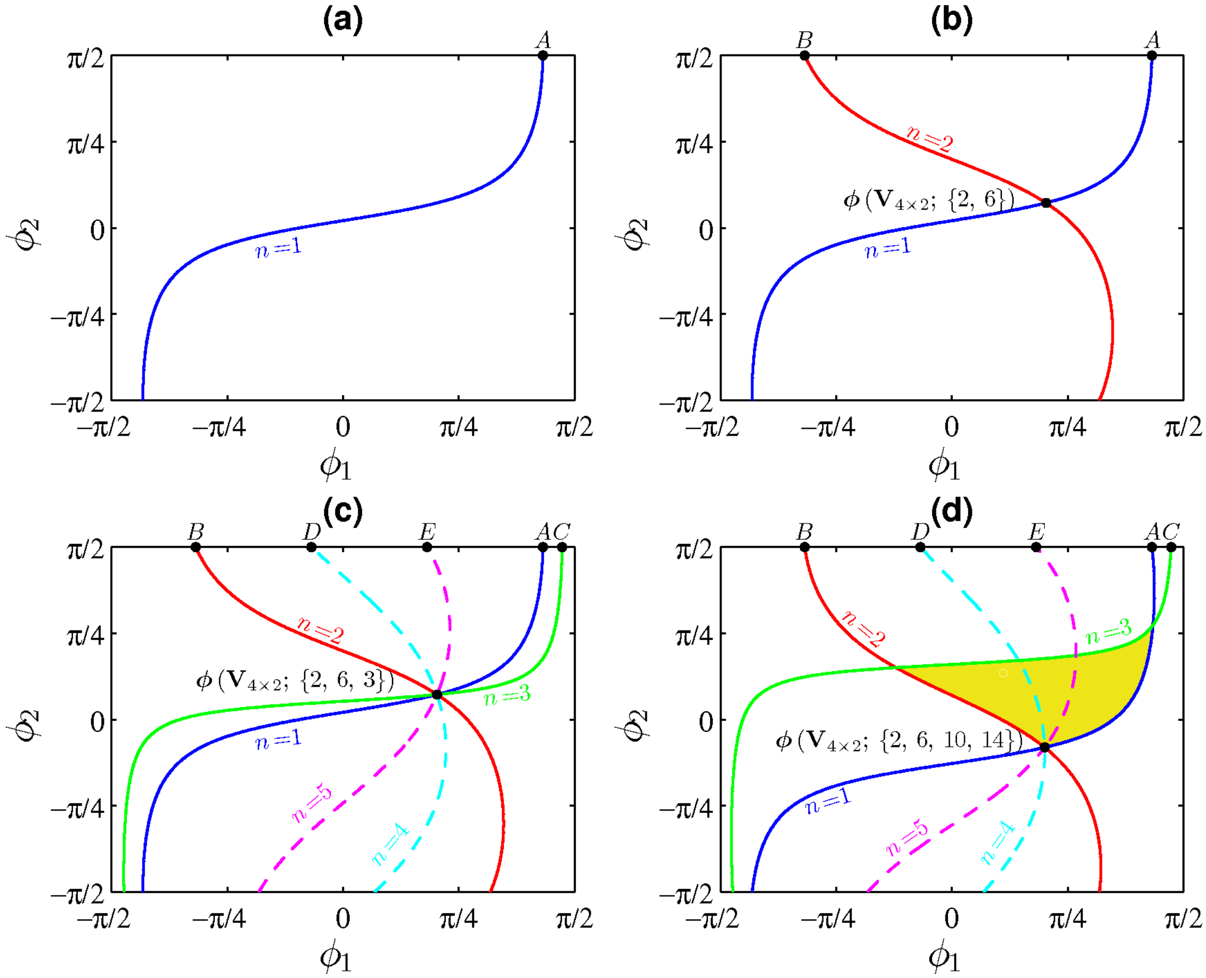


Fig. 4.

axis, we observe that all $\frac{M}{2}$ hypersurfaces $\mathcal{H}(\mathcal{G}^{(2)}_{:,1:4}) = \{\mathcal{H}(\tilde{\mathbf{V}}_{2,1:4}), \mathcal{H}(\tilde{\mathbf{V}}_{6,1:4}), \mathcal{H}(\tilde{\mathbf{V}}_{10,1:4}), \mathcal{H}(\tilde{\mathbf{V}}_{14,1:4})\}$ originating from the second row of $\mathbf{V}_{4\times2}$ pass through the intersection point $\boldsymbol{\phi}(\mathbf{V}_{4\times2}; \{2,6,3\})$; these additional curves are depicted with dashed lines in Fig.4(c) [see $n = 4$ and $n = 5$ respectively]. As ϕ_3 increases, the $\frac{M}{2}$ hypersurfaces continue to intersect to a single point $\boldsymbol{\phi}(\mathbf{V}_{4\times2}; \{2,6,10,14\})$ but the hypersurface $n = 3$ curves away, thus creating $\frac{M}{2} - 1$ new cells. These cells are highlighted in Fig. 4(d).

Since each cell is associated with a distinct M-PSK candidate vector, we can collect all these vectors into

$$\mathcal{J}(\mathbf{V}_{N\times D}) \triangleq \bigcup_{\substack{\mathcal{I}_{2D-1}\subset\{1,2,\ldots,\frac{MN}{2}\},\\ \boldsymbol{\phi}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]}} \left\{\mathbf{s}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})\right\} \subseteq \mathcal{A}_M^N. \tag{42}$$

Several properties of the decision function $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-1})$ are presented in the following proposition. The proof is provided in the Appendix.

Proposition 3: For any $\boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ the following hold true:

(i) $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-2},-\frac{\pi}{M})=e^{j\frac{2\pi}{M}}\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}^{'}_{1:2D-2},\frac{\pi}{M})$ for some $\boldsymbol{\phi}^{'}_{1:2D-2}\in\Phi^{2D-2}$,

(ii) $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-3},\frac{\pi}{2},\phi_{2D-1})=\mathbf{s}(\mathbf{V}_{N\times(D-1)};\boldsymbol{\phi}_{1:2D-3})$,

(iii) $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-3},-\frac{\pi}{2},\phi_{2D-1})=-\mathbf{s}(\mathbf{V}_{N\times D};-\boldsymbol{\phi}_{1:2D-3},\frac{\pi}{2},\phi^{'}_{2D-1}),\forall\phi^{'}_{2D-1}\in(-\frac{\pi}{M},\frac{\pi}{M}]$,

(iv) $\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-3},\pm\frac{\pi}{2},\phi_{2D-1})=\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-3},\pm\frac{\pi}{2},\phi^{'}_{2D-1}),\forall\phi^{'}_{2D-1}\in(-\frac{\pi}{M},\frac{\pi}{M}]$.

■

Taking into consideration only cells into the region of interest $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$, we observe that

$$|\mathcal{J}(\mathbf{V}_{N\times D})|=\sum_{i=0}^{D-1}\binom{N}{i}\binom{N-i}{2(D-i)-1}\frac{M}{2}^{2(D-i)-2}\left(\frac{M}{2}-1\right)^{i}, \tag{43}$$

i.e. there are $|\mathcal{J}(\mathbf{V}_{N\times D})|$ candidate vectors $\mathbf{s}$ in $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$, associated with cells each of which minimizes ϕ_{2D-1} component at a single point that constitutes the intersection of the corresponding $2D-1$ hypersurfaces. We also note that there exist cells that are not associated with such a vertex and contain uncountably many points of the form $(\phi_1,\dots,\phi_{2D-2},-\frac{\pi}{M})$. However, according to Proposition 3, Part (i), every such a cell can be ignored since there exists another cell that contains points of the form $(\phi^{'}_1,\dots,\phi^{'}_{2D-2},\frac{\pi}{M})$, is associated with a rotated equivalent candidate vector and is "led" by a vertex-intersection that lies in $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ (thus, it belongs to $\mathcal{J}(\mathbf{V}_{N\times D})$) unless the initial cell contains a point with $\phi_{2D-2}=\pm\frac{\pi}{2}$, as Proposition 3, Part (iv) mentions. For example, in Fig. 1(c) such cells are identified for $\phi_3=-\frac{\pi}{M}\overset{M=8}{=}-\frac{\pi}{8}$. These M-phase candidate vectors are equivalent rotated versions of the vectors determined for $\phi_3=\frac{\pi}{8}$, hence the former ones can be ignored.

In addition, if $\phi_{2D-2}=\pm\frac{\pi}{2}$ for a particular cell, then this cell "exists" for any $\phi_{2D-1}\in(-\frac{\pi}{M},\frac{\pi}{M}]$, implying that we can ignore ϕ_{2D-1} (or set it to an arbitrary value $\phi^{'}_{2D-1}$), set ϕ_{2D-2} to $\pm\frac{\pi}{2}$, and consider cells defined on $\Phi^{2D-3}\times\{\pm\frac{\pi}{2}\}\times\{\phi^{'}_{2D-1}\}$. Finally, due to Proposition 3, Part (iii), the cells that are defined when $\phi_{2D-2}=-\frac{\pi}{2}$ are associated with vectors which are opposite to the vectors that are associated with cells defined when $\phi_{2D-2}=\frac{\pi}{2}$. Therefore, we can ignore the case $\phi_{2D-2}=-\frac{\pi}{2}$, set $\phi_{2D-2}=\frac{\pi}{2}$, ignore ϕ_{2D-1}, and, according to Proposition 3, Part (ii), identify the cells that are determined by the *reduced-size* matrix $\mathbf{V}_{N\times(D-1)}$ over the hypercube $\Phi^{2D-4}\times(-\frac{\pi}{M},\frac{\pi}{M}]$. As an example, in Fig. 1(c) we set $\phi_3=\frac{\pi}{8}$ and $\phi_2=\frac{\pi}{2}$ and examine the cells that appear on the leftmost vertical edge of the cube for $\phi_1\in(-\frac{\pi}{8},\frac{\pi}{8}]$.

Hence, $\mathcal{S}(\mathbf{V}_{N\times D})=\mathcal{J}(\mathbf{V}_{N\times D})\cup\mathcal{S}(\mathbf{V}_{N\times(D-1)})$ and by induction,

$$\mathcal{S}(\mathbf{V}_{N\times d})=\mathcal{J}(\mathbf{V}_{N\times d})\cup\mathcal{S}(\mathbf{V}_{N\times(d-1)}),\quad d=2,3,\dots,D, \tag{44}$$

which implies that

$$\mathcal{S}(\mathbf{V}_{N\times D})=\mathcal{J}(\mathbf{V}_{N\times D})\cup\mathcal{J}(\mathbf{V}_{N\times(D-1)})\cup\cdots\cup\mathcal{J}(\mathbf{V}_{N\times 1}) \tag{45}$$

$$=\bigcup_{d=0}^{D-1}\mathcal{J}(\mathbf{V}_{N\times(D-d)}). \tag{46}$$

As a result, the cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$ is

$$|\mathcal{S}(\mathbf{V}_{N\times D})| = |\mathcal{J}(\mathbf{V}_{N\times D})| + |\mathcal{J}(\mathbf{V}_{N\times(D-1)})| + \cdots + |\mathcal{J}(\mathbf{V}_{N\times 1})| \tag{47}$$

$$= \sum_{d=1}^{D}\sum_{i=0}^{d-1}\binom{N}{i}\binom{N-i}{2(d-i)-1}\left(\frac{M}{2}\right)^{2(d-i)-2}\left(\frac{M}{2}-1\right)^{i} \tag{48}$$

$$= \mathcal{O}\left(\left(\frac{MN}{2}\right)^{2D-1}\right). \tag{49}$$

We observe that in case $\mathbf{Q}$ is full rank, i.e. $D = N$, the expression (48) returns as many elements as the cardinality of the set $\mathcal{A}_M^{N-1}$, i.e. $|\mathcal{A}_M^{N-1}| = |\mathcal{S}(\mathbf{V}_{N\times D})|$, as the following proposition states.[10] The proof is provided in the Appendix.

Proposition 4: If $D = N$, then the computation of $\mathbf{s}_{\text{opt}}$ is $\mathcal{NP}$-hard and can be implemented by applying exhaustive search or the proposed algorithm among all elements of $\mathcal{A}_M^{N-1}$ since $|\mathcal{A}_M^{N-1}| = |\mathcal{S}(\mathbf{V}_{N\times D})| = M^{N-1}$. ■

To summarize the results, we have utilized $2D-1$ auxiliary hyperspherical coordinates, partitioned the hypercube $\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ into a finite number of cells that are associated with distinct M-phase vectors which constitute $\mathcal{S}(\mathbf{V}_{N\times D}) \subseteq \mathcal{A}_M^N$ and proved that $\mathbf{s}_{\text{opt}} \in \mathcal{S}(\mathbf{V}_{N\times D})$. Therefore, the initial problem in (8) has been converted into numerical maximization of $\|\mathbf{V}^{\mathcal{H}}\mathbf{s}\|$ among all vectors $\mathbf{s} \in \mathcal{S}(\mathbf{V}_{N\times D})$.

IV. Algorithmic Developments

In this section, we present the basic steps of the proposed algorithm for the construction of $|\mathcal{S}(\mathbf{V}_{N\times D})|$ for arbitrary $N, D \in \mathbb{N}, D \leq N$ and $M \in \{2^k \mid k = 1, 2\ldots\}$. Let $\mathcal{C}_d$ be the set that contains all $|\mathcal{J}(\mathbf{V}_{N\times d})|$ combinations of $2d-1$ hypersurfaces that intersect to a single intersection point in $\Phi^{2d-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ for $d = 1, 2, \ldots, D$, i.e. $\mathcal{I}_{2d-1} = \{i_1, i_2, \ldots, i_{2d-1}\} \in \mathcal{C}_d$ if and only if the intersection of hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d})$, $\ldots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d})$ constitutes a vertex of one or more cells in $\Phi^{2d-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$. Furthermore, we define $\mathcal{N}_{\mathcal{I}_{2d-1}} \subset \{1, 2, \ldots, N\}$ as the set of indices of rows from $\mathbf{V}$ related with the $2d-1$ hypersurfaces that participate in the intersection point $\phi(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})$. From eq. (46), we observe that the initial problem of the determination of $|\mathcal{S}(\mathbf{V}_{N\times D})|$ can be divided into smaller parallel construction problems of $|\mathcal{J}(\mathbf{V}_{N\times d})|$ for $d = 1, \ldots, D$. Moreover, the construction of $|\mathcal{J}(\mathbf{V}_{N\times d)})|$ can be fully parallelized since the candidate vector(s) $\mathbf{s}(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})$ can be computed independently for each $\mathcal{I}_{2d-1} \in \mathcal{C}_d$. Therefore, in the sequel we concentrate only on the computation of the candidate vector(s) $\mathbf{s}(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1}), \forall \mathcal{I}_{2d-1} \in \mathcal{C}_d$ and $\forall d \in \{1, 2, \ldots, D\}$.

For the following statements, we assume a certain value for $d \in \{1, 2, \ldots, D\}$ and a certain set of indices $\mathcal{I}_{2d-1} = \{i_1, i_2, \ldots, i_{2d-1}\} \in \mathcal{C}_d$. According to the derivations in the previous section, the combination of hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d})$, $\ldots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d})$ intersects at a single point $\phi(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})$ that "leads" $\mathcal{Q}$ cells $C_1(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})$, $C_2(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})$, $\ldots$, $C_{\mathcal{Q}}(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})$ associated with $\mathcal{Q}$ different M-phase candidate

[10] According to the cardinality derivation, we have ignored rotated candidate vectors and thus the cardinality of the candidate set drops from $|\mathcal{A}_M^N|$ to $|\mathcal{A}_M^{N-1}|$.

vectors $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}),\ q=1,2,\ldots,\mathcal{Q}$. As already stated, the number of cells $\mathcal{Q}$ depends on the number p of pairs of participating hypersurfaces passing through $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ that originate from the same row of $\mathbf{V}$ and equals to $(\frac{M}{2}-1)^p$. To identify $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}),\ q\in\{1,2,\ldots,\mathcal{Q}\}$, we detect one-by-one its N elements separately according to the following rules[11]:

(i) For any $n\in\{1,2,\ldots,N\}$ and $n\notin\mathcal{N}_{\mathcal{I}_{2d-1}}$, the corresponding element of candidate vector $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ maintains its value at $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$, hence it is determined by

$$s_{q,n}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})=s(\mathbf{V}_{n,1:d};\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})), \tag{50}$$

for $\forall q\in\{1,2,\ldots,\mathcal{Q}\}$.

(ii) For any $n\in\mathcal{N}_{\mathcal{I}_{2d-1}}$ such that there is only one hypersurface, say $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$, that is related with the n-th row of $\mathbf{V}$ and participates in the intersection, the corresponding element of $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ cannot be determined at $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$. However, it maintains its value at the intersection of the remaining $2d-2$ hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d-1}),\mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d-1}),\ \ldots,\mathcal{H}(\tilde{\mathbf{V}}_{i_{k-1},1:2d-1}),\mathcal{H}(\tilde{\mathbf{V}}_{i_{k+1},1:2d-1})\ \ldots,\mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d-1})$, hence it is determined by

$$s_{q,n}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})=s(\mathbf{V}_{n,1:d};\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}-\{i_k\})), \tag{51}$$

where $\Im\{\mathbf{V}_{n,d}\}=0$ for $\forall q\in\{1,2,\ldots,\mathcal{Q}\}$.

(iii) For any $n\in\mathcal{N}_{\mathcal{I}_{2d-1}}$ such that there is a pair of hypersurfaces with indices $i_k,i_m\in\mathcal{I}_{2d-1},i_k\neq i_m$, say $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d}),\mathcal{H}(\tilde{\mathbf{V}}_{i_m,1:2d})\in\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$ that participate in the intersection, the corresponding element of $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ cannot be determined at $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$. However, it is observed that the value of s_n in the associated cell $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ can be evaluated by computing the ambiguities at the intersection points of the remaining $2d-3$ hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d-1})$, $\mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d-1})$, $\ldots,\mathcal{H}(\tilde{\mathbf{V}}_{i_{k-1},1:2d-1})$, $\mathcal{H}(\tilde{\mathbf{V}}_{i_{k+1},1:2d-1})$ $\mathcal{H}(\tilde{\mathbf{V}}_{i_{m-1},1:2d-1})$, $\mathcal{H}(\tilde{\mathbf{V}}_{i_{m+1},1:2d-1})\ \ldots,\mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d-1})$ with the hypersurfaces that "construct" the cell $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$, say[12] $\mathcal{H}(\tilde{\mathbf{V}}_{i'_k,1:2d})$, $\mathcal{H}(\tilde{\mathbf{V}}_{i'_m,1:2d})\in\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$, and finding the common M-PSK element of these ambiguities. Hence it is determined by

$$\begin{aligned}&s_{q,n}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})=\\&\quad s\Big(\mathbf{V}_{n,1:d};\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}-\{i_k,i_m\})+\{i'_k\}\Big)\bigcap s\Big(\mathbf{V}_{n,1:d};\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}-\{i_k,i_m\})+\{i'_m\}\Big),\end{aligned} \tag{52}$$

where $\Im\{\mathbf{V}_{n,d}\}=0$ for $\forall q\in\{1,2,\ldots,\mathcal{Q}\}$.

The above statements suggest the following construction of $\mathbf{s}_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}),q\in\{1,2,\ldots,\mathcal{Q}\}$. Assuming distinct intersections of hypersurfaces, the $2d-1$ participating hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d}),\mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d}),\ldots,\mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d})$ pass through the "leading" vertex $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ of cell $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$. If $n\in\{1,2,\ldots,N\}\setminus\mathcal{N}_{\mathcal{I}_{2d-1}}$, i.e., none of the $2d-1$ hypersurfaces originates from the n-th row of $\mathbf{V}$, then none of the hypersurfaces $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$

[11]In the following rules, we denote the n-th element of $\mathbf{s}_q$ as $s_{q,n}$.

[12]We underline that i'_k,i'_m may be equal to i_k,i_m.

passes through $\phi(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$, implying that the decision for $s_{q,n}$ with respect to the partition of the hypercube $\Phi^{2d-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ by $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$ is the same as the decision for $s_{q,n}$ at any point of the cell of interest $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ with respect to the same partition. As a result, the value of the corresponding M-phase element $s_{q,n}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ is well-determined at the "leading" vertex, as (50) states. For example, in Fig. 1(c), $s_4(\mathbf{V}_{4\times 2};\{1,2,3\})$ is well determined at $\phi(\mathbf{V}_{4\times 2};\mathcal{I}_{2d-1})$ through (50) and maintains its value in the associated cell $C(\mathbf{V}_{4\times 2};\{1,2,3\})$.

On the other hand, if $n\in\mathcal{N}_{\mathcal{I}_{2d-1}}$ such that there is only one hypersurface, say $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$ related to the n-th row of $\mathbf{V}$, then, $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$ passes through $\phi(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ leading to an ambiguous decision about $s(\mathbf{V}_{n,1:d};\phi(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}))$ between two neighbouring M-phase elements of the M-PSK alphabet, separated by the decision boundary $\mathcal{B}$ related to $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$. For example, in Fig. 1(c), the hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{1,1:4})$, $\mathcal{H}(\tilde{\mathbf{V}}_{2,1:4})$ and $\mathcal{H}(\tilde{\mathbf{V}}_{3,1:4})$ pass through $\phi(\mathbf{V}_{4\times 2};\{1,2,3\})$ leading to ambiguous decisions of $s(\mathbf{V}_{1,1:2};\phi(\mathbf{V}_{4\times 2};\{1,2,3\}))$, $s(\mathbf{V}_{2,1:2};\phi(\mathbf{V}_{4\times 2};\{1,2,3\}))$ and $s(\mathbf{V}_{3,1:2};\phi(\mathbf{V}_{4\times 2};\{1,2,3\}))$. In such case, ambiguity is resolved if we exclude $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$ and consider the intersection of the remaining $2d-2$ hypersurfaces at $\phi_{2d-1}=\frac{\pi}{M}$. Indeed, the value of $s_{q,n}$ at any point of the cell of interest $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ with respect to $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$ is the same as the value of $s_{q,n}$ at $\phi(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}-\{i_k\})$ with respect to the same hypersurface. Therefore, the value of the corresponding M-PSK element $s_{q,n}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ is well-determined through (51). For example, in Fig. 1(c) the ambiguity with respect to $s_1(\mathbf{V}_{4\times 2};\{1,2,3\})$, $s_2(\mathbf{V}_{4\times 2};\{1,2,3\})$ and $s_3(\mathbf{V}_{4\times 2};\{1,2,3\})$ at intersection $\phi(\mathbf{V}_{4\times 2};\{1,2,3\})$ is resolved through (51) at $C=\phi(\mathbf{V}_{4\times 2};\{2,3\})$, $A=\phi(\mathbf{V}_{4\times 2};\{1,3\})$ and $B=\phi(\mathbf{V}_{4\times 2};\{1,2\})$, respectively.

Finally, if $n\in\mathcal{N}_{\mathcal{I}_{2d-1}}$ such that there is a pair of hypersurfaces, say $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$,$\mathcal{H}(\tilde{\mathbf{V}}_{i_m,1:2d})$, originating from the n-th row of $\mathbf{V}$, then, according to corollary 1, all hypersurfaces of $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$ pass through $\phi(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$. Thus, $\phi(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ is a point on the common intersection axis of the family of hypersurfaces $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$ and thus we have ambiguity for $s_{q,n}$ among all elements of the M-PSK alphabet (there is no preference between two specific M-PSK symbols like in case (ii)). The ambiguity is resolved if we exclude hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$,$\mathcal{H}(\tilde{\mathbf{V}}_{i_m,1:2d})$ and compute the intersection point of the remaining $2d-3$ hypersurfaces with each of the surfaces from $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$ that "construct" the cell $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ at $\phi_{2d-1}=\frac{\pi}{M}$. Since each cell $C_q(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}), q=1,2,\ldots,\mathcal{Q}$, is "constructed" by hypersurfaces from $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(n)}_{:,1:2d})$ rotated by consecutive decision boundaries $\mathcal{B}$, these interesection points lead to ambiguous decision sets about $s_{q,n}$ between neighbouring elements of the M-PSK alphabet. The intersection of these sets determines the value of the corresponding M-phase element $s_{q,n}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$, as (52) states. For example, in Fig. 1(d), we use an arbitraty rank-2 complex matrix $\mathbf{V}_{4\times 2}$ for $M=8$ and depict the intersection of hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{4,1:4})$, $\mathcal{H}(\tilde{\mathbf{V}}_{8,1:4})$ and $\mathcal{H}(\tilde{\mathbf{V}}_{2,1:4})$ [curves $n=1, n=2$ and $n=3$ respectively] where $\mathcal{H}(\tilde{\mathbf{V}}_{4,1:4})$, $\mathcal{H}(\tilde{\mathbf{V}}_{8,1:4})\in\mathcal{H}(\boldsymbol{\mathcal{G}}^{(4)}_{:,1:4})$. According to corollary 1, the remaining hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{12,1:4})$, $\mathcal{H}(\tilde{\mathbf{V}}_{16,1:4})$ of the set $\mathcal{H}(\boldsymbol{\mathcal{G}}^{(4)}_{:,1:4})$ pass through $\phi(\mathbf{V}_{4\times 2};\{2,4,8\})$ [depicted as $n=4, n=5$ respectively]. We observe that $\phi(\mathbf{V}_{4\times 2};\{2,4,8\})$ "leads" $(\frac{M}{2}-1)$ cells, described by the points $\{A,B,E,\phi(\mathbf{V}_{4\times 2};\{2,4,8\})\}$, $\{B,C,E,\phi(\mathbf{V}_{4\times 2};\{2,4,8\})\}$ and $\{C,D,E,\phi(\mathbf{V}_{4\times 2};\{2,4,8\})\}$. Each one of the aforementioned cells is related with a different M-PSK candidate vector. Taking as example the cell that contains the points $\{A,B,E,\phi(\mathbf{V}_{4\times 2};\{2,4,8\})\}$, the ambiguity of $s_4(\mathbf{V}_{4\times 2};\{2,4,8\})$ in this cell is resolved by computing the ambiguity decision sets at points A and B with respect to s_4 and finding the common M-PSK element of these sets, as (52) states. The same procedure is repeated for the other two cells.

To obtain the vector of hyperspherical coordinates $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}), \mathcal{I}_{2d-1} \in \mathcal{C}_d$ efficiently, we just need to compute the zero right singular vector of $\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}$ and calculate its spherical coordinates. More specifically, according to the proof of Proposition 2, Part (i), for a full-rank $(2d-1)\times 2d$ real matrix, the system that represents the intersection of $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2d}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2d}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2d-1},1:2d})$, i.e.

$$\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2d-1}) = \mathbf{0}_{(2d-1)\times 1}, \tag{53}$$

has a unique solution $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1}) \in \Phi^{2d-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]$ which consists of the hyperspherical coordinates of the zero right singular vector of $\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}$. Therefore, to obtain $\boldsymbol{\phi}(\mathbf{V}_{N\times d};\mathcal{I}_{2d-1})$ we just need to compute the zero right singular vector of $\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}$ and calculate its hyperspherical coordinates.

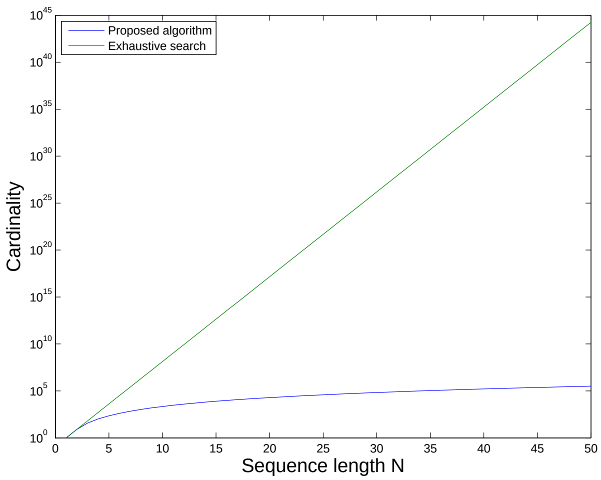
The complete algorithm for the construction of $\mathcal{S}(\mathbf{V}_{N\times D})$ is provided in Table I. The algorithm visits independently $|\mathcal{S}(\mathbf{V}_{N\times D})| = \mathcal{O}\big((\frac{MN}{2})^{2D-1}\big)$ intersections and computes the candidate M-phase vector(s) associated with each intersection. The calculation of the zero right singular vector of $\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}$ costs $\mathcal{O}(d^2)$, the conversion to spherical coordinates costs $\mathcal{O}(d)$ and the operation $\max_{s\in\mathcal{A}_M}\Re\{s^*\mathbf{V}_{n,1:d}\mathbf{c}(\boldsymbol{\phi}_{1:2d-1})\}$ costs $\mathcal{O}(d)$. For each $\mathcal{I}_{2d-1} \in \mathcal{C}_d$, the cost of the algorithm is $\mathcal{O}\big(\frac{MN}{2}\big)$. Therefore the overall complexity of the algorithm for the computation of $\mathcal{S}(\mathbf{V}_{N\times D})$ with fixed $D \le N$ becomes $\mathcal{O}\big((\frac{MN}{2})^{2D-1}\big)\mathcal{O}\big(\frac{MN}{2}\big) = \mathcal{O}\big((\frac{MN}{2})^{2D}\big)$. We recall that the corresponding complexity of the algorithm proposed in [22] is of the order $\mathcal{O}(N^{2D})$ and $\mathcal{O}\big((2N)^{2D}\big)$ for BPSK and QPSK, respectively.

In Fig. 5 we draw the complexity of the proposed algorithm (blue line) and the complexity of the exhaustive search (green line) for $D = 2, M = 8, 32$ (Fig. 5(a)-(b)) and for $D = 3, M = 8, 32$ (Fig.5(c)-(d)). We observe that as the sequence length grows, the complexity of the exhaustive search grows exponentially and becomes impractically large even for moderate sequence lengths. Whereas, the complexity of the proposed algorithm grows polynomially with respect to N, is much faster than exhaustive search and remains practical for moderate sequence lengths.

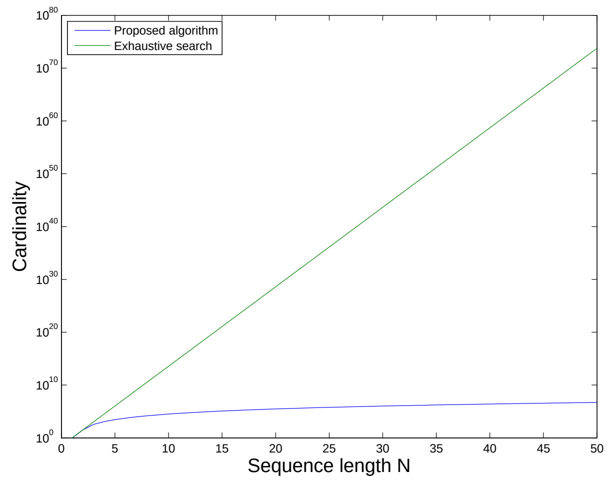
We observe that the computation of the candidate vectors of $\mathcal{S}(\mathbf{V}_{N\times D})$ is performed independently from cell to cell, which implies that there is no need to store the data that have been used for each candidate and we only have to store the "best" vector that has been met. Therefore, the proposed method is fully parallelizable and its memory utilization is efficiently minimized, in constrast to the incremental algorithm in [5]. We also mention that if the initial problem is of a high rank that makes the optimization intractable, then the matrix $\mathbf{Q}$ in (2) can be approximated by keeping the D strongest principal components of it. In such a case, as seen in (46), the proposed method is rank-scalable.

V. Conclusion

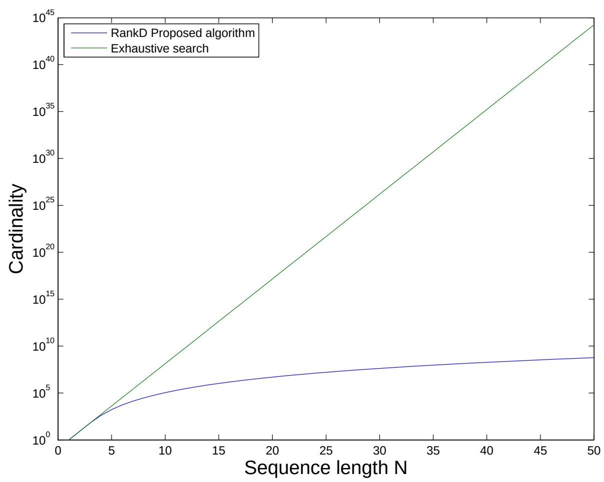
In this paper, we presented a more generalized algorithm for the computation of the maximizing argument of a rank-deficient quadratic form over any M-ary PSK alphabet $\mathcal{A}_M^N$. We do this by utilizing auxiliary hyperspherical coordinates that separate the multidimensional space into a polynomial-size set of cells, each of which is associated



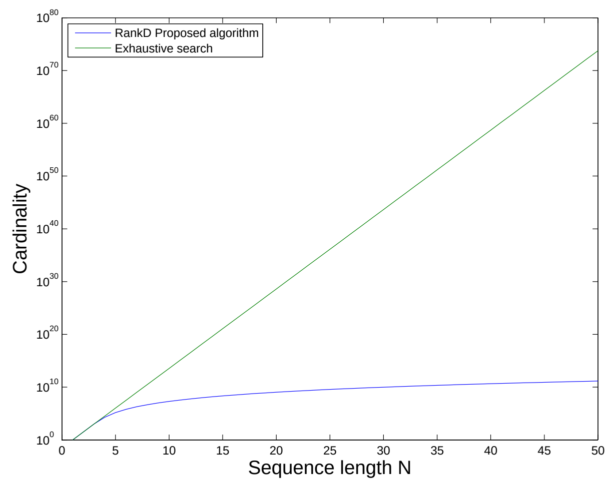
(a) $D = 2, M = 8$



(b) $D = 2, M = 32$



(c) $D = 3, M = 8$



(d) $D = 3, M = 32$

Fig. 5. Proposed algorithm complexity (blue line) Vs. exhaustive search complexity (green line).

with a distinct candidate vector. We have showed that its computational complexity is polynomial in the length N of the maximizing argument if the rank of the observation matrix $\mathbf{Q}$ in the quadratic form $\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$ is independent of N. We continued by developing an algorithm that computes the set $\mathcal{S}(\mathbf{V}_{N\times D})$ of candidate vectors in polynomial time, is fully parallelizable, rank-scalable and time and memory efficient. Thus, without loss of optimality, the proposed algorithm serves as an efficient alternative approach to exhaustive search for the computation of the maximizing M-ary phase vector $\mathbf{s}$ in the quadratic form $\mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$.

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APPENDIX

A. *Proof of Lemma 1*

Assume arbitrary vector $\mathbf{x} \in \mathbb{R}^{2D}$, $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_{2D}]^T$ with radial distance $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_{2D}^2} = r, r \geq 0$. We want to prove that there are unique hyperspherical coordinates $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ that can fully describe vector $\mathbf{x}$. At first, assume that the spherical coordinates of $\mathbf{x}$ are

$$\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \begin{bmatrix} r\sin\phi_1 \\ r\cos\phi_1\sin\phi_2 \\ r\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ r\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin\phi_{2D-1} \\ r\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos\phi_{2D-1} \end{bmatrix}_{2D\times 1} \tag{54}$$

Then

$$\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) = \mathbf{x} \quad \Rightarrow \quad \begin{bmatrix} r\sin\phi_1 \\ r\cos\phi_1\sin\phi_2 \\ r\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ r\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin\phi_{2D-1} \\ r\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos\phi_{2D-1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{2D-1} \\ x_{2D} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{2D-1} \\ \phi_{2D-1} \end{bmatrix} = \begin{bmatrix} \sin^{-1}\left(\frac{x_1}{r}\right) \\ \sin^{-1}\left(\frac{x_2}{r\cos\phi_1}\right) \\ \sin^{-1}\left(\frac{x_3}{r\cos\phi_1\cos\phi_2}\right) \\ \vdots \\ \sin^{-1}\left(\frac{x_{2D-1}}{r\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]}\right) \\ \cos^{-1}\left(\frac{x_{2D}}{r\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]}\right) \end{bmatrix} \tag{55}$$

We observe that given the fact that coordinates $\boldsymbol{\phi}_{1:2D-2}$ lie in $(-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2}$, the values of $\phi_1, \phi_2, \dots, \phi_{2D-2}$ can be uniquely determined using the equations in (55) sequentially, i.e. ϕ_1 can be determined from $\sin^{-1}\left(\frac{x_1}{r}\right)$, ϕ_2 can be determined from $\sin^{-1}\left(\frac{x_2}{r\cos\phi_1}\right)$ using previously computed ϕ_1, etc. But the value of ϕ_{2D-1} cannot be determined uniquely just using one of the two equations of ϕ_{2D-1}. More specifically, using equation $\phi_{2D-1} = \sin^{-1}\left(\frac{x_{2D-1}}{r\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]}\right)$, we have ambiguity between two solutions of ϕ_{2D-1} since $\phi_{2D-1} \in (-\pi, \pi]$. But the ambiguity is resolved if we simultaneously solve the last equation in (55) and choose as ϕ_{2D-1} the intersection of the solutions provided by the last two equations. Therefore, any vector $\mathbf{x} \in \mathbb{R}^{2D}$ can be uniquely described using unique hyperspherical coordinates as defined in (54) and thus $\tilde{\mathbf{c}}(\phi_{2D-1})$ is the hyperspherical coordinate vector that describes the whole $(2D)$-dimensional Euclidean space. ∎

B. *Proof of Proposition 1*

Let $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$ be a hyperspherical coordinate complex vector as defined in (12) for arbitrary hyperspherical coordinates $\boldsymbol{\phi}_{1:2D-1} \in (-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$. We are interested in proving that there always exists an angle $\alpha \in \arg\{\mathcal{A}_{\mathrm{M}}\}$ that relocates the angular coordinate ϕ_{2D-1} of the hyperspherical vector $\left\{\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha}\right\}$ in the interval $(\frac{\pi}{M}, \frac{\pi}{M}]$. Since we are interested only in coordinate ϕ_{2D-1} of the rotated complex coordinate vector

$$\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha} = \begin{bmatrix} c_1(\boldsymbol{\phi}_{1:2}) \\ c_2(\boldsymbol{\phi}_{1:4}) \\ \vdots \\ c_D(\boldsymbol{\phi}_{1:2D-1}) \end{bmatrix} e^{j\alpha} = \begin{bmatrix} \cos\phi_1 \sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2(D-1)}\cos\phi_i\right]\cos\phi_{2D-1} + j\left[\prod_{i=1}^{2(D-1)}\cos\phi_i\right]\sin\phi_{2D-1} \end{bmatrix} e^{j\alpha}$$

we focus on the last row $c_D(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha} = \left(\left[\prod_{i=1}^{2(D-1)}\cos\phi_i\right]\cos\phi_{2D-1} + j\left[\prod_{i=1}^{2(D-1)}\cos\phi_i\right]\sin\phi_{2D-1}\right)e^{j\alpha}$. Using Euler's formula, $c_D(\boldsymbol{\phi}_{1:2D-1})$ can be transformed into $c_D(\boldsymbol{\phi}_{1:2D-1}) = \Re\{c_D(\boldsymbol{\phi}_{1:2D-1})\} + j\Im\{c_D(\boldsymbol{\phi}_{1:2D-1})\} = re^{j\omega}$ where $rgeq0$ and

$$\begin{aligned} \omega &= \arctan\left(\frac{\Im\{c_D(\boldsymbol{\phi}_{1:2D-1})\}}{\Re\{c_D(\boldsymbol{\phi}_{1:2D-1})\}}\right) = \arctan\left(\frac{\left[\prod_{i=1}^{2(D-1)}\cos\phi_i\right]\sin\phi_{2D-1}}{\left[\prod_{i=1}^{2(D-1)}\cos\phi_i\right]\cos\phi_{2D-1}}\right) \\ &= \arctan\left(\frac{\sin\phi_{2D-1}}{\cos\phi_{2D-1}}\right) = \arctan\left(\tan\phi_{2D-1}\right) = \phi_{2D-1}. \end{aligned}$$

Thus, $c_D(\boldsymbol{\phi}_{1:2D-1}) = re^{j\phi_{2D-1}}$. Multiplying this term with $e^{j\alpha}$ we get: $c_D(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha} = re^{j(\phi_{2D-1}+\alpha)} = re^{j(\hat{\phi}_{2D-1})}$. Now, since $\alpha \in \arg\{\mathcal{A}_{\mathrm{M}}\} = \left\{\frac{2\pi \mathrm{m}}{\mathrm{M}} \mid \mathrm{m} = 0, 1, \ldots, \mathrm{M}-1\right\}$, we observe that multiplication of $c_D(\boldsymbol{\phi}_{1:2D-1})$ with $e^{j\alpha}$ leads to relocation of ϕ_{2D-1} of $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$ into one of the M slices of length $\frac{2\pi}{M}$ described in the set

$$\left\{\left(-\frac{\pi}{M}, \frac{\pi}{M}\right], \left(\frac{\pi}{M}, \frac{3\pi}{M}\right], \ldots, \left(\frac{(2M-5)\pi}{M}, \frac{(2M-3)\pi}{M}\right], \left(\frac{(2M-3)\pi}{M}, \frac{(2M-1)\pi}{M}\right]\right\}.$$

W.l.o.g., we choose the value of α such that the last coordinate of the rotated hyperspherical complex vector $\left\{\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})e^{j\alpha}\right\}$ belongs in the interval $\left(-\frac{\pi}{M}, \frac{\pi}{M}\right]$. ■

C. *Proof of Proposition 2*

(i) Consider $\mathcal{I}_{2D-1} = \{i_1, i_2, \ldots, i_{2D-1}\}$ and $2D-1$ hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2D}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2D-1},1:2D})$ that correspond to $2D-1$ rows of $\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}$. Since each hypersurface $\mathcal{H}(\tilde{\mathbf{V}}_{i,1:2D})$ is described by equation $\tilde{\mathbf{V}}_{i,1:2D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = 0, i \in \{i_1, i_2, \ldots, i_{2D-1}\}$, their intersection(s) will satisfy the system of equations

$$\begin{pmatrix} \tilde{\mathbf{V}}_{i_1,1:2D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = 0 \\ \tilde{\mathbf{V}}_{i_2,1:2D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = 0 \\ \vdots \\ \tilde{\mathbf{V}}_{i_{2D-1},1:2D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = 0 \end{pmatrix}.$$

The above system is rewritten as $\tilde{\mathbf{V}}_{\mathcal{I}_{2D-1},1:2D}\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = \mathbf{0}_{(2D-1)\times 1}$. Therefore, the solution $\boldsymbol{\phi}_{1:2D-1}$ is such that $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$ belongs to the null space of $\tilde{\mathbf{V}}_{\mathcal{I}_{2D-1},1:2D}$, denoted by $\mathcal{N}(\tilde{\mathbf{V}}_{\mathcal{I}_{2D-1},1:2D})$, and has dimension greater than or equal to one, since $\mathrm{rank}(\tilde{\mathbf{V}}_{\mathcal{I}_{\mathrm{2D-1}},\mathrm{1:2D}}) \leq \mathrm{2D-1}$. Let

$$\tilde{\mathbf{V}}_{\mathcal{I}_{2D-1},1:2D} = \tilde{\mathbf{U}}_{(2D-1)\times(2D-1)}\mathbf{\Lambda}_{(2D-1)\times 2D}\mathbf{U}^T_{(2D)\times(2D)}$$

be the singular value decomposition of $\tilde{\mathbf{V}}_{\mathcal{I}_{2D-1},1:2D}$, where $\tilde{\mathbf{U}}$ and $\mathbf{U}$ are orthogonal matrices,

$$\mathbf{\Lambda} = [\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{2D-1}})\ \mathbf{0}_{(\mathrm{2D-1})\times 1}],$$

and w.l.o.g. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2D-1} \geq 0$.

We consider two cases.

(i) If $\lambda_{2D-1} > 0$, then $\mathcal{N}(\tilde{\mathbf{V}}_{\mathcal{I}_{2D-1},1:2D}) = \{\alpha\mathbf{U}_{1:2D,2D}\ :\ \alpha \in \mathbb{R}\}$, which implies that $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = \frac{\mathbf{U}_{1:2D,2D}}{\|\mathbf{U}_{1:2D,2D}\|}$ or $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = -\frac{\mathbf{U}_{1:2D,2D}}{\|\mathbf{U}_{1:2D,2D}\|}$. Since we require $\boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, only one solution $\pm\frac{\mathbf{U}_{1:2D,2D}}{\|\mathbf{U}_{1:2D,2D}\|}$ is valid and the hyperspherical coordinate vector we look for is uniquely determined by the hyperspherical coordinates of $\frac{\mathbf{U}_{1:2D,2D}}{\|\mathbf{U}_{1:2D,2D}\|}$ or $-\frac{\mathbf{U}_{1:2D,2D}}{\|\mathbf{U}_{1:2D,2D}\|}$.

(ii) If $\lambda_{2D-1} = 0$, then $\mathrm{dim}\mathcal{N}(\tilde{\mathbf{V}}_{\mathcal{I}_{\mathrm{2D-1}},\mathrm{1:2D}}) \geq 2$ which implies that there are uncountably many solutions for $\mathbf{c}(\boldsymbol{\phi}_{1:2D-1})$ that satisfy the requirement $\boldsymbol{\phi}_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$.

(ii) For arbitrary D, M, N and given complex matrix $\mathbf{V}_{N\times D}$, we construct $\hat{\mathbf{V}}_{\frac{MN}{2}\times D}$ as in (28). Then, $\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}$ is given by

$$\tilde{\mathbf{V}} = \left[\Re(\hat{\mathbf{V}}_{:,1})\ \ \Im(\hat{\mathbf{V}}_{:,1})\ \ \Re(\hat{\mathbf{V}}_{:,2})\ \ \Im(\hat{\mathbf{V}}_{:,2})\ \ \ldots\ \ \Re(\hat{\mathbf{V}}_{:,D})\ \ \Im(\hat{\mathbf{V}}_{:,D})\right].$$

Let $K \in \{3, 4, \ldots, \frac{M}{2}\}$ and $\mathcal{I}_K \triangleq \{i_1, i_2, \ldots i_K\} \subset \{0, 1, \ldots, \frac{M}{2} - 1\}, i_k \neq i_m$ for $k \neq m$, denote a set of K hypersurfaces which are rotated versions of the same, arbitrary chosen, row of $\mathbf{V}$. W.l.o.g., we choose surfaces originating from the first row of the observation matrix $\mathbf{V}$. Motivated by the definition of groups in (34), we define group $\boldsymbol{\mathcal{G}}^{(1)}$

$$\boldsymbol{\mathcal{G}}^{(1)} \triangleq \begin{bmatrix} \tilde{\mathbf{V}}_{1,1:2D} \\ \tilde{\mathbf{V}}_{(1+N),1:2D} \\ \vdots \\ \tilde{\mathbf{V}}_{\left(1+(\frac{M}{2}-1)N\right),1:2D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{G}}^{(1)}_{1,1:2D} \\ \boldsymbol{\mathcal{G}}^{(1)}_{2,1:2D} \\ \vdots \\ \boldsymbol{\mathcal{G}}^{(1)}_{\frac{M}{2},1:2D} \end{bmatrix}_{\frac{M}{2}\times 2D}$$

which is related with the first row of $\mathbf{V}$. To prove proposition $2(ii)$, we define the following system of equations for arbitrary K,

$$\begin{pmatrix} \boldsymbol{\mathcal{G}}^{(1)}_{i_1,1:2D} \\ \boldsymbol{\mathcal{G}}^{(1)}_{i_2,1:2D} \\ \vdots \\ \boldsymbol{\mathcal{G}}^{(1)}_{i_K,1:2D} \end{pmatrix} \tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) = \mathbf{0}_{\mathcal{I}_K\times 1} \Rightarrow \tag{56}$$

$$\underbrace{\begin{pmatrix} \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}\big) & \Im\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}\big) & \dots & \Re\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}\big) & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}\big) \\ \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_2}{M}}\big) & \Im\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_2}{M}}\big) & \dots & \Re\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_2}{M}}\big) & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}\big) \\ & & \vdots & & \\ \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_K}{M}}\big) & \Im\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_K}{M}}\big) & \dots & \Re\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_K}{M}}\big) & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_K}{M}}\big) \end{pmatrix}}_{(A)} \tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) = \mathbf{0}_{\mathcal{I}_K \times 1}. \quad (57)$$

W.l.o.g., we assume $i_1 < i_2 < \cdots < i_K$. Therefore, expression (A) in (57) can be further transformed into[13]

$$\begin{pmatrix} \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}\big) & \dots & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}\big) \\ \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}e^{-j\frac{2\pi(i_2-i_1)}{M}}\big) & \dots & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}e^{-j\frac{2\pi(i_2-i_1)}{M}}\big) \\ & \vdots & \\ \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}e^{-j\frac{2\pi(i_K-i_1)}{M}}\big) & \dots & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}e^{-j\frac{2\pi(i_K-i_1)}{M}}\big) \end{pmatrix} =$$

$$\begin{pmatrix} \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}\big) & \dots & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}\big) \\ \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}[\cos\frac{2\pi(i_2-i_1)}{M} - j\sin\frac{2\pi(i_2-i_1)}{M}]\big) & \dots & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}[\cos\frac{2\pi(i_2-i_1)}{M} - j\sin\frac{2\pi(i_2-i_1)}{M}]\big) \\ & \vdots & \\ \Re\big(\mathbf{V}_{(1,1)}e^{-j\frac{2\pi i_1}{M}}[\cos\frac{2\pi(i_K-i_1)}{M} - j\sin\frac{2\pi(i_K-i_1)}{M}]\big) & \dots & \Im\big(\mathbf{V}_{(1,D)}e^{-j\frac{2\pi i_1}{M}}[\cos\frac{2\pi(i_K-i_1)}{M} - j\sin\frac{2\pi(i_K-i_1)}{M}]\big) \end{pmatrix}$$

and by letting $\alpha_d \triangleq \Re\big(\mathbf{V}_{(1,d)}e^{-j\frac{2\pi i_1}{M}}\big), \beta_d \triangleq \Im\big(\mathbf{V}_{(1,d)}e^{-j\frac{2\pi i_1}{M}}\big)$ for $d \in \{1, 2, \dots, D\}$ and $\omega_l \triangleq \frac{2\pi(i_l-i_1)}{M}$ for $l \in \{1, 2, \dots, K\}$ we get

$$\begin{pmatrix} [\alpha_1\cos\omega_1 + \beta_1\sin\omega_1] & \dots & [\beta_D\cos\omega_1 - \alpha_D\sin\omega_1] \\ [\alpha_1\cos\omega_2 + \beta_1\sin\omega_2] & \dots & [\beta_D\cos\omega_2 - \alpha_D\sin\omega_2] \\ & \vdots & \\ [\alpha_1\cos\omega_K + \beta_1\sin\omega_K] & \dots & [\beta_D\cos\omega_K - \alpha_D\sin\omega_K] \end{pmatrix}.$$

Thus, eq. (57) becomes

$$\begin{pmatrix} [\alpha_1\cos\omega_1 + \beta_1\sin\omega_1] & \dots & [\beta_D\cos\omega_1 - \alpha_D\sin\omega_1] \\ [\alpha_1\cos\omega_2 + \beta_1\sin\omega_2] & \dots & [\beta_D\cos\omega_2 - \alpha_D\sin\omega_2] \\ & \vdots & \\ [\alpha_1\cos\omega_K + \beta_1\sin\omega_K] & \dots & [\beta_D\cos\omega_K - \alpha_D\sin\omega_K] \end{pmatrix} \begin{bmatrix} \sin\phi_1 \\ \cos\phi_1\sin\phi_2 \\ \cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin\phi_{2D-1} \\ \Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos\phi_{2D-1} \end{bmatrix} = \mathbf{0}_{\mathcal{I}_K \times 1}.$$

Solving the above system of equations with respect to ϕ_1, we get:

[13]For the sake of simplicity and space, from now on we shall only include the first and last element of each row in the equations.

$$\begin{pmatrix} \tan\phi_1 = -\frac{(\beta_1\cos\omega_1-\alpha_1\sin\omega_1)\sin\phi_2+\cdots+(\alpha_D\cos\omega_1+\beta_D\sin\omega_1)\prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1}+(\beta_D\cos\omega_1-\alpha_D\sin\omega_1)\prod_{i=2}^{2D-1}\cos\phi_i}{(\alpha_1\cos\omega_1+\beta_1\sin\omega_1)} \\ \vdots \\ \tan\phi_1 = -\frac{(\beta_1\cos\omega_2-\alpha_1\sin\omega_2)\sin\phi_2+\cdots+(\alpha_D\cos\omega_2+\beta_D\sin\omega_2)\prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1}+(\beta_D\cos\omega_2-\alpha_D\sin\omega_2)\prod_{i=2}^{2D-1}\cos\phi_i}{(\alpha_1\cos\omega_2+\beta_1\sin\omega_2)} \\ \vdots \\ \tan\phi_1 = -\frac{(\beta_1\cos\omega_K-\alpha_1\sin\omega_K)\sin\phi_2+\cdots+(\alpha_D\cos\omega_K+\beta_D\sin\omega_K)\prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1}+(\beta_D\cos\omega_K-\alpha_D\sin\omega_K)\prod_{i=2}^{2D-1}\cos\phi_i}{(\alpha_1\cos\omega_K+\beta_1\sin\omega_K)} \end{pmatrix}. \quad (58)$$

Running over the whole domain for $\boldsymbol{\phi}_{2:2D-1} \in \Phi^{2D-3} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$, the equations in (58) define K different hypersurfaces as functions of ϕ_1. To show that two or more hypersurfaces originating from the same row of the observation matrix $\mathbf{V}$ have the same intersection, it suffices to show that the intersection of any combination of two equations from the above system is independent of their phase rotation difference $\omega_l, l \in \{1, 2, \dots, K\}$. Choosing randomly two equations $m, n \in \{1, 2, \dots, K\}, m \neq n$ from the above system of equations and computing their intersection, we get

$$-\frac{\begin{bmatrix} \beta_1\cos\omega_m - \alpha_1\sin\omega_m \\ \vdots \\ \alpha_D\cos\omega_m + \beta_D\sin\omega_m \\ \beta_D\cos\omega_m - \alpha_D\sin\omega_m \end{bmatrix}^T \begin{bmatrix} \sin\phi_2 \\ \vdots \\ \prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1} \\ \prod_{i=2}^{2D-1}\cos\phi_i\cos\phi_{2D-1} \end{bmatrix}}{(\alpha_1\cos\omega_m + \beta_1\sin\omega_m)} = -\frac{\begin{bmatrix} \beta_1\cos\omega_n - \alpha_1\sin\omega_n \\ \vdots \\ \alpha_D\cos\omega_n + \beta_D\sin\omega_n \\ \beta_D\cos\omega_n - \alpha_D\sin\omega_n \end{bmatrix}^T \begin{bmatrix} \sin\phi_2 \\ \vdots \\ \prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1} \\ \prod_{i=2}^{2D-1}\cos\phi_i\cos\phi_{2D-1} \end{bmatrix}}{(\alpha_1\cos\omega_n + \beta_1\sin\omega_n)} \Rightarrow$$

$$\begin{bmatrix} (\alpha_1^2+\beta_1^2)(\cos\omega_m\sin\omega_n - \cos\omega_n\sin\omega_m) \\ \vdots \\ (\beta_1\alpha_D - \alpha_1\beta_D)(\cos\omega_m\sin\omega_n - \cos\omega_n\sin\omega_m) \\ (\alpha_1\alpha_D + \beta_1\beta_D)(\cos\omega_m\sin\omega_n - \cos\omega_n\sin\omega_m) \end{bmatrix}^T \begin{bmatrix} \sin\phi_2 \\ \vdots \\ \prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1} \\ \prod_{i=2}^{2D-1}\cos\phi_i\cos\phi_{2D-1} \end{bmatrix} = 0 \Rightarrow$$

$$\sin(\omega_m - \omega_n)\begin{bmatrix} \alpha_1^2+\beta_1^2 \\ \vdots \\ \beta_1\alpha_D - \alpha_1\beta_D \\ \alpha_1\alpha_D + \beta_1\beta_D \end{bmatrix}^T \begin{bmatrix} \sin\phi_2 \\ \vdots \\ \prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1} \\ \prod_{i=2}^{2D-1}\cos\phi_i\cos\phi_{2D-1} \end{bmatrix} = 0 \Rightarrow$$

$$\begin{bmatrix} \alpha_1^2+\beta_1^2 \\ \vdots \\ \beta_1\alpha_D - \alpha_1\beta_D \\ \alpha_1\alpha_D + \beta_1\beta_D \end{bmatrix}^T \begin{bmatrix} \sin\phi_2 \\ \vdots \\ \prod_{i=2}^{2D-2}\cos\phi_i\sin\phi_{2D-1} \\ \prod_{i=2}^{2D-1}\cos\phi_i\cos\phi_{2D-1} \end{bmatrix} = 0$$

Since the above equation is satisfied for any combination of two equations in the system (56), we ought to assume that the intersection of any two or more rotated hypersurfaces is common and independent of their phase rotation difference $\omega_l, l \in \{1, 2, \dots, K\}$. $\blacksquare$

D. ***Proof of Proposition 3***

Given that

$$\mathbf{c}(\boldsymbol{\phi}_{1:2D-1}) = \begin{bmatrix} \cos\phi_1 \sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1 \cos\phi_2 \cos\phi_3 \sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\cos\phi_{2D-1} + j\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\sin\phi_{2D-1} \end{bmatrix}_{D\times 1} \tag{59}$$

we use the following lemma in the subsequent derivations:

Lemma 2: Let $\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M})$ be a hyperspherical complex vector where $\boldsymbol{\phi}_{1:2D-2} \in \Phi^{2D-2}$. Then, the following expression holds true

$$e^{j\frac{2\pi}{M}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M}) = \mathbf{c}(\boldsymbol{\phi}^{'}_{1:2D-2}, \frac{\pi}{M}), \text{ for some } \boldsymbol{\phi}^{'}_{1:2D-2} \in \Phi^{2D-2}. \tag{60}$$

□

Proof: According to (59), $\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M})$ equals to

$$\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M}) = \begin{bmatrix} \cos\phi_1 \sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1 \cos\phi_2 \cos\phi_3 \sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\cos(-\frac{\pi}{M}) + j\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\sin(-\frac{\pi}{M}) \end{bmatrix}.$$

Rotating $\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M})$ by phasor $e^{j\frac{2\pi}{M}}$, we get

$$\begin{aligned} e^{j\frac{2\pi}{M}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M}) &= \begin{bmatrix} e^{j\frac{2\pi}{M}}(\cos\phi_1 \sin\phi_2 + j\sin\phi_1) \\ e^{j\frac{2\pi}{M}}(\cos\phi_1 \cos\phi_2 \cos\phi_3 \sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3) \\ \vdots \\ e^{j\frac{2\pi}{M}}\left(\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\cos(-\frac{\pi}{M}) + j\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\sin(-\frac{\pi}{M})\right) \end{bmatrix} \\ &= \begin{bmatrix} \cos\phi^{'}_1 \sin\phi^{'}_2 + j\sin\phi^{'}_1 \\ \cos\phi^{'}_1 \cos\phi^{'}_2 \cos\phi^{'}_3 \sin\phi^{'}_4 + j\cos\phi^{'}_1\cos\phi^{'}_2\sin\phi^{'}_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2}\cos\phi^{'}_i\right]\cos\phi^{'}_{2D-1} + j\left[\prod_{i=1}^{2D-2}\cos\phi^{'}_i\right]\sin\phi^{'}_{2D-1} \end{bmatrix} \\ &= \mathbf{c}(\boldsymbol{\phi}^{'}_{1:2D-1}) \end{aligned}$$

i.e. $\boldsymbol{\phi}^{'}_{1:2D-1}$ are the hyperspherical coordinates of complex vector $e^{j\frac{2\pi}{M}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-2}, -\frac{\pi}{M})$. Calculating explicitly the values of $\phi^{'}_{1:2D-2}$, we have

$$
\begin{aligned}
\sin\phi^{'}_1 &= \Im\{e^{j\frac{2\pi}{M}}(\cos\phi_1\sin\phi_2 + j\sin\phi_1)\} \Rightarrow \\
\phi^{'}_1 &= \sin^{-1}\big(\cos\frac{2\pi}{M}\sin\phi_1 + \sin\frac{2\pi}{M}\cos\phi_1\sin\phi_2\big) \in (-\frac{\pi}{2},\frac{\pi}{2}], \qquad (61)\\
\sin\phi^{'}_2 &= \Re\{e^{j\frac{2\pi}{M}}(\cos\phi_1\sin\phi_2 + j\sin\phi_1)\} \Rightarrow \\
\phi^{'}_2 &= \sin^{-1}\Big(\frac{\cos\frac{2\pi}{M}\cos\phi_1\sin\phi_2 - \sin\frac{2\pi}{M}\sin\phi_1}{\cos\phi^{'}_1}\Big) \in (-\frac{\pi}{2},\frac{\pi}{2}], \qquad (62)\\
\sin\phi^{'}_3 &= \Im\{e^{j\frac{2\pi}{M}}(\cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3)\} \Rightarrow \\
\phi^{'}_3 &= \sin^{-1}\Big(\frac{\cos\frac{2\pi}{M}\cos\phi_1\cos\phi_2\sin\phi_3 + \sin\frac{2\pi}{M}\cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4}{\cos\phi^{'}_1\cos\phi^{'}_2}\Big) \in (-\frac{\pi}{2},\frac{\pi}{2}], \qquad (63)\\
&\dots\\
\sin\phi^{'}_{2D-2} &= \Re\Bigg\{e^{j\frac{2\pi}{M}}\Bigg(\Big[\prod_{i=1}^{2D-3}\cos\phi_i\Big]\sin\phi_{2D-2} + j\Big[\prod_{i=1}^{2D-4}\cos\phi_i\Big]\sin\phi_{2D-3}\Bigg)\Bigg\} \Rightarrow \\
\phi^{'}_{2D-2} &= \sin^{-1}\Big(\frac{\cos\frac{2\pi}{M}[\prod_{i=1}^{2D-3}\cos\phi_i]\sin\phi_{2D-2} - \sin\frac{2\pi}{M}[\prod_{i=1}^{2D-4}\cos\phi_i]\sin\phi_{2D-3}}{\prod_{i=1}^{2D-3}\cos\phi^{'}_i}\Big) \in (-\frac{\pi}{2},\frac{\pi}{2}]. \qquad (64)
\end{aligned}
$$

For the last hyperspherical coordinate $\phi^{'}_{2D-1}$ we have the following system of equations

$$
\begin{pmatrix}
\sin\phi^{'}_{2D-1} = \Im\Bigg\{e^{j\frac{2\pi}{M}}\Bigg(\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos(-\frac{\pi}{M}) + j\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin(-\frac{\pi}{M})\Bigg)\Bigg\}\\
\cos\phi^{'}_{2D-1} = \Re\Bigg\{e^{j\frac{2\pi}{M}}\Bigg(\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos(-\frac{\pi}{M}) + j\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin(-\frac{\pi}{M})\Bigg)\Bigg\}
\end{pmatrix} \Rightarrow
$$

$$
\begin{pmatrix}
\sin\phi^{'}_{2D-1} = \frac{\cos\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin(-\frac{\pi}{M})+\sin\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos(-\frac{\pi}{M})}{\prod_{i=1}^{2D-2}\cos\phi^{'}_i}\\
\cos\phi^{'}_{2D-1} = \frac{\cos\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos(-\frac{\pi}{M})-\sin\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin(-\frac{\pi}{M})}{\prod_{i=1}^{2D-2}\cos\phi^{'}_i}
\end{pmatrix}.
$$

Dividing these two equations, we have

$$
\begin{aligned}
\frac{\sin\phi^{'}_{2D-1}}{\cos\phi^{'}_{2D-1}} &= \frac{\frac{\cos\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin(-\frac{\pi}{M})+\sin\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos(-\frac{\pi}{M})}{\prod_{i=1}^{2D-2}\cos\phi^{'}_i}}{\frac{\cos\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\cos(-\frac{\pi}{M})-\sin\frac{2\pi}{M}\Big[\prod_{i=1}^{2D-2}\cos\phi_i\Big]\sin(-\frac{\pi}{M})}{\prod_{i=1}^{2D-2}\cos\phi^{'}_i}} \Rightarrow \\
\tan\phi^{'}_{2D-1} &= \frac{\big[\prod_{i=1}^{2D-2}\cos\phi_i\big]\big(\cos\frac{2\pi}{M}\sin(-\frac{\pi}{M}) + \sin\frac{2\pi}{M})\cos(-\frac{\pi}{M})\big)}{\big[\prod_{i=1}^{2D-2}\cos\phi_i\big]\big(\cos\frac{2\pi}{M}\cos(-\frac{\pi}{M}) - \sin\frac{2\pi}{M})\sin(-\frac{\pi}{M})\big)} \Rightarrow \\
\tan\phi^{'}_{2D-1} &= \frac{\sin(\frac{2\pi}{M} - \frac{\pi}{M})}{\cos(\frac{2\pi}{M} - \frac{\pi}{M})} = \tan\frac{\pi}{M} \Rightarrow \\
\phi^{'}_{2D-1} &= \frac{\pi}{M}
\end{aligned}
$$

Therefore, $\phi^{'}_{1:2D-2} \in \Phi^{2D-2}$ and $\phi^{'}_{2D-1} = \frac{\pi}{M}$ and, hence, we have proved the equation (60). $\square$

(i) $$\mathbf{s}(\mathbf{V}_{\mathrm{N\times D}};\boldsymbol{\phi}_{1:2\mathrm{D}-2},-\frac{\pi}{\mathrm{M}})=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-2},-\frac{\pi}{M})\}$$
$$\overset{\text{Lem. 2}}{=}\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\left\{(\mathbf{s}e^{j\frac{2\pi}{M}})^{\mathcal{H}}\mathbf{V}\,\underbrace{e^{j\frac{2\pi}{M}}\mathbf{c}(\boldsymbol{\phi}_{1:2D-2},-\frac{\pi}{M})}_{\mathbf{c}(\boldsymbol{\phi}'_{1:2D-2},\frac{\pi}{M})}\right\}$$
$$=e^{-j\frac{2\pi}{M}}\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}'_{1:2D-2},\frac{\pi}{M})\right\}$$
$$=e^{-j\frac{2\pi}{M}}\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}'_{1:2D-2},\frac{\pi}{M}),\text{for some }\boldsymbol{\phi}'_{1:2D-2}\in\Phi^{2D-2}.$$

(ii) $$\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-3},\frac{\pi}{2},\phi_{2D-1})=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-3},\frac{\pi}{2},\phi_{2D-1})\}$$
$$=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\begin{bmatrix}\cos\phi_1\sin\phi_2+j\sin\phi_1\\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4+j\cos\phi_1\cos\phi_2\sin\phi_3\\ \vdots\\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\sin\frac{\pi}{2}+j\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\sin\phi_{2D-3}\\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos\frac{\pi}{2}\cos\phi_{2D-1}+j\left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos\frac{\pi}{2}\sin\phi_{2D-1}\end{bmatrix}\right\}$$
$$=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}_{N\times(D-1)}\begin{bmatrix}\cos\phi_1\sin\phi_2+j\sin\phi_1\\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4+j\cos\phi_1\cos\phi_2\sin\phi_3\\ \vdots\\ \left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\cos\phi_{2D-3}+j\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\sin\phi_{2D-3}\end{bmatrix}\right\}$$
$$=\mathbf{s}(\mathbf{V}_{N\times(D-1)};\boldsymbol{\phi}_{1:2D-3}).$$

(iii) $$\mathbf{s}(\mathbf{V}_{N\times D};\boldsymbol{\phi}_{1:2D-3},-\frac{\pi}{2},\phi_{2D-1})=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-3},-\frac{\pi}{2},\phi_{2D-1})\}$$
$$=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\begin{bmatrix}\cos\phi_1\sin\phi_2+j\sin\phi_1\\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4+j\cos\phi_1\cos\phi_2\sin\phi_3\\ \vdots\\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\sin(-\frac{\pi}{2})+j\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\sin\phi_{2D-3}\\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos(-\frac{\pi}{2})\cos\phi_{2D-1}+j\left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos(-\frac{\pi}{2})\sin\phi_{2D-1}\end{bmatrix}\right\}$$
$$=\arg\max_{\mathbf{s}\in\mathcal{A}_M^N}\Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}_{N\times(D-1)}\begin{bmatrix}-\cos(-\phi_1)\sin(-\phi_2)-j\sin(-\phi_1)\\ -\cos(-\phi_1)\cos(-\phi_2)\cos(-\phi_3)\sin(-\phi_4)-j\cos(-\phi_1)\cos(-\phi_2)\sin(-\phi_3)\\ \vdots\\ -\left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]-j\left[\prod_{i=1}^{2D-4}\cos(-\phi_i)\right]\sin(-\phi_{2D-3})\end{bmatrix}\right\}$$

$$= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\left\{ \mathbf{s}^{\mathcal{H}}\mathbf{V}_{N\times D}\begin{bmatrix} -\cos(-\phi_1)\sin(-\phi_2) - j\sin(-\phi_1) \\ -\cos(-\phi_1)\cos(-\phi_2)\cos(-\phi_3)\sin(-\phi_4) - j\cos(-\phi_1)\cos(-\phi_2)\sin(-\phi_3) \\ \vdots \\ -\left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]\sin\frac{\pi}{2} - j\left[\prod_{i=1}^{2D-4}\cos(-\phi_i)\right]\sin(-\phi_{2D-3}) \\ -\left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]\cos\frac{\pi}{2}\cos\phi'_{2D-1} - j\left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]\cos\frac{\pi}{2}\sin\phi'_{2D-1} \end{bmatrix} \right\}$$

$$= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\left\{ -\mathbf{s}^{\mathcal{H}}\mathbf{V}_{N\times D}\begin{bmatrix} \cos(-\phi_1)\sin(-\phi_2) + j\sin(-\phi_1) \\ \cos(-\phi_1)\cos(-\phi_2)\cos(-\phi_3)\sin(-\phi_4) + j\cos(-\phi_1)\cos(-\phi_2)\sin(-\phi_3) \\ \vdots \\ \left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]\sin\frac{\pi}{2} + j\left[\prod_{i=1}^{2D-4}\cos(-\phi_i)\right]\sin(-\phi_{2D-3}) \\ \left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]\cos\frac{\pi}{2}\cos\phi'_{2D-1} + j\left[\prod_{i=1}^{2D-3}\cos(-\phi_i)\right]\cos\frac{\pi}{2}\sin\phi'_{2D-1} \end{bmatrix} \right\}$$

$$= -\mathbf{s}(\mathbf{V}_{N\times D}; -\boldsymbol{\phi}_{1:2D-3}, \frac{\pi}{2}, \phi'_{2D-1}).$$

(iv) $\mathbf{s}(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{1:2D-3}, \pm\frac{\pi}{2}, \phi_{2D-1}) = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\boldsymbol{\phi}_{1:2D-3}, \pm\frac{\pi}{2}, \phi_{2D-1})\}$

$$= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\left\{ \mathbf{s}^{\mathcal{H}}\mathbf{V}\begin{bmatrix} \cos\phi_1\sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\sin(\pm\frac{\pi}{2}) + j\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\sin\phi_{2D-3} \\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos(\pm\frac{\pi}{2})\cos\phi_{2D-1} + j\left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos(\pm\frac{\pi}{2})\sin\phi_{2D-1} \end{bmatrix} \right\}$$

$$= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\left\{ \mathbf{s}^{\mathcal{H}}\mathbf{V}_{N\times(D-1)}\begin{bmatrix} \cos\phi_1\sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \pm\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\cos\phi_{2D-3} + j\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\sin\phi_{2D-3} \end{bmatrix} \right\}$$

$$= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\left\{ \mathbf{s}^{\mathcal{H}}\mathbf{V}\begin{bmatrix} \cos\phi_1\sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1\cos\phi_2\cos\phi_3\sin\phi_4 + j\cos\phi_1\cos\phi_2\sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\sin(\pm\frac{\pi}{2}) + j\left[\prod_{i=1}^{2D-4}\cos\phi_i\right]\sin\phi_{2D-3} \\ \left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos(\pm\frac{\pi}{2})\cos\phi'_{2D-1} + j\left[\prod_{i=1}^{2D-3}\cos\phi_i\right]\cos(\pm\frac{\pi}{2})\sin\phi'_{2D-1} \end{bmatrix} \right\}$$

$$= \mathbf{s}(\mathbf{V}_{N\times D}; \boldsymbol{\phi}_{1:2D-3}, \pm\frac{\pi}{2}, \phi'_{2D-1}).$$

$\blacksquare$

E. ***Proof of Proposition 4***

Asssuming $D = N$, observation matrix $\mathbf{V}$ becomes a full rank symmetric complex matrix and thus $\mathbf{V}\mathbf{V}^{\mathcal{H}} = \sum_{n=1}^{N} \mathbf{v}_n \mathbf{v}_n^{\mathcal{H}} = \sum_{n=1}^{N} \lambda_n \mathbf{q}_n \mathbf{q}_n^{\mathcal{H}} = \mathbf{Q}$. In this case, the cardinality expression in (48) can be rewritten as

$$|\mathcal{S}(\mathbf{V}_{N\times N})| = \sum_{d=1}^{N}\sum_{i=0}^{d-1} \binom{N}{i}\binom{N-i}{2(d-i)-1}\Big(\frac{M}{2}\Big)^{2(d-i)-2}\Big(\frac{M}{2}-1\Big)^{i}.$$

Interchanging summations and making some variable substitutions, we can transform the above equation in the following equivalent form[14]

$$\begin{aligned}
|\mathcal{S}(\mathbf{V}_{N\times N})| &= \sum_{i=0}^{N}\sum_{\substack{d=0,\\ d \text{ odd}}}^{N-i} \binom{N}{i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d-1}\Big(\frac{M}{2}-1\Big)^{i} \\
&= \Big(\frac{M}{2}\Big)^{-1}\sum_{i=0}^{N}\sum_{\substack{d=0,\\ d \text{ odd}}}^{N-i} \binom{N}{i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d}\Big(\frac{M}{2}-1\Big)^{i} \\
&= \Big(\frac{M}{2}\Big)^{-1}\sum_{i=0}^{N}\binom{N}{i}\Big(\frac{M}{2}-1\Big)^{i}\sum_{\substack{d=0,\\ d \text{ odd}}}^{N-i} \binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d}.
\end{aligned} \tag{65}$$

Definition: According to binomial formula, for $\forall a, b \in \mathbb{R}$ and $n, k \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}. \tag{66}$$

$\square$

Corollary 2: The sum of the coeffcients of the odd terms of the expansion $(a+b)^n$ is equal to the sum of the coefficients the even terms. $\square$

Using the binomial formula and according to the corollary 2,

$$\sum_{\substack{d=0,\\ d \text{ odd}}}^{N-i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d} = \frac{1}{2}\sum_{d=0}^{N-i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d}.$$

Therefore, (65) becomes:

$$\begin{aligned}
|\mathcal{S}(\mathbf{V}_{N\times N})| &= \Big(\frac{M}{2}\Big)^{-1}\sum_{i=0}^{N}\binom{N}{i}\Big(\frac{M}{2}-1\Big)^{i}\sum_{\substack{d=0,\\ d \text{ odd}}}^{N-i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d} \\
&= \Big(\frac{M}{2}\Big)^{-1}\sum_{i=0}^{N}\binom{N}{i}\Big(\frac{M}{2}-1\Big)^{i}\frac{1}{2}\sum_{d=0}^{N-i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d} \\
&= (M)^{-1}\sum_{i=0}^{N}\binom{N}{i}\Big(\frac{M}{2}-1\Big)^{i}\sum_{d=0}^{N-i}\binom{N-i}{d}\Big(\frac{M}{2}\Big)^{d}
\end{aligned}$$

[14]As in most binomial and multinomial proofs, quantities of the form 0^0 are assumed to be equal to 1.

$$\overset{(66)}{=} (M)^{-1} \sum_{i=0}^{N} \binom{N}{i} \Big(\frac{M}{2} - 1\Big)^{i} \Big(\frac{M}{2} + 1\Big)^{N-i}$$
$$= (M)^{-1} \Big(\frac{M}{2} - 1 + \frac{M}{2} + 1\Big)^{N}$$
$$= (M)^{-1} M^{N} = M^{N-1}.$$

■

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