Contents

TECHNICAL UNIVERSITY OF CRETE

DEPARTMENT OF SCIENCES

Graduate Program
APPLIED SCIENCES AND TECHNOLOGY

MSc Thesis
in APPLIED AND COMPUTATIONAL MATHEMATICS

On Quantum Channels: Special Constructions Of Random and Optimally Unitary Channel Maps

ERASMIA VARIKOU

Supervisor: Professor Demosthenes Ellinas

Chania, 2014

Abstract

The study of quantum channels constitutes a main area of Quantum Information Science. Quantum channel maps represent any dynamic changes and/or erroneous modifications affecting quantum signals in the course of their processing within the context of various computational or communicational algorithms. In the colloquial language of quantum information it is said that "a quantum channel acts on the quantum signal", or stated in precise mathematical terms, we have that "a positive and completely positive trace preserving map acts on the Hermitian, positive and trace one state operator". The representation theory of such channel maps provides the so called "operator sum representation" for them, in terms of the so called Kraus generators. This Thesis puts forward a construction technique for some new families of particular channels of the type of random and optimally unitary channels. This is done by working in the space of particular classes of circulant matrices acting on finite dimensional Hilbert spaces. The resulting channels are featuring unital maps which act on state matrices of signals via some convex combinations of the adjoint action of their unitary Kraus generators. The effect of these channels on quantum signals is further investigated by their induced action on the spectrum of their associated density matrices. This task is carried out for finite dimensional signals by determining the bi-stochastic matrices associated with the constructed channels. Basic convex geometric properties of bi-stochastic matrices (Birkhoff's theorem) provide means for studying the effects on the probabilistic eigenvalues of quantum signals, hence to account for e.g. entropic transformations exercised by the new maps upon quantum signals.

Acknowledgements

First of all, I would like to thank my professor Demosthenes Ellinas for his unlimited patience, support, guidance and precious help. I also would like to thank my colleagues Christos Konstandakis, Alexandros Dimitriadis and George Moutzianou for their help as well as the members of the examination committee Ellinas Demosthenes, prof. Manousakis Antonios and prof. Petrakis Minos as well as the Technical University of Crete.

This thesis is submitted for the partial fulfilment of the requirements of the MSc degree in Applied and Computational Mathematics of the Science Department, Technical University of Crete.

Dedicated to my family and friends

1 Mathematical Tools

1.1 Qubits

As modern-day computers use bits as their basic unit of information processing, which can assume one of two states that we label 0 and 1, we can define *qubits* or *quantum bits* as the information processing units for quantum computation. Like a bit, a quantum bit can also be found in one of two states which can be denoted as $|0\rangle$ and $|1\rangle$. In quantum theory an object enclosed using the notation $|\cdot\rangle$ can be called a *state*, a *vector* or a *ket* using P.A.M. Dirac's *brak-ket* notation. This, represents a column vector. The respective row vector is denoted by $\langle\cdot|$ and is called *bra.*

While in an ordinary computer a bit can be found in the states 0 or 1, a qubit can exist in the state $|0\rangle$ or in the state $|1\rangle$, but it can also exist in a so called *superposition state.* This is a linear combination of the states $|0\rangle$ and $|1\rangle$. If we label a superposition state as $|\psi\rangle$, then it is written as[17]

$$|\psi\rangle = a\,|0\rangle + \beta\,|1\rangle\,, \tag{1}$$

where a, β are complex numbers of the form $z = x + iy$ and $i = \sqrt{-1}$.

While a qubit can be found in a superposition of the states $|0\rangle$ and $|1\rangle$, whenever we make a measurement, we are not going to find it like that. In fact we are going to find it in only one of the two states. The laws of quantum mechanics tell us that the modulus squared of a, β in (1) gives us the probability of finding the qubit in state $|0\rangle$ or $|1\rangle$, respectively. That is,

$$|a|^2 : \text{is the probability of finding } |\psi\rangle \text{ in state } |0\rangle\,,$$

$$|\beta|^2 : \text{is the probability of finding } |\psi\rangle \text{ in state } |1\rangle\,.$$

Due to the fact that the probabilities must sum to one the multiplicative coefficients in (1) have some constraints in what they can be. Since the squares of these coefficients are related to the probability of obtaining a given measurement result, a and β are constrained by the requirement that

$$|a|^2 + |\beta|^2 = 1. \tag{2}$$

When this condition is satisfied for the squares of the coefficients of a qubit, we say that the qubit is *normalized.*

1.2 Hilbert Space $\mathcal{H}$

Definition 1 *An inner-product space $\mathcal{H}$ is a complex vector space, provided with an inner product $\langle\cdot\,|\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ satisfying the following axioms for any vectors $\phi, \psi, \phi_1, \phi_2 \in \mathcal{H}$, and any $c_1, c_2 \in \mathbb{C}$ [21],[22].*

$$\langle\phi\,|\psi\rangle = \langle\psi\,|\phi\rangle^*\,, \tag{3}$$

$$\langle\psi\,|\psi\rangle \geq 0 \textit{ and } \langle\psi\,|\psi\rangle = 0 \textit{ if and only if } \psi = 0, \tag{4}$$

$$\langle\psi\,|c_1\phi_1 + c_2\phi_2\rangle = c_1\,\langle\psi\,|\phi_1\rangle + c_2\,\langle\psi\,|\phi_2\rangle\,. \tag{5}$$

The inner-product introduces on $\mathcal{H}$ the norm

$$\|\psi\| = \sqrt{\langle \psi \,|\psi\rangle}, \tag{6}$$

and the Euclidean distance

$$d\,(\phi, \psi) = \|\phi - \psi\|\,. \tag{7}$$

Definition 2 *A Hilbert space $\mathcal{H}$ is an abstract vector space processing the structure of an inner product that allows length and angle to be measured.*

Some properties of the norm

$$\|\phi\| \geq 0 \text{ for all } \phi \in \mathcal{H} \text{ and } \|\phi\| = 0 \text{ if and only if } \phi = 0, \tag{8}$$

$$\|\phi + \psi\| \leq \|\phi\| + \|\psi\| \text{ (triangle inequality)}, \tag{9}$$

$$\|a\phi\| = |a|\,\|\phi\| \text{ for } a \in \mathbb{R}, \tag{10}$$

$$|\langle \phi \,|\psi\rangle| \leq \|\phi\|\,\|\psi\| \text{ (Schwarz inequality)}. \tag{11}$$

1.3 Tensor Products

In quantum mechanics sometimes we need to work in larger groups than the Hilbert space. In these situations we construct a large group made of 2 or more Hilbert spaces with the use of a tool called Kronecker or tensor product. This tool is denoted by the symbol $\otimes$. For example, if we need to construct the Hilbert space $\mathcal{H}$ from two other Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with dimensions N_1 and N_2, respectively, then the new Hilbert space will be

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \tag{12}$$

and its dimention will be the product of the dimentions of the other two Hilbert spaces

$$\dim \mathcal{H} = \dim \mathcal{H}_1 \dim \mathcal{H}_2 = N_1 N_2. \tag{13}$$

A state vector of the Hilbert space $\mathcal{H}$ is the tensor product of the state vectors of the other two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Let's take the state vectors $|\phi\rangle \in \mathcal{H}_1$, $|\chi\rangle \in \mathcal{H}_2$ and $|\psi\rangle \in \mathcal{H}$ then

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle\,. \tag{14}$$

We know also that the tensor product of two vectors is linear[22]

$$|\phi\rangle \otimes [|\chi_1\rangle + |\chi_2\rangle] = |\phi\rangle \otimes |\chi_1\rangle + |\phi\rangle \otimes |\chi_2\rangle\,, \tag{15}$$

$$[|\phi_1\rangle + |\phi_2\rangle] \otimes |\chi\rangle = |\phi_1\rangle \otimes |\chi\rangle + |\phi_2\rangle \otimes |\chi\rangle\,, \tag{16}$$

as well as that the tensor product is linear with respect to scalars

$$|\phi\rangle \otimes (\alpha\,|\chi\rangle) = \alpha\,|\phi\rangle \otimes |\chi\rangle\,, \tag{17}$$

and vice versa. In order to construct a basis for Hilbert space $\mathcal{H}$ we will use the tensor product of the basis vectors of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If we denote the basis vectors of the Hilbert space $\mathcal{H}_1$ by $|u_i\rangle$ and $\mathcal{H}_2$ by $|v_i\rangle$ then we can construct the basis vectors $|w_i\rangle$ of $\mathcal{H}$ using

$$|w_i\rangle = |u_i\rangle \otimes |v_i\rangle \,. \tag{18}$$

Furthermore it is important to mention that the order of tensor product is not relevant

$$|\phi\rangle \otimes |\chi\rangle = |\chi\rangle \otimes |\phi\rangle \,. \tag{19}$$

To compute the inner product of two vectors of the Hilbert space $\mathcal{H}$ we take the inner products of the vectors belonging to $\mathcal{H}_1$ and $\mathcal{H}_2$ and multiply them together

$$|\psi_1\rangle = |\phi_1\rangle \otimes |\chi_1\rangle \,, \tag{20}$$

$$|\psi_2\rangle = |\phi_2\rangle \otimes |\chi_2\rangle \,. \tag{21}$$

Thus,

$$\langle\psi_1|\,\psi_2\rangle = (\langle\phi_1| \otimes \langle\chi_1|)\,(|\phi_2\rangle \otimes |\chi_2\rangle) = \langle\phi_1|\,\phi_2\rangle\,\langle\chi_1|\,\chi_2\rangle \,. \tag{22}$$

Now, in order to go from $\mathbb{C}^2 \to \mathbb{C}^4$ and compute the tensor product of column vectors, if we consider having

$$|\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } |\chi\rangle = \begin{pmatrix} c \\ d \end{pmatrix} , \tag{23}$$

then the tensor product is

$$|\phi\rangle \otimes |\chi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} . \tag{24}$$

There is something else we need to know about tensor products. The way they interact with tensor products. In this case, if we have the previous vectors $|\phi\rangle \in \mathcal{H}_1$, $|\chi\rangle \in \mathcal{H}_2$ and $|\psi\rangle \in \mathcal{H}$ and the operators A and B that act on $|\phi\rangle$ and $|\chi\rangle$, respectively, then we have

$$(A \otimes B)\,|\psi\rangle = (A \otimes B)\,(|\phi\rangle \otimes |\chi\rangle) = (A\,|\phi\rangle) \otimes (B\,|\chi\rangle) \,. \tag{25}$$

To conclude, we will mention how to compute tensor products of matrices. If we have the 2×2 matrices A and B and we want to produce a new matrix that acts on a four-dimentional Hilbert space then we take their tensor product

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} , \tag{26}$$

and their tensor product is,

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \tag{27}$$

The reason why we mentioned the tensor product is because we are going to use it to compute the collective channel version of the most common quantum channels.

1.4 Trace

Definition 3 *Let $\mathcal{H}$ be an $n-$dimentional Hilbert space and $\mathcal{B}$ be an orthogonal basis on $\mathcal{H}$. The trace operator of a linear mapping $M : \mathcal{H} \to \mathcal{H}$ is defined by*

$$Tr\,(M) = \sum_{\phi \in \mathcal{B}} \langle \phi | \, M \, | \phi \rangle \,. \tag{28}$$

Moreover, if A is the matrix representation of M in the basis $\mathcal{B}$, then

$$Tr\,(M) = Tr\,(A) = \sum_{i=1}^{n} a_{ii}, \tag{29}$$

where a_{ii} are the diagonal elements of A.

Furthermore, trace of an $n \times n$ matrix $A \in M_n$ can be obtained as well, by the sum of the eigenvalues $\lambda_i \in \mathbb{C}$ of A,

$$Tr\,(A) = \lambda_1 + \cdots + \lambda_n.$$

Trace has some basic properties such as, the trace is linear mapping

$$Tr\,(A + B) = Tr\,(A) + Tr\,(B)\,, \tag{30}$$

$$Tr\,(cA) = cTr\,(A)\,. \tag{31}$$

for all square matrices A and B, and all scallars c. A matrix and its transpose have the same trace,

$$Tr\,(A) = Tr\,(A^{\intercal})\,. \tag{32}$$

However, there are properties about product of matrices of which we will mention only the ones in need of this paper. Matrices $A_{m\times n}$ and $B_{n\times m}$ in a trace of a product can be switched,

$$Tr\,(AB) = Tr\,(BA)\,. \tag{33}$$

Another property is that the trace is invariant under cyclic permutations, known as the cyclic property. For example for matrices A, B, C and D we have that

$$Tr\,(ABCD) = Tr\,(BCDA) = Tr\,(CDAB) = Tr\,(DABC)\,. \tag{34}$$

It is important to note that not all permutations are allowed. For instance $Tr\,(ABC) \neq Tr\,(ACB)$. A property for the trace of a tensor product is

$$Tr\,(A \otimes B) = Tr\,(A)\, Tr\,(B)\,. \tag{35}$$

Furthermore,

$$Tr\,(|\psi\rangle\,\langle\phi|) = \langle\phi\;|\psi\rangle\,. \tag{36}$$

1.5 Matrices and Operators

1.5.1 Hermitian Operator

Of paramount importance are adjoint and self-adjoint operators. An adjoint operator T^* to a bounded operator T is an operator that for every $\psi,\, \phi \in \mathcal{H}$,

$$\langle\psi\;|T\phi\rangle = \langle T^*\psi\;|\phi\rangle\,. \tag{37}$$

An operator T is called self-adjoint if $T = T^*$. We can also denote $\langle\psi\;|T\phi\rangle$ as $\langle\psi|\,T\,|\phi\rangle$, as a concequence (37) can be written

$$\langle\psi\;|T\phi\rangle = \langle\psi|\,T\,|\phi\rangle = \langle T^*\psi\;|\phi\rangle\,. \tag{38}$$

Hermitian matrices belong to self-adjoint matrices, i.e, if A Hermitian matrix, then $A = A^*$.

Theorem 4 *Hermitian matrices have the following properties.*

1. $\lambda_i \in \mathbb{R}$, where λ_i the eigenvalues of a Hermitian matrix.

2. All eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal.

   Proof. (Property 1). If $A\phi = \lambda\phi$, then

   $$\lambda^*\,\langle\phi\;|\phi\rangle = \langle\lambda\phi\;|\phi\rangle = \langle A\phi\;|\phi\rangle = \langle\phi\;|A\phi\rangle = \lambda\,\langle\phi\;|\phi\rangle\,. \tag{39}$$

   As a result, $\lambda^* = \lambda$. ■

   Proof. (Property 2). Let $\lambda \neq \lambda'$, $A\phi = \lambda\phi$, $A\phi' = \lambda'\phi'$. Since, $\lambda,\, \lambda' \in \mathbb{R}$, it holds

   $$\lambda'\,\langle\phi'\;|\phi\rangle = \langle A\phi'\;|\phi\rangle = \langle\phi'\;|A\phi\rangle = \lambda\,\langle\phi'\;|\phi\rangle\,, \tag{40}$$

   and therefore $\langle\phi'\;|\phi\rangle = 0$. ■

A self-adjoint operator A of a finite dimentional Hilbert space $\mathcal{H}$ has the spectral representation. If $\lambda_1, \ldots, \lambda_n$ are its distinct eigenvalues, then A can be expressed

$$A = \sum_{i=1}^{n} \lambda_i P_i, \tag{41}$$

where P_i is the projection operator in the subspace of $\mathcal{H}$ spanned by the eigenvectors corresponding to λ_i.

When all eigenvalues are distinct and $|\phi_i\rangle$ is the eigenstate/eigenvector corresponding to the eigenvalue λ_i, then

$$A = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle \langle\phi_i| . \tag{42}$$

Since $P_i P_j = 0$ for two different projections, it holds for any polynomial p

$$p(A) = \sum_{i=1}^{n} p(\lambda_i) P_i. \tag{43}$$

This can be generalized to define for any function $f : \mathbb{R} \to \mathbb{C}$ by

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) P_i. \tag{44}$$

Example 5 *The eigenvalues of the X Pauli matrix are 1 and -1 and the corresponding eigenvectors $|0'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1'\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Since $|0'\rangle\langle 0'| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $|1'\rangle\langle 1'| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$, the matrix*

$$X = 1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - 1 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{45}$$

and

$$\sqrt{X} = \sqrt{1} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \sqrt{-1} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{46}$$

that is, $\sqrt{X} = \sqrt{1} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + i \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ which gives,

$$\sqrt{X} = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}. \tag{47}$$

1.5.2 Unitary Operator

Each self-adjoint operator A has spectral decomposition $A = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle \langle\phi_i|$ and therefore,

$$e^{iA} = \sum_{j=1}^{n} e^{i\lambda_j} |\phi_j\rangle \langle\phi_j| , \tag{48}$$

hence,

$$\left(e^{iA}\right)^* = \sum_{j=1}^{n} e^{-i\lambda_j} |\phi_j\rangle \langle\phi_j| = \left(e^{iA}\right)^{-1}, \tag{49}$$

which implies that the matrix e^{iA} is unitary.

We show now that every unitary matrix $U = e^{iH}$ for a self-adjoint operator H. Actually, if U is written in the form $U = A + iB$, then A and B are both self-adjoint and have spectral decompositions

$$A = \sum_{i=1}^{n} \lambda_i \left|\phi_i\right\rangle \left\langle\phi_i\right|, \tag{50}$$

$$B = \sum_{i=1}^{n} \mu_i \left|\phi_i\right\rangle \left\langle\phi_i\right|. \tag{51}$$

All eigenvalues of a unitary matrix have an absolute value 1 and self-adjoint matrices have eigenvalues real. Therefore, for each j a $\theta_j \in [0, 2\pi]$ has to exist such that $\lambda_j + i\mu_j = e^{i\theta_j}$. Hence, for

$$H = \sum_{i=1}^{n} \theta_j \left|\phi_j\right\rangle \left\langle\phi_j\right|, \tag{52}$$

we have $U = e^{iH}$.

Example 6 *Hadamard transform* $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *has eigenvalues* 1 *and* -1 *and the corresponding eigenvectors are* $\phi_1 = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$ *and* $\phi_{-1} = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$. *The spectral decomposition of* H *is*

$$H = 1 \left|\phi_1\right\rangle \left\langle\phi_1\right| - 1 \left|\phi_{-1}\right\rangle \left\langle\phi_{-1}\right|, \tag{53}$$

and as a consequence, $H = e^{iA}$, *where*

$$A = 0 \left|\phi_1\right\rangle \left\langle\phi_1\right| + \pi \left|\phi_{-1}\right\rangle \left\langle\phi_{-1}\right|. \tag{54}$$

1.6 Matrix Vector Spaces

This chapter presents some prerequisities of vector space formed by matrices (see [7]).

1.6.1 The Inner Product Space $\mathbb{C}^n$

For $n \in \mathbb{N}$ let $\mathbb{C}^n$ denote the vector space over the complex field $\mathbb{C}$ of complex column vectors with n entries $x = \left(\xi_j\right)_{j=1}^{n}$. On this vector space consider the *inner product*

$$\langle x, y\rangle = \sum_{j=1}^{n} \xi_j \overrightarrow{\eta_j}, x = \left(\xi_j\right)_{j=1}^{n},\ y = \left(\eta_j\right)_{j=1}^{n}. \tag{55}$$

Take into consideration that in this notation, the inner product $\langle \cdot, \cdot \rangle$ is linear in the first variable x and conjugate linear in the second variable y. We denote by $\|\cdot\|$ the associated unitary *norm*, which is,

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2 \right)^{1/2}, x = (\xi_j)_{j=1}^{n}, \tag{56}$$

and by $\left\{ e_i^{(n)} \right\}_{i=1}^{n}$ the *canonical basis* of $\mathbb{C}^n$ where, $e_i^{(n)}$ is the $n-$tuple with 1 on the $i-$th position and 0 elsewhere.

1.6.2 The Vector Space $M_{k,k}$

For arbitrary numbers $k \in \mathbb{N}$ we denote by $M_{k,k}$ the vector space over the field of complex numbers $\mathbb{C}$ of $k \times k$ matrices with complex entries. We identify in a natural way $M_{k,k}$ (i.e. the space of rectangular complex matrices $k \times k$) with the vector space $\mathcal{L}\left(\mathbb{C}^k, \mathbb{C}^k\right)$ of linear transformations $A : \mathbb{C}^k \to \mathbb{C}^k$, by means of the canonical bases $\left\{ e_i^{(k)} \right\}_{i=1}^{k}$ and $\left\{ e_j^{(k)} \right\}_{j=1}^{k}$. More precisely, the identification is $A = |a_{i,j}|_{i=1,\ldots,k,j=1,\ldots,k}$ where

$$a_{i,j} = \left\langle Ae_j^{(k)}, e_i^{(k)} \right\rangle, i = 1, \ldots, k, j = 1, \ldots, k. \tag{57}$$

By this identification, on $M_{k,k}$ there exists the *operator norm,* more explicitly,

$$\|A\| = \sup \left\{ \|Ax\| \,| x \in \mathbb{C}^k, \ \|x\| \leq 1 \right\} \tag{58}$$

$$= \inf \left\{ t \geq 0 | \ \|Ax\| \leq t \|x\| \text{ for all } x \in \mathbb{C}^k \right\}. \tag{59}$$

This norm renders the matrix $M_{k,k}$ a normed space.

On $M_{k,k}$ we regard the *adjoint operation*, $M_{k,k} \ni A \longmapsto A^* \in M_{k,k}$, where the matrix of A^* is obtained by changing rows into columns in the matrix of A and taking the complex conjugate. In terms of the identification of $M_{k,k}$ with the vector space $\mathcal{L}\left(\mathbb{C}^k, \mathbb{C}^k\right)$, this implies

$$\langle Ax, y \rangle = \langle x, A^* y \rangle, x \in \mathbb{C}^k, \ y \in \mathbb{C}^k. \tag{60}$$

The map $M_{k,k} \ni A \longmapsto A^* \in M_{k,k}$ has the following properties:

- $(aA + \beta B)^* = \overline{a} A^* + \overline{\beta} B^*$, $A, B \in M_{k,k}$, $a, \beta \in \mathbb{C}$;
- $(AB)^* = B^* A^*$, $A \in M_{k,k}$ and $B \in M_{k,m}$;
- $(A^*)^* = A$, $A \in M_{k,k}$.

With respect to the canonical base of $\mathbb{C}^k$, for $k \in \mathbb{N}$, we consider the matrix units $\left\{ E_{i,j}^{(k)} | i = 1, \ldots, k, \ j = 1, \ldots, k \right\} \subset M_{k,k}$ of size $k \times k$, that is, $E_{i,j}^{(k)}$ is the $k \times k$ matrix with all entries 0 except the $(i, j)-$th entry which is 1.

We also record the following direct consequences of the definitions: for all $j = 1, \ldots, n$ and $i = 1, \ldots, k$ we have

$$E_{i,j}^{(k)*} = E_{j,i}^{(k)}, \tag{61}$$

and if, in addition, $p \in \mathbb{N}$, $r = 1, \ldots, k$, and $s = 1, \ldots, p$, then

$$E_{i,j}^{(n,k)} E_{r,s}^{(k,p)} = \delta_{j,r} E_{i,s}^{(n,p)}. \tag{62}$$

1.6.3 The Matrix Algebra M_k

We let $M_k = M_{k,k}$ and note that it is an algebra over the complex field. On M_k we consider the adjoint operation A^* which now is internal $M_k \ni A \longmapsto A^* \in M_k$. Thus, M_k is a unital $*-$algebra; we denote by I_k its unit, that is, the matrix with 1 on the main diagonal and 0 elsewhere.

A matrix $A \in M_k$ is called *selfadjoint (hermitian)* if $A = A^*$. If A is selfadjoint then all its eigenvalues are simple and real. A matrix $A \in M_k$ is called *positive* if it is selfadjoint and all its eigenvalues are nonnegative. We denote by M_k^+ the set of positive matrices from M_k. We state the following proposition without proof **[7]**:

Proposition 7 *Let $A \in M_k$. The following assertions are equivalent:*

(i) A is positive.
(ii) $A = B^* B$ for some $B \in M_k$.
(iii) $A = B^2$ for some $B \in M_k^+$.
(iv) $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{C}^k$.

The operator norm $\|A\|$ makes M_k a unital normed algebra, that is,

$$\|AB\| \leq \|A\| \, \|B\|, \;\; A, B \in M_k, \;\; \|I_k\| = 1. \tag{63}$$

With respect to the involution A^* the norm has an important property:

$$\|A^* A\| = \|A\|^2, \;\; A \in M_k. \tag{64}$$

In particular, the involution is isometric, that is, $\|A^*\| = \|A\|$ for all $A \in M_k$.

On M_k there is a special linear form, the *trace* $Tr : M_k \to \mathbb{C}$ defined as the sum of the entries from the main diagonal

$$Tr\,(A) = \sum_{j=1}^{k} a_{j,j}, A = [a_{i,j}]_{i,j=1}^{k} \in M_k. \tag{65}$$

In addition to linearity, the trace has two remarkable properties:

$$Tr\,(AB) = Tr\,(BA)\,, \;\; A, B \in M_k \text{ and } Tr\,(A) \geq 0, \;\; A \in M_k^+. \tag{66}$$

The trace is *faithful* in the sense that if $A \in M_k^+$ and $Tr\,(A) = 0$ then $A = 0$.

From now on and for the rest of the paper we will no longer denote the conjugate transpose by a star $*$ but with a dagger $\dagger$.

1.7 Density Matrix

1.7.1 Definition

Pure states are fundamental objects for quantum mechanics in the sense that the evolution of any closed quantum system can be seen as a unitary evolution of pure states.

However, to deal with unisolated and composed quantum systems the concept of mixed states is of great importance. A probability distribution $\{(p_i, \phi_i)\,|1 \leq i \leq n\}$ on pure states $\{\phi_i\}_{i=1}^n$ with probabilities $0 \leq p_i \leq 1$, $\sum_{i=1}^n p_i = 1$ is called a mixed state and is denoted by $[\psi\rangle = \{(p_i, \phi_i)\,|1 \leq i \leq n\}$. For example, a mixed state is created if a source produces pure state $|\phi_i\rangle$ with probability p_i and $\sum_{i=1}^n p_i = 1$. To each mixed state $[\psi\rangle = \{(p_i, \phi_i)\,|1 \leq i \leq n\}$ corresponds a density operator

$$\rho_{[\psi\rangle} = \sum_{i=1}^n p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|. \tag{67}$$

Example 8 *The density matrix corresponding to the mixed state*

$$\left(\frac{1}{2}, |0\rangle\right) \oplus \left(\frac{1}{2}, |1\rangle\right), \tag{68}$$

is

$$\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \frac{1}{2}\mathbf{1}. \tag{69}$$

Example 9 *For every one qubit state of the form $\alpha\,|0\rangle + \beta\,|1\rangle$, to the mixed state*

$$\left(\frac{1}{4}, \alpha\,|0\rangle + \beta\,|1\rangle\right) \oplus \left(\frac{1}{4}, \alpha\,|0\rangle - \beta\,|1\rangle\right) \oplus \left(\frac{1}{4}, \beta\,|0\rangle + \alpha\,|1\rangle\right) \oplus \left(\frac{1}{4}, \beta\,|0\rangle - \alpha\,|1\rangle\right), \tag{70}$$

corresponds the density matrix

$$\frac{1}{4}\begin{pmatrix}a\\\beta\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix} + \frac{1}{4}\begin{pmatrix}a\\-\beta\end{pmatrix}\begin{pmatrix}\alpha & -\beta\end{pmatrix} + \frac{1}{4}\begin{pmatrix}\beta\\\alpha\end{pmatrix}\begin{pmatrix}\beta & \alpha\end{pmatrix} + \frac{1}{4}\begin{pmatrix}\beta\\-\alpha\end{pmatrix}\begin{pmatrix}\beta & -\alpha\end{pmatrix} = \frac{1}{4}\mathbf{1}. \tag{71}$$

If ρ is a density matrix and in a basis $\{\beta_i\}_{i=1}^n$

$$\rho = \left\{\rho_{i,j}\right\}_{i,j=1}^n, \tag{72}$$

then

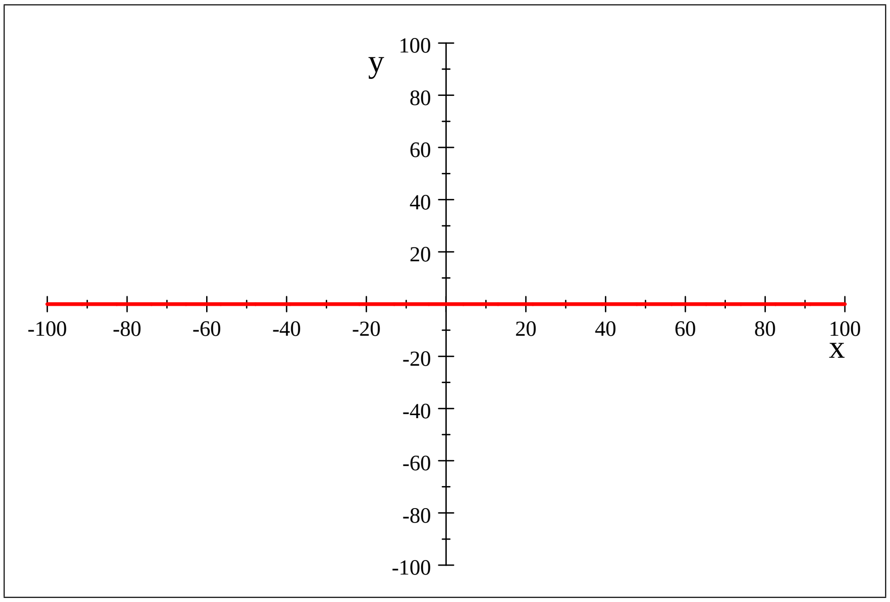
$$\rho = \sum_{i,j=1}^n \rho_{i,j} \left|\beta_i\right\rangle \left\langle\beta_j\right|. \tag{73}$$

As a consequence, for every k, l, $\langle\beta_k|\,\rho\,|\beta_l\rangle = \rho_{k,l}$.

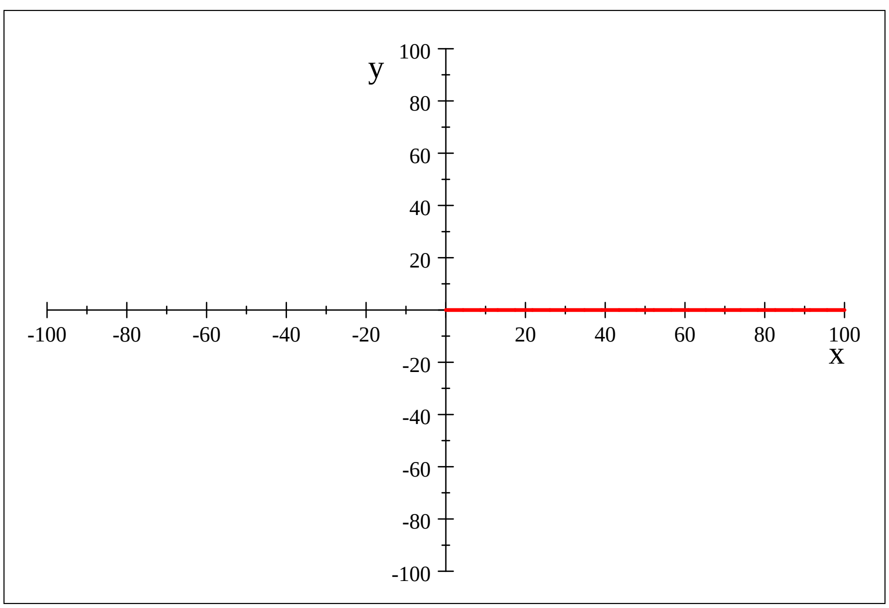
1.7.2 Properties

1. If $\rho^2 = \rho$ for a density matrix ρ, then ρ is a pure state, i.e. $\rho = |\phi\rangle\langle\phi|$ for a pure state $|\phi\rangle$.

2. A matrix ρ is a density matrix if it is Hermitian, i.e. $\rho = \rho^\dagger$,

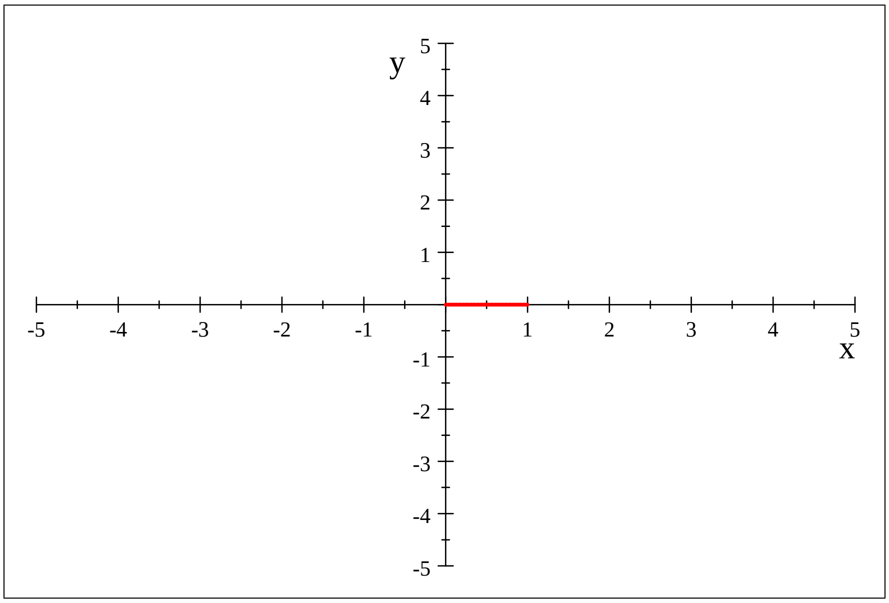


nonnegative,



and

$$Tr\left(\rho\right)=\sum_i \left\langle u_i\right|\rho\left|u_i\right\rangle=\sum_i \left\langle u_i\right|\left.\psi\right\rangle\left\langle\psi\right|\left.u_i\right\rangle=\sum_i c_i c_i^*=\sum_i \left|c_i\right|^2=1. \tag{74}$$



If ρ_1 is a density matrix on a Hilbert space $\mathcal{H}_1$ and ρ_2 is a density matrix on a Hilbert space $\mathcal{H}_2$, then $\rho_1 \otimes \rho_2$ is a density matrix on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

If ρ is a density matrix, then so is the matrix $\rho^{\intercal}$.

If ρ_1, ρ_2 are density matrices on a Hilbert space $\mathcal{H}$, then $p\rho_1+(1-p)\,\rho_2$, $0\leq p\leq 1$ is a density matrix on $\mathcal{H}$.

In general, suppose there are n possible states. For a mixed state $\left|\psi_i\right\rangle$ the density operator is written as $\rho_i = \left|\psi_i\right\rangle\left\langle\psi_i\right|$. Denote the probability that a member of the ensemble has been prepared in the state $\left|\psi_i\right\rangle$ as p_i. Then the density operator for the entire system is

$$\rho=\sum_{i=1}^{n} p_i\rho_i=\sum_{i=1}^{n} p_i\left|\psi_i\right\rangle\left\langle\psi_i\right|. \tag{75}$$

Suppose, for example, that the evolution of a closed quantum system is described by the unitary operator U. If the system was initially in the state $\left|\psi_i\right\rangle$ with probability p_i then after the evolution has occurred the system will be in the state $U\left|\psi_i\right\rangle$ with probability p_i. Thus, the evolution of the density operator is described by the equation

$$\rho=\sum_{i=1}^{n} p_i\left|\psi_i\right\rangle\left\langle\psi_i\right| \xrightarrow{U} \sum_{i=1}^{n} p_i U\left|\psi_i\right\rangle\left\langle\psi_i\right| U^{\dagger}=U\rho U^{\dagger}. \tag{76}$$

A quantum system whose state $|\psi\rangle$ is known exactly is said to be in a *pure state.* In this case the density operator is simply $\rho = |\psi\rangle\langle\psi|$ and it satisfies $Tr\left(\rho^2\right) = 1$. Otherwise, ρ is in a *mixed state*; it is said to be a *mixture* of the different pure states in the ensemble of ρ and it satisfies $Tr\left(\rho^2\right) < 1$.[21]

1.8 Probabilities in Quantum Mechanics

In order to understand the purpose of this thesis we need to begin with the basics. Quantum mechanics is a branch of physics which deals with physical phenomena at microscopic scales. The formalism of quantum mechanics can be derived from a few postulates which are justified by experiments. The set of axioms defining the quantum theory differs depending on the author. However, some features occur common in every formulation, either as axioms or as their consequences. One of such key features is the *superposition principle.* It is satisfied by several experimental data as interference pattern in double slit experiment with electrons or interference of a single photon in the Mach Zender interferometer. The superposition principle states that the state of a quantum system, which is denoted in Dirac notation by $|\psi\rangle$, can be represented by a coherent combination of several states $|\psi_i\rangle$ with complex coefficients a_i [17], [21], [22], [23],

$$|\psi\rangle = \sum_i a_i |\psi_i\rangle . \tag{77}$$

The quantum state $|\psi\rangle$ of an N level system is represented by a vector from the complex Hilbert space $\mathcal{H}_N$. The inner product $\langle\psi_i|\psi\rangle$ defines the coefficients a_i in (1). The square norm of a_i is interpreted as the probability that the system described by $|\psi\rangle$ is in the state $|\psi_i\rangle$. To provide a proper probabilistic interpretation a vector used in quantum mechanics is normalized by the condition $\langle\psi|\psi\rangle = \|\psi\|^2 = \sum_i |a_i|^2 = 1$.

Quantum mechanics is a probabilistic theory. One single measurement does not provide much information on the prepared system. However, several measurements on identically prepared quantum systems allow one to characterize the quantum state.

A physical quantity is represented by a linear operator called an *observable.* An observable A is a Hermitian operator, $A = A^\dagger$, which can be constructed by a set of real numbers λ_i and a set of states $|\phi_i\rangle$ determined by the measurement, $A = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$. The physical value corresponds to the average of the observable in the state $|\psi\rangle$ [23],

$$\langle A\rangle_\psi = \sum_i \lambda_i |\langle\psi|\phi_i\rangle|^2 = \langle\psi| A |\psi\rangle . \tag{78}$$

One can consider the situation in which a state $|\psi\rangle$ is not known exactly. Only a statistical mixture of several quantum states $|\phi_i\rangle$ which occur with probabilities

p_i is given. In this case the average value of an observable has the form

$$\langle A \rangle_{\{p_i,\phi_i\}} = \sum_i p_i \langle \phi_i | A | \phi_i \rangle , \tag{79}$$

which can be written in terms of an operator on $\mathcal{H}_N$ called a *density matrix* $\rho = \sum_i p_i |\phi_i\rangle \langle \phi_i|$ as

$$\langle A \rangle_{\{p_i,\phi_i\}} = Tr \rho A. \tag{80}$$

A density matrix describes a so called *mixed state.* In a specific basis the density matrices characterizing an N level quantum system are represented by $N \times N$ matrices ρ which are Hermitian, have trace equal to unity and are positive. Let us denote the set of all such matrices by $\mathcal{M}_N$ [23],[12],

$$\mathcal{M}_N = \left\{ \rho : \dim \rho = N, \ \rho = \rho^{\dagger} \ , \rho \geq 0, \ Tr \rho = 1 \right\}. \tag{81}$$

This set is convex. External points of this set are formed by projectors of the form $|\psi\rangle \langle \psi|$ called *pure states*, which correspond to vectors $|\psi\rangle$ of the Hilbert space.

The state of composed quantum system which consists of one N_1 level system and one N_2 level system is represented by a vector of size $N_1 N_2$ from the Hilbert space which has a tensor product structure, $\mathcal{H}_{N_1 N_2} = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$. Such a space contains also states which cannot be written as tensor products of vectors from separate spaces,

$$|\psi_{12}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle , \tag{82}$$

and are called *entangled states.* States with a tensor product structure are called *product states*. If the state of only one subsystem is considered one has to take an average over the second subsystem and leads to a reduced density matrix,

$$\rho_1 = Tr_2 \rho_{12}. \tag{83}$$

A density matrix describes therefore the state of an *open quantum system.*

The evolution of a normalized vector in the Hilbert space is determined by a unitary operator $\left|\psi'\right\rangle = U \ |\psi\rangle$. The transformation U is related to Hamiltonian evolution due to the Schrodinger equation [23],

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle , \tag{84}$$

where H denotes the Hamiltonian operator of the system, while t represents time and $2\pi\hbar$ is the Planck constant. A discrete time evolution of an open quantum system characterized by a density operator ρ is described by a *quantum operator* which will be considered later.

According to a general approach to quantum measurement , it can be defined by a set of k operators $\left\{E^i\right\}_{i=1}^k$ forming a *positive operator valued measure* (POVM). The index i is related to a possible measurement result, for instance the value of the measured quantity. The operators E^i are positive and satisfy the identity resolution,

$$\sum_{i=1}^{k} E^i = \mathbf{1}. \tag{85}$$

The quantum state is changing during the measurement process. After the measurement process that gives the outcome i as a result, the quantum state ρ is transformed into[23]

$$\rho_i' = \frac{K^i \rho K^{i\dagger}}{Tr\left(K^i \rho K^{i\dagger}\right)}, \tag{86}$$

where $K^{i\dagger} K^i = E^i \geq 0$. The probability p_i of the outcome i is given by $p_i = Tr\left(K^i \rho K^{i\dagger}\right)$. Due to relation (2), the probabilities of all outcomes sum up to unity.

1.9 Schmidt Decomposition

The theorem known as *Schmidt decomposition* provides a useful representation of a pure state of a bi partite quantum system.

Theorem 10 *(Schmidt). Any quantum state $|\psi_{12}\rangle$ from the Hilbert space composed of the tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ of dimensions d_1 and d_2, respectively, can be represented as*

$$|\psi_{12}\rangle = \sum_{i=1}^{d} \lambda_i |i_1\rangle \otimes |i_2\rangle, \tag{87}$$

where $\{|i_1\rangle\}_{i=1}^{d_1}$ and $\{|i_2\rangle\}_{i=1}^{d_2}$ are orthogonal basis of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and $d = \min\{d_1, d_2\}$[23], [21].

Theorem 11 *Choose any orhogonal basis $\left\{\left|\phi_1^k\right\rangle\right\}_{k=1}^{d_1}$ of $\mathcal{H}_1$ and any orthogonal basis $\left\{\left|\phi_2^k\right\rangle\right\}_{k=1}^{d_2}$ of $\mathcal{H}_2$. In this product basis, the bi partite state $|\psi_{12}\rangle$ reads*

$$|\psi_{12}\rangle = \sum_{0 \leq k \leq d_1,\ 0 \leq j \leq d_2} a_{kj} \left|\phi_1^k\right\rangle \otimes \left|\phi_2^j\right\rangle. \tag{88}$$

Singular value decomposition of a matrix A of size $d_1 \times d_2$ with entries a_{kj} gives $a_{kj} = \sum_i u_{ki} \lambda_i v_{ij}$. Here u_{ki} and v_{ij} are entries of two unitary matrices, while λ_i are singular values of A. Summation over indexes k and j cause changes of two orthogonal bases into

$$|i_1\rangle = \sum_k u_{ki} \left|\phi_1^k\right\rangle, \tag{89}$$

$$|i_2\rangle = \sum_k v_{ki} \left|\phi_2^j\right\rangle. \tag{90}$$

The number of nonzero singular values is not larger than the smaller one of the numbers (d_1, d_2)[23], [21].
Proof. *The Schmidt decomposition implies that both partial traces of any bi partite pure state have the same nonzero part of the spectrum:*

$$Tr_1 |\psi_{12}\rangle \langle\psi_{12}| = \sum_{i=1}^{d} \lambda_i^2 |i_2\rangle \langle i_2| \,, \tag{91}$$

$$Tr_2 |\psi_{12}\rangle \langle\psi_{12}| = \sum_{i=1}^{d} \lambda_i^2 |i_1\rangle \langle i_1| \,. \tag{92}$$

The Schmidt coefficients λ_i are invariant under local unitary transformations $U_1 \otimes U_2$ applied to $|\psi_{12}\rangle$. The number of non zero coefficients λ_i is called the Schmidt number. Any pure state which has the Schmidt number greater than 1 is called entangled state. A pure state for which all Schmidt coefficients λ_i are equal to $\frac{1}{\sqrt{d}}$ is called a maximally entangled state. ■

Another important consequence of the Schmidt decomposition is that for any mixed state ρ there is a pure state $|\psi\rangle$ of a higher dimentional Hilbert space such that ρ can be obtained by taking the partial trace,

$$\rho = Tr_1 |\psi\rangle \langle\psi| \,. \tag{93}$$

Such state $|\psi\rangle$ is called a *purification* of ρ. The Schmidt decomposition gives the recipe for the purification procedure. It is enough to take square roots of eigenvalues of ρ in place of λ_i and its eigenvectors in place of $|i_1\rangle$. Any orthogonal basis in $\mathcal{H}_2$ provides a purification of ρ, which can be written as

$$|\psi\rangle = \sum_i (U \otimes \sqrt{\rho}) |i_1\rangle \otimes |i_2\rangle \,, \tag{94}$$

where U is an arbitrary unitary tranformation and $\sqrt{\rho}\,|i_2\rangle = \lambda i\,|i_2\rangle$[23], [21].

1.10 Optimal Polar Decomposition

Theorem 12 *Each matrix $A \in \mathcal{M}_N$ can be written as*

$$A = PU \tag{95}$$

where P is a positive semi-definite hermitian matrix and U is a unitary matrix.

This matrix decomposition is called *polar decomposition.* The matrix P exists always and is computed from the equation $P = \left(AA^\dagger\right)^{\frac{1}{2}}$. If A is invertible then P will be invertible as well. As a consequence, U will be unique due to $U = P^{-1}A$. This also means that the factorization $A = PU$ is also unique.

In general, it holds that $U = VW^\dagger$ and $P = V\Sigma V^\dagger$, with V, W unitary matrices from the Singular Value Decomposition of A and Σ the respective matrix with nonnegative real numbers on the diagonal.

The matrix A can also be written as $A = U'P'$, but now it is $U' = VW^\dagger = U$ and $P' = \left(A^\dagger A\right)^{\frac{1}{2}}$, $P' = W^\dagger \Sigma W$.

Polar Decomposition $A = PU$ is the same writing of a complex number in polar coordinates $z = |z|\, e^{i\phi}$. It is easy to be shown that if a complex number is illustrated on the spot $M\,(z)$ in $2-$dimentions then from all complex numbers of unit circle with center $O\,(0,0)$, $e^{i\phi}$ is the one with the closest image of $M\,(z)$ ([15]).

For a normal matrix $A \in \mathcal{M}_N$, the closest unitary matrix is that of the polar decomposition of A.

Proposition 13 ***Proof.*** *Since $A \in \mathcal{M}_N$ (finite dimention) all matrix norms are equivalent. We will use the one that is obtained from the trace of a matrix $X \in \mathcal{M}_N$, i.e.*

$$\|X\| = \sqrt{Tr(XX^\dagger)}. \tag{96}$$

Namely, it suffices to find for which unitary matrix $Q \in \mathcal{M}_N$ the non negative value $\|A - Q\|^2$ becomes minimum. If A is unitary then obviously the closest unitary matrix is itself and from the polar decomposition we have that

$$A = \left(AA^\dagger\right)^{\frac{1}{2}} U = I^{\frac{1}{2}} U = U. \tag{97}$$

If A is not unitary we have that

$$\begin{aligned}
\|A - Q\|^2 &= Tr\left[(A - Q)\left(A^\dagger - Q^\dagger\right)\right] \\
&= Tr\left(AA^\dagger - AQ^\dagger - QA^\dagger + QQ^\dagger\right) \\
&= Tr\left(AA^\dagger - AQ^\dagger - QA^\dagger + \mathbf{1}\right) \\
&= Tr\left(AA^\dagger\right) - Tr\left(AQ^\dagger\right) - Tr\left(QA^\dagger\right) + Tr\mathbf{1}
\end{aligned}$$

or

$$\|A - Q\|^2 = Tr\left(AA^\dagger\right) - 2\Re e\left[Tr\left(QA^\dagger\right)\right] + n. \tag{98}$$

From the SVD of matrix A we have that $A = V\Sigma W^\dagger$ according to the notation of the previous theorem

$$\begin{aligned}
\Re e\left[Tr\left(AQ^\dagger\right)\right] &= \Re e\left[Tr\left(QA^\dagger\right)\right] \\
&= \Re e\left[Tr\left(Q^\dagger V\Sigma W^\dagger\right)\right] \\
&= \Re e\left[Tr\left(W^\dagger Q^\dagger V\Sigma\right)\right]
\end{aligned}$$

or

$$\Re e\left[Tr\left(AQ^\dagger\right)\right] = \Re e\left[Tr\left(\Psi\Sigma\right)\right], \tag{99}$$

where $\Psi = W^\dagger Q^\dagger V$. We notice that the matrix Ψ is unitary because, V, W, Q are unitary and $\Psi\Psi^\dagger = W^\dagger Q^\dagger VV^\dagger QW = \mathbf{1}$. Therefore, $Tr\left(\Psi\Psi^\dagger\right) = n$ or equivalently, $\sum_{i=1}^{n} |\psi|_{ii}^2 = n$. Furthermore for arbitrary complex numbers

$$z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i, \ldots, z_n = x_n + y_n i,$$

and n *arbitrary non negative numbers* $\theta_1, \theta_2, \ldots, \theta_n \geq 0$ *it holds*

$$\Re e\left[\sum_{k=1}^{n}\theta_k z_k\right] = \sum_{k=1}^{n}\theta_k x_k \leq \sum_{k=1}^{n}\theta_k \left|z\right|_k . \tag{100}$$

The equality will be valid if and only if $z_k = x_k \geq 0$ *for every* $k = 1, 2, \ldots, n$. *From (99) and (100) we obtain*

$$\begin{aligned}\Re e\left[Tr\left(AQ^{\dagger}\right)\right] &= \Re e\left[Tr\left(\Psi\Sigma\right)\right] \\ &= \Re e\left[\sum_{i=1}^{n}\psi_{ii}\sigma_{ii}\right] \\ &\leq \sum_{i=1}^{n}\left|\psi_{ii}\right|\sigma_{ii} \qquad (since\ \sigma_{ii} \geq 0)\,. \end{aligned} \tag{101}$$

The equality in (101) holds if and only if $\psi_{ii} \geq 0$. *But,* $\Psi\Psi^{\dagger} = \mathbf{1}$, *so* $\psi_{ii}\psi_{ii}^{*} = 1$ *for every* $i = 1, 2, \cdots, n$. *This is translated into that on one hand the minimum value of the norm* $\left\|A - Q\right\|^2$ *is*

$$\min\left\|A - Q\right\|^2 = Tr\left(AA^{\dagger}\right) - 2\sum_{i=1}^{n}\sigma_{ii} + n,$$

and on the other hand that this happens when $\Psi = \mathbf{1}$, *i.e.,* $\Psi = W^{\dagger}Q^{\dagger}V = \mathbf{1}$. *From this and because* V, W *are unitary yields that* $Q^{\dagger} = WV^{\dagger}$ *or* $Q = VW^{\dagger}$ *and SVD of* A *will be*

$$A = V\Sigma W^{\dagger} = \left(V\Sigma V^{\dagger}\right)\left(VW^{\dagger}\right). \tag{102}$$

According to the previous observation, (102) is the polar decomposition of A *therefore its closest unitary is the one comes from the polar decomposition of* A. *We have inferred that* A *is normal so, the previous polar decomposition is unique[15].* ■

2 Quantum Channels : An Introduction

2.1 Quantum Channel

In this chapter we are going to give a more intuitive introduction to the idea of quantum channels and their uses in the general context of quantum mechanics and in the next chapter we are going to present a more formal and mathematically complete presentation.

In order to determine the state of an open quantum system at a certain time evolution there are two methods. The criterion to choose which method is more effective to use is based on whether the physical model is known. When the physical model is concrete we only need a Hamiltonian which describes it and determines the Schrodinger's equation or the master equation. Their solution will provide us the desired quantum system state at any time moment. The other method to determine in which state the quantum system exists is called "black box" because the physical model is not defined. To accomplice our goal we are obliged to construct a quantum map $\rho' = \mathcal{E}(\rho)$, no further than the laws of quantum mechanics are respected. This quantum map is as well used if someone wants to look for all the possible functions of a quantum state whether the physical model is known or unknown. Main features and some representations of the map $\mathcal{E}$, which describes a "black box" model of non unitary quantum evolution, are given below.

The quantum positive map $\mathcal{E}$ describes the dynamics of a quantum system ρ which interacts with an environment. It is given by a nonunitary quantum map $\mathcal{E} : \rho \rightarrow \rho'$. Any such map is completely positive, and trace preserving . "Complete positivity" means that an extended map $\mathcal{E}\otimes\mathbf{1}_M$, which is a trivial extension of $\mathcal{E}$ on the space M of any dimension, transforms the set of positive operators into itself. A completely positive and trace preserving quantum map is called *quantum operation* or *quantum channel.*

Due to the theorem of Jamiolkowski and Choi the complete positivity of a map $\mathcal{E}$ is equivalent to positivity of a state $D_{\mathcal{E}}$ corresponding via the so called *Jamiolkowski isomorphism (or CJ transform).* This isomorphism determines the correspondence between a quantum operation $\mathcal{E}$ acting on N dimentional matrices and density matrix $\rho_{_\varepsilon}$ of dimention N^2 which is called Choi matrix or the Jamiolkowski state. Explicitly let a CP map $\mathcal{E} = \sum_i AdA_i$ and $|\mathbf{1}\rangle\rangle = \sum_k |k\rangle\otimes|k\rangle$ a maximally entangled state, define the CJ transform $\mathcal{T} : End(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}\otimes\mathcal{H})$, from maps $\mathcal{E} \in End(\mathcal{H})$, to density matrices

$$\mathcal{T}[\mathcal{E}] = (\mathcal{E}\otimes id)\,|\mathbf{1}\rangle\rangle\langle\langle\mathbf{1}| \equiv \rho_{_\varepsilon} \tag{103}$$

The dynamical matrix $\rho_{_\varepsilon}$ corresponding to a trace preserving operation satisfies the partial trace condition

$$Tr_2\rho_{_\varepsilon} = \mathbf{1}. \tag{104}$$

The quantum operation $\mathcal{E}$ can be represented as *superoperator matrix.* It is a matrix which acts on the vector of length N^2, which contains the entries ρ_{ij} of the density matrix ordered lexicographically. Thus, the superoperator $\mathcal{E}$ is represented by a square matrix of size N^2. The superoperator in some orthogonal product basis $\{|i\rangle\otimes|j\rangle\}$ is represented by a matrix $\mathcal{E}'$ indexed by four indexes,

$$\mathcal{E}'_{ijkl} = \langle i|\otimes\langle j|\,\mathcal{E}\,|k\rangle\otimes|l\rangle\,. \tag{105}$$

The matrix representation of the dynamical matrix is related to the superoperator matrix by the reshuffling formula as follows

$$\langle i|\otimes\langle j|\,\rho_{\mathcal{E}}\,|k\rangle\otimes|l\rangle = \langle i|\otimes\langle k|\,\mathcal{E}'\,|j\rangle\otimes|l\rangle\,. \tag{106}$$

To describe a quantum operation, one may use the Stinespring's dilation theorem as follows. Consider a quantum system, described by the state ρ on

$\mathcal{H}_N$, interacting with its environment characterized by a state on $\mathcal{H}_M$. The joint evolution of the two states is described by a unitary operation U. Usually it is assumed that the joint state of the system and the environment is initially not entangled. Moreover, due to the possibility to purification the environment, its initial state is given by a pure one. The evolving joint state is therefore:

$$\omega = U\left(|1\rangle\langle 1| \otimes \rho\right) U^{\dagger}, \tag{107}$$

where $|1\rangle \in \mathcal{H}_M$ and U is a unitary matrix of size NM. The state of the system after the operation is obtained by tracing out the environment,

$$\rho' = \mathcal{E}(\rho) = Tr_M\left[U\left(|1\rangle\langle 1| \otimes \rho\right) U^{\dagger}\right] = \sum_{i=1}^{M} K^i \rho K^{i\dagger}, \tag{108}$$

where the Kraus operators read, $K^i = \langle i| U |1\rangle$. In matrix representation the Kraus operators are formed by successive blocks of the first block column of the unitary evolution matrix U. Here the state ω can be equivalently given as

$$\omega = \sum_{i,j=1}^{M} K^i \rho K^{j\dagger} \otimes |i\rangle\langle j|. \tag{109}$$

Due to the Kraus theorem any completely positive map $\mathcal{E}$ can be written in the Kraus form,

$$\rho' = \mathcal{E}(\rho) = \sum_{i=1}^{M} K^i \rho K^{i\dagger}. \tag{110}$$

The opposite relation is also true, any map of the Kraus form is completely positive.

2.2 Complementary Channel

Consider a quantum channel $\mathcal{E}$ described by the Kraus operators K^i,

$$\mathcal{E}(\rho) = Tr_M \omega = \sum_{i=1}^{M} K^i \rho K^{i\dagger}. \tag{111}$$

The channel $\widetilde{\mathcal{E}}$ *complementary* to $\mathcal{E}$ is defined by

$$\widetilde{\mathcal{E}}(\rho) = Tr_N \omega = \sum_{i=1}^{M} \widetilde{K}^i \rho \widetilde{K}^{i\dagger}, \tag{112}$$

and it describes the state of the M dimentional environment after the interaction with the principal system ρ. One can derive the relation between operators $\left\{\widetilde{K}^j\right\}_{j=1}^{N}$ and $\left\{K^i\right\}_{i=1}^{M}$ from the last equation by substituting ω with

$\sum_{i,j=1}^{M} K^i \rho K^{j\dagger} \otimes |i\rangle \langle j|$. This relation can be written as

$$\sum_{i,j=1}^{M} \left(TrK^i \rho K^{j\dagger}\right) |i\rangle \langle j| = \sum_{i=1}^{N} \widetilde{K}^i \rho \widetilde{K}^{i\dagger}. \tag{113}$$

Comparison of the matrix elements of both sides gives

$$\sum_{a=1}^{N} \widetilde{K}^a_{im} \rho_{mn} \widetilde{K}^{a\dagger}_{nj} = \sum_{a=1}^{N} K^i_{am} \rho_{mn} K^{j\dagger}_{na}, \tag{114}$$

where matrix elements are indicated by lower indexes and the Einstein summation convention is applied. Hence, for any quantum channel $\mathcal{E}$ given by a set of Kraus operators K^i, one can define the Kraus operators $\widetilde{K}^a$ representing the complementary channel $\widetilde{\mathcal{E}}$ as

$$\widetilde{K}^a_{ij} = K^i_{aj}, i = 1, \ldots, M, j, a = 1, \ldots, N. \tag{115}$$

2.3 One Qubit Quantum Channels

One qubit quantum channels acting on density matrices of size 2 have many special features which cause that the set of these channels is well understood. However, many properties of one qubit maps are not shared with the quantum channels are often considered in this thesis, the following section presents a brief review of their basic properties.

A quantum two level state is called *quantum bit* or *qubit.* It is represented by a 2×2 density matrix. Any Hermitian matrix of size two can be represented in the basis of identity matrix and the three Pauli matrices $\overrightarrow{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{116}$$

The Pauli matrices have complex entries and they are Hermitian and unitary matrices which have the properties $\det \sigma_i = -1$ and $Tr\sigma_i = 0$ also the eigenvalues of each σ_i are ± 1.

The Pauli matrices σ_1 and σ_3 satisfy the following[26]

$$\begin{aligned} \sigma_1^2 &= \sigma_3^2 = \mathbf{1}, & (117) \\ \sigma_1\sigma_3 &= -\sigma_3\sigma_1 = e^{i\pi}\sigma_3\sigma_1. & (118) \end{aligned}$$

The so-called Walsh-Hadamard conjugation matrix is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{119}$$

Like the Pauli matrices, H is both Hermitian and unitary. The matrices σ_1, σ_3 and H satisfy the relation

$$\sigma_1 = H\sigma_3 H^\dagger \tag{120}$$

One qubit state ρ decomposed in the mentioned basis is given by the formula

$$\rho = \frac{1}{2}\left(\mathbf{1} + \overrightarrow{r}\cdot\overrightarrow{\sigma}\right), \overrightarrow{r} \in \mathbb{R}^3. \tag{121}$$

Positivity condition, $\rho \geq 0$, implies that $|\overrightarrow{r}| \leq 1$. The vector $\overrightarrow{r}$ is called the *Bloch vector.* All possible Bloch vectors representing quantum states form the *Bloch ball.* Pure one qubit states form a sphere of radius $|\overrightarrow{r}| = 1$.

Any linear one qubit quantum operation $\mathcal{E}$ transforms the Bloch ball into the ball or into an ellipsoid inside the ball. The channel $\mathcal{E}$ transforms the Bloch vector $\overrightarrow{r}$ representing the state ρ into $\overrightarrow{r}'$ which corresponds to ρ'. This transformation is described by

$$\overrightarrow{r}' = W\overrightarrow{r} + \overrightarrow{\kappa}. \tag{122}$$

Here the matrix W is a square real matrix of size 3. A procedure analogous to the singular value decomposition of the matrix W gives $W = O_1 D O_2$, where O_i represents an orthogonal rotation and D is diagonal. Up to two orthogonal rotations, one before the transformation $\mathcal{E}$ and one after it, the one qubit map $\mathcal{E}$ can be represented by the following matrix [24]

$$\mathcal{E}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \kappa_1 & \eta_1 & 0 & 0 \\ \kappa_2 & 0 & \eta_2 & 0 \\ \kappa_3 & 0 & 0 & \eta_3 \end{pmatrix}. \tag{123}$$

The absolute values of the parameters η_i are interpreted as the lengths of the axes of the ellipsoid which is the image of the Bloch ball transformed by the map. The parameters κ_i form the vector $\overrightarrow{\kappa}$ of translation of the center of the ellipsoid with respect to the center of the Bloch ball.

Due to complete positivity of the map $\mathcal{E}$ and the trace preserving property, the vectors $\overrightarrow{\eta}$ and $\overrightarrow{\kappa}$ are subjected to several constraints. They can be derived from the positivity condition of a dynamical matrix given by

$$\rho_{\mathcal{E}} = \frac{1}{2}\begin{pmatrix} 1+\eta_3+\kappa_3 & 0 & \kappa_1+i\kappa_2 & \eta_1+\eta_2 \\ 0 & 1-\eta_3+\kappa_3 & \eta_1-\eta_2 & \kappa_1+i\kappa_2 \\ \kappa_1-i\kappa_2 & \eta_1-\eta_2 & 1-\eta_3-\kappa_3 & 0 \\ \eta_1+\eta_2 & t_1-i\kappa_2 & 0 & 1+\eta_3-\kappa_3 \end{pmatrix}. \tag{124}$$

3 Completely Positive Trace Preserving Maps (CPTP) - Operator Sum Representation

In this chapter, in reference to the subsection 1.6 we will give a more mathematically accurate definition on CPTP maps.

3.1 Definition and Examples

To begin with we will determine what positivity and complete positive means for a map and we present some examples for better understanding of the definitions[7].

Let n be a natural number and $\varphi : M_n \to M_n$ a linear map. The map φ is called *positive* if it maps positive matrices into positive matrices, i.e. if $M_n^+ = \{A \in M_n; A > 0\} \subseteq M_n$, then $\varphi(M_n^+) \subseteq M_n^+$.

Example 14 *The transpose map $\tau : M_n \to M_n$ that maps each $n \times n$ matrix into its transpose is positive.*

Let, in addition, m be a natural number. A linear map $\varphi : M_n \to M_n$ always induces a linear map $\varphi_m = id_m \otimes \varphi : M_m \otimes M_n \to M_m \otimes M_n$; more precisely, with the identification $M_m \otimes M_n \simeq M_m(M_n)$, the matrix algebra of all $m \times m$ matrices with entries from M_n, then for $A(A_{ij}) \in M_m \otimes M_n$ we have that

$$\varphi_m(A) \equiv \varphi_m \left(\begin{array}{ccc} A_{11} & ... & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & ... & A_{mm} \end{array} \right) = \left(\begin{array}{ccc} \varphi(A_{11}) & ... & \varphi(A_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(A_{n1}) & ... & \varphi(A_{nn}) \end{array} \right). \tag{125}$$

Then if $A_{ij} \in M_n$ and for each $A_{ij} > 0$, and φ is *positive* i.e. $\varphi(A_{ij}) > 0$, then if $\varphi_m(A) > 0$ for up to some $m \in \mathbb{N}$, then φ is called m- positive. If φ is m- positive for any $m \in \mathbb{N}$ the φ is called *completly positive*, otherwise φ is positive but not completely positive.

An example demonstrating a positive but non-completely positive map is given by the transpose of a matrix : Choose $n = 2$, $m = 2$. Let φ be the transpose of matrices in M_2

$$\varphi_2(A) = \varphi_2 \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{126}$$

which is not completely positive because $spec\,(\varphi_2(A)) = \{-1, 1\}$.

More general we have the following definition: A linear map $\varphi : M_n \to M_n$ is called *completely positive* if it is $m-$positive for all natural numbers m. We denote by $CP\,(M_n, M_n)$ the set of all completely positive maps from M_n to M_n.

The following examples are generic.

Examples : $*-$*Morphisms.* Let $\pi : M_n \to M_n$ be a morphism of $*-$algebras, for every n. Then π is completely positive.

Stinespring Representation. Let $\pi : M_n \to M_m$ be a morphism of $*-$algebras, for $n \leq m$ and $V \in M_{m,n}$. Then $\varphi = V^* \pi(\cdot) V \in CP\,(M_n, M_n)$.

Kraus Representation. Given $n \times n$ matrices $V_1, V_2, ..., V_m \in M_{n,n}$ define $\varphi : M_n \to M_n$ by

$$\varphi(A) = V_1^* A V_1 + V_2^* A V_2 + \cdots + V_m^* A V_m \text{ for all } A \in M_n. \tag{127}$$

Definition 15 *Denoting the space of $n \times n$ matrices with complex entries by M_n, we call a matrix $B \in M_n$ positive if it is positive-semidefinite, that is if it satisfies $x^* Bx \geq 0$ for all $x \in \mathbb{C}^n$. Otherwise, a matrix is positive if it is Hermitian and all its eigenvalues are non-negative, or if there exists some matrix B such that it can be written $B = C^* C$.*

The map $\mathcal{E}\colon\ M_n \to M_n$ is called *positive* if for all positive $A \in M_n$, $\mathcal{E}(B)$ is also positive.

The following theorem, due originally to Man-Duon Choi is commonly known as "Choi's Theorem" and it classifies all completely positive maps.

Theorem 16 *For all A_i, where A_i are Kraus operators, the map $\mathcal{E}\colon M_n \to M_n$ given by*

$$\mathcal{E}(\rho) = \sum_{i=0}^{n} A_i \rho A_i^{\dagger}, \tag{128}$$

is completely positive.

We need $\mathcal{E}$ to be trace preserving, which means $Tr\mathcal{E}(\rho) = Tr\rho$. Let $\mathcal{E}\colon M_n \to M_n$ be a map as described above with the additional constraint for the $\{A_i\}$ that

$$\sum_{i=0}^{n} A_i^{\dagger} A_i = \mathbf{1}, \tag{129}$$

where $\mathbf{1}$ is the identity matrix on M_n.

The operators $\{A_i\}_{i=1}^{n}$ which generate the transformation $\mathcal{E}$ are called Kraus generators of $\mathcal{E}$. The Kraus operators are named after mathematician Kraus whose contribution in quantum measurement and completely positive maps was influential. Kraus operators need not be unique.

3.2 Transformations of Density Matrix

Closed quantum systems with density matrices $\rho \in M_n$ evolve by transitions $\mathcal{E} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ specified by a unitary matrix U

$$\mathcal{E}(\rho) = U \rho U^{\dagger}. \tag{130}$$

In the case of density matrices, the constraints that the matrices are positive and have trace 1 ensure that the eigenvalues form a probability distribution, as the trace is equal to the sum of the eigenvalues for positive matrices. Since conjugation by a unitary matrix preserves eigenvalues, this probability distribution remains the same in closed quantum systems. In general, we would like to consider a broader range of transmitions, the full manner in which one density matrix may be mapped onto another. Applying the operator to states separately and then mixing them should be the same as applying the operation to states mixed first. Consequently, we mix a set of density matrices $\{A_i\}$ by associating a probability distribution $\{p_i\}$, where p_i indicates the probability of finding the state A_i in the new ensemble.

This means that if $\rho \in \mathcal{D}(\mathcal{H})$ for $\mathcal{E} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ then the three following properties must hold

$$\mathcal{E}(\rho) > 0, \tag{131}$$

$$(\mathcal{E}(\rho))^{\dagger} = \mathcal{E}(\rho), \tag{132}$$

and

$$Tr\mathcal{E}(\rho) = Tr\rho. \tag{133}$$

If we call $\mathcal{E}(\rho) = \rho'$ then $\rho' \in \mathcal{D}(\mathcal{H})$. Then, a form of $\mathcal{E}$ is

$$\mathcal{E}(\rho) = \sum_{i=1}^{n} p_i A_i \rho A_i^{\dagger}, \tag{134}$$

and is called Operator Sum Representation of CPTP map $\mathcal{E}$, where A_i operators are positive and $0 \leq p_i \leq 1$, $\sum_{i=1}^{n} p_i = 1$.

The first property demands $\mathcal{E}(\rho) > 0$. To prove that this is true it is sufficient to show that for every vector $|\psi\rangle \in \mathcal{H}$, it holds that $\langle\psi| \mathcal{E}(\rho) |\psi\rangle > 0$. From (134) and $\varrho = \sum_a \lambda_a |u_a\rangle \langle u_a|$, for probability $\lambda_a \geq 0$ and $|u_a\rangle \in \mathcal{H}$ we obtain,

$$\begin{aligned}
\langle\psi| \mathcal{E}(\rho) |\psi\rangle &= \sum_i \langle\psi| A_i \varrho A_i^{\dagger} |\psi\rangle \\
&= \sum_i \langle\psi| A_i (\sum_a \lambda_a |u_a\rangle \langle u_a|) A_i^{\dagger} |\psi\rangle \\
&= \sum_i \sum_a \lambda_a \langle\psi| A_i |u_a\rangle \langle u_a| A_i^{\dagger} |\psi\rangle \\
&= \sum_i \sum_a \lambda_a \langle\psi| A_i |u_a\rangle \langle\psi| A_i^{*} |u_a\rangle^{*} \\
&= \sum_i \sum_a \lambda_a |\langle\psi| A_i |u_a\rangle|^2 > 0. \qquad (135)
\end{aligned}$$

For the second property we apply the complex conjugate to $\sum_{i=1}^{n} p_i A_i \rho A_i^{\dagger}$ and we have

$$\begin{aligned}
(\mathcal{E}(\rho))^{\dagger} &= \left(\sum_{i=1}^{n} p_i A_i \rho A_i^{\dagger} \right)^{\dagger} \\
&= \sum_{i=1}^{n} p_i \left(A_i \rho A_i^{\dagger} \right)^{\dagger} \\
&= \sum_{i=1}^{n} p_i A_i \rho A_i^{\dagger} \qquad (136) \\
&= \mathcal{E}(\rho), \qquad (137)
\end{aligned}$$

for the third property of trace preservation we take the trace of $\mathcal{E}(\rho)$

$$\begin{aligned} Tr\mathcal{E}(\rho) &= Tr\left(\sum_{i=1}^{n} p_i A_i \rho A_i^{\dagger}\right) \\ &= \sum_{i=1}^{n} p_i Tr\left(A_i \rho A_i^{\dagger}\right) \\ &= \sum_{i=1}^{n} p_i Tr\left(A_i^{\dagger} A_i \rho\right) \\ &= Tr\left(\left(\sum_{i=1}^{n} p_i A_i^{\dagger} A_i\right)\rho\right) \\ &= Tr\mathbf{1}\rho = Tr\rho. \end{aligned} \tag{138}$$

The last equation is valid due to the normalization constrain $\sum_{i=1}^{n} p_i A_i^{\dagger} A_i = \mathbf{1}$ imposed upon the A_i generators. A particular case of this property is the unitary generators i.e. $A_i = \sqrt{p_i} U_i$ where U_i unitary.

4 One Qubit Quantum Channels

In this chapter we will introduce the most common one qubit quantum channels and we will use them as Kraus operators in order to compute the corresponding CPTP map. The quantum channels are the following,

Quantum Channel	Kraus Generators
X	$\{\sqrt{p}\mathbf{1}, \sqrt{1-p}X\}$
Y	$\{\sqrt{p}\mathbf{1}, \sqrt{1-p}Y\}$
Z	$\{\sqrt{p}\mathbf{1}, \sqrt{1-p}Z\}$
H	$\{\sqrt{p}\mathbf{1}, \sqrt{1-p}H\}$
Depolarizing	$\{\sqrt{p}\mathbf{1}, \frac{\sqrt{1-p}}{3}X, \frac{\sqrt{1-p}}{3}Y, \frac{\sqrt{1-p}}{3}Z\}$
Amplitude-Damping	$\{\sqrt{p}\begin{pmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{pmatrix}, \sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{pmatrix}\}$

.

To compute the CPTP maps $\mathcal{E} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ for each and every one of the above channels we will use the denstiy matrix ρ of the form $\begin{pmatrix}\mu & 0\\ 0 & 1-\mu\end{pmatrix}$, where $0 \leq \mu \leq 1$ and the Choi's theorem where $\mathcal{E}(\rho) = \sum_i p_i A_i \rho A_i^{\dagger}$ and A_i are the Kraus operators. Therefore, the CPTP map of channel X is

$$
\begin{aligned}
\mathcal{E}_X(\rho) &= \sum_{i=0}^{1} p_i A_i \rho A_i^{\dagger} \\
&= p\mathbf{1}\rho\mathbf{1}^{\dagger} + (1-p)\, X\rho X^{\dagger} \\
&= p\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix} + (1-p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} p\mu + (1-p)(1-\mu) & 0 \\ 0 & p(1-\mu) + (1-p)\mu \end{pmatrix},
\end{aligned}
\tag{139}
$$

of channel Y is

$$
\begin{aligned}
\mathcal{E}_Y(\rho) &= \sum_{i=0}^{1} p_i A_i \rho A_i^{\dagger} \\
&= p\mathbf{1}\rho\mathbf{1}^{\dagger} + (1-p)\, Y\rho Y^{\dagger} \\
&= p\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix} + (1-p)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \begin{pmatrix} p\mu + (1-p)(1-\mu) & 0 \\ 0 & p(1-\mu) + (1-p)\mu \end{pmatrix},
\end{aligned}
\tag{140}
$$

of channel Z is

$$
\begin{aligned}
\mathcal{E}_Z(\rho) &= \sum_{i=0}^{1} p_i A_i \rho A_i^{\dagger} \\
&= p\mathbf{1}\rho\mathbf{1}^{\dagger} + (1-p)\, Z\rho Z^{\dagger} \\
&= p\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix} + (1-p)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix},
\end{aligned}
\tag{141}
$$

of Hadamard channel is

$$
\begin{aligned}
\mathcal{E}_H(\rho) &= \sum_{i=0}^{1} p_i A_i \rho A_i^{\dagger} \\
&= p\mathbf{1}\rho\mathbf{1}^{\dagger} + (1-p)\, H\rho H^{\dagger} \\
&= p\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix} + \frac{(1-p)}{2}\begin{pmatrix} 1 & 2\mu-1 \\ 2\mu-1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} p\mu + \frac{1-p}{2} & \frac{(1-p)(2\mu-1)}{2} \\ \frac{(1-p)(2\mu-1)}{2} & p(1-\mu) + \frac{1-p}{2} \end{pmatrix},
\end{aligned}
\tag{142}
$$

of Depolarizing channel is

$$
\begin{aligned}
\mathcal{E}_{DP}\left(\rho\right) &= \sum_{i=0}^{3} p_i A_i \rho A_i^{\dagger} \\
&= p\mathbf{1}\rho\mathbf{1}^{\dagger} + \frac{(1-p)}{3} X\rho X^{\dagger} + \frac{(1-p)}{3} Y\rho Y^{\dagger} + \frac{(1-p)}{3} Z\rho Z^{\dagger} \\
&= \begin{pmatrix} \left(p+\frac{1-p}{3}\right)\mu + \frac{2(1-p)}{3}\left(1-\mu\right) & 0 \\ 0 & \left(p+\frac{1-p}{3}\right)\left(1-\mu\right) + \frac{2(1-p)}{3}\mu \end{pmatrix} \quad (143)
\end{aligned}
$$

and last, of Amplitude-Damping channel is

$$
\begin{aligned}
\mathcal{E}_{A-D}\left(\rho\right) &= \sum_{i=0}^{1} p_i A_i \rho A_i^{\dagger} \\
&= p\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix} + (1-p)\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\
&= \begin{pmatrix} p\gamma\left(1-\mu\right) + \mu\left(1-p\right) & 0 \\ 0 & \left(1-p\right)\left(1-\gamma\right)\left(1-\mu\right) \end{pmatrix}. \quad (144)
\end{aligned}
$$

A CPTP map is characterized as unital if it verifies $\mathcal{E}(\mathbf{1}) = \mathbf{1}$, otherwise it is called non-unital. We will check for the above channels.

$$
\begin{aligned}
\mathcal{E}\left(\mathbf{1}\right) &= \sum_{i=0}^{n-1} p_i A_i \mathbf{1} A_i^{\dagger} \\
&= \sum_{i=0}^{n-1} p_i A_i A_i^{\dagger} \\
&= \mathbf{1}, \quad (145)
\end{aligned}
$$

for the channels X, Y, Z, Hadamard and Depolarization because their Kraus operators are unitary matrices so, it holds that $\sum_{i=0}^{1} A_i A_i^{\dagger} = \mathbf{1}$. Also, since p_i is the probability we find each state their sum adds up to 1. On the other hand, it is easy to check that the Amplitude-Damping channel is non-unital i.e,

$$
\begin{aligned}
\mathcal{E}_{A-D}\left(\mathbf{1}\right) &= \sum_{i=0}^{n-1} p_i A_i \mathbf{1} A_i^{\dagger} \\
&= p\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \mathbf{1} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix} + (1-p)\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \mathbf{1} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\
&= p\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} + (1-p)\begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} \\
&= \begin{pmatrix} p\gamma + 1 - p & 0 \\ 0 & \left(1-p\right)\left(1-\gamma\right) \end{pmatrix} \neq \mathbf{1}.
\end{aligned}
$$

.

5 Collective Quantum Channels

In this chapter we will introduce the idea of *collective quantum channels.* Collective quantum channels is a wider perspective of one qubit quantum channels as they act on multidimensional systems of quantum mechanics.

5.1 Definition

A *collective channel* or *collective operator* is a normalized sum of individual operators which are represented by block matrices [14],

$$(\mathcal{E} \otimes \mathcal{E})(\rho_1 \otimes \rho_2) = \mathcal{E}(\rho_1) \otimes \mathcal{E}(\rho_2). \tag{146}$$

More specifically,

$$\rho_1 \otimes \rho_2 \xrightarrow{\mathcal{E}\otimes\mathcal{E}} \mathcal{E}(\rho_1) \otimes \mathcal{E}(\rho_2) = \sum_k \lambda_k A_k \rho_1 A_k^\dagger \otimes \sum_l \lambda_l A_l \rho_2 A_l^\dagger, \tag{147}$$

from the properties of tensor product yields that (147) is equal to

$$\begin{aligned} & \sum_{kl} \lambda_k \lambda_l (A_k \otimes A_l)(\rho_1 \otimes \rho_2)(A_k \otimes A_l)^\dagger & (148) \\ = & \sum_{kl} \lambda_{kl} (A_k \otimes A_l)(\rho_1 \otimes \rho_2)(A_k \otimes A_l)^\dagger. & (149) \end{aligned}$$

where $\lambda_{kl} = \lambda_k \lambda_l$, is joint probability distribution which is factorized. This implies statistical independence of the action of Kraus generators on the density matrix $\rho_1 \otimes \rho_2$. If instead of the statistical independence $\lambda_{kl} = \lambda_k \lambda_l$ (two coin tossing), we choose one single coin tossing i.e. $\lambda_{kl} = \lambda_k \delta_{kl}$, then we have a collective channel action on the density matrix $\rho_1 \otimes \rho_2$. The general form of a collective channel $\mathcal{E}^{(n)} : \mathcal{D}(\mathcal{H}^{\otimes n}) \to \mathcal{D}(\mathcal{H}^{\otimes n})$ is

$$\mathcal{E}^{(n)}(\rho) = \sum_k \lambda_k A_k^{\otimes n} \rho \left(A_k^{\otimes n}\right)^\dagger, \tag{150}$$

where λ_k is the probability of each occasion to take place, k states the number of Kraus generators that correspond to each channel and A_k are the Kraus generators.

5.2 Common Collective Channels

Hence, the collective channel X is

$$\begin{aligned} \mathcal{E}_X^{(n)}(\rho) &= \sum_{k=0}^{1} \lambda_k A_k^{\otimes n} \rho \left(A_k^{\otimes n}\right)^\dagger \\ &= \lambda_1 \mathbf{1}^{\otimes n} \rho \mathbf{1}^{\otimes n} + \lambda_2 \sigma_1^{\otimes n} \rho \left(\sigma_1^{\otimes n}\right)^\dagger, \end{aligned} \tag{151}$$

the collective channel Y is

$$\begin{aligned}\mathcal{E}_Y^{(n)}(\rho) &= \sum_{k=0}^{1}\lambda_k A_k^{\otimes n}\rho\left(A_k^{\otimes n}\right)^\dagger \\ &= \lambda_0\mathbf{1}^{\otimes n}\rho\mathbf{1}^{\otimes n}+\lambda_1\sigma_2^{\otimes n}\rho\left(\sigma_2^{\otimes n}\right)^\dagger,\end{aligned}\tag{152}$$

the collective channel Z is

$$\begin{aligned}\mathcal{E}_Z^{(n)}(\rho) &= \sum_{k}\lambda_k A_k^{\otimes n}\rho\left(A_k^{\otimes n}\right)^\dagger \\ &= \lambda_0\mathbf{1}^{\otimes n}\rho\mathbf{1}^{\otimes n}+\lambda_1\sigma_3^{\otimes n}\rho\left(\sigma_3^{\otimes n}\right)^\dagger,\end{aligned}\tag{153}$$

the collective Hadamard channel is

$$\begin{aligned}\mathcal{E}_H^{(n)}(\rho) &= \sum_{k}\lambda_k A_k^{\otimes n}\rho\left(A_k^{\otimes n}\right)^\dagger \\ &= \lambda_0\mathbf{1}^{\otimes n}\rho\mathbf{1}^{\otimes n}+\lambda_1 H^{\otimes n}\rho\left(H^{\otimes n}\right)^\dagger,\end{aligned}\tag{154}$$

the collective Amplitude-Damping channel is

$$\begin{aligned}\mathcal{E}_{A-D}^{(n)}(\rho) &= \sum_{k}\lambda_k A_k^{\otimes n}\rho\left(A_k^{\otimes n}\right)^\dagger \\ &= \lambda_0 A_1^{\otimes n}\rho\left(A_1^{\otimes n}\right)^\dagger+\lambda_1 A_2^{\otimes n}\rho\left(A_2^{\otimes n}\right)^\dagger,\end{aligned}\tag{155}$$

where $A_1=\begin{pmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{pmatrix}$ and $A_2=\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{pmatrix}$, and finally the collective Depolarizing channel is

$$\begin{aligned}\mathcal{E}_{DP}^{(n)}(\rho) &= \sum_{k=0}^{3}\lambda_k A_k^{\otimes n}\rho\left(A_k^{\otimes n}\right)^\dagger \\ &= \lambda_0\mathbf{1}^{\otimes n}\rho\mathbf{1}^{\otimes n}+\lambda_1\sigma_1^{\otimes n}\rho\left(\sigma_1^{\otimes n}\right)^\dagger+\lambda_2\sigma_2^{\otimes n}\rho\left(\sigma_2^{\otimes n}\right)^\dagger+\lambda_3\sigma_3^{\otimes n}\rho\left(\sigma_3^{\otimes n}\right)^\dagger\end{aligned}\tag{156}$$

Examples for $n=2$

In order to be more explicit we will give an example for every channel when $n=2$ and $\lambda_1=p$ and $\lambda_2=1-p$ and A_i are the Kraus generators of each channel. That is, for X

$$\begin{aligned}\mathcal{E}_X^{(2)}(\rho) &= \sum_{k=0}^{1}\lambda_k A_k^{\otimes 2}\rho\left(A_k^{\otimes 2}\right)^\dagger \\ &= p\left(\mathbf{1}\otimes\mathbf{1}\right)\rho\left(\mathbf{1}\otimes\mathbf{1}\right)^\dagger+(1-p)\left(X\otimes X\right)\rho\left(X\otimes X\right)^\dagger \\ &= p\begin{pmatrix}\mathbf{1} & \\ & \mathbf{1}\end{pmatrix}\rho\begin{pmatrix}\mathbf{1} & \\ & \mathbf{1}\end{pmatrix}^\dagger+(1-p)\begin{pmatrix} & X\\ X & \end{pmatrix}\rho\begin{pmatrix} & X\\ X & \end{pmatrix}^\dagger,\end{aligned}\tag{157}$$

for Y

$$
\begin{aligned}
\mathcal{E}_{Y}^{(2)}(\rho) &= \sum_{k=0}^{1} \lambda_k A_k^{\otimes 2} \rho \left(A_k^{\otimes 2}\right)^{\dagger} \\
&= p\left(\mathbf{1}\otimes\mathbf{1}\right)\rho\left(\mathbf{1}\otimes\mathbf{1}\right)^{\dagger} + (1-p)\left(Y\otimes Y\right)\rho\left(Y\otimes Y\right)^{\dagger} \\
&= p\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}\rho\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}^{\dagger} + (1-p)\begin{pmatrix} & -iY \\ iY & \end{pmatrix}\rho\begin{pmatrix} & -iY \\ iY & \end{pmatrix}^{\dagger}, \qquad (158)
\end{aligned}
$$

for Z

$$
\begin{aligned}
\mathcal{E}_{Z}^{(2)}(\rho) &= \sum_{k=0}^{1} \lambda_k A_k^{\otimes 2} \rho \left(A_k^{\otimes 2}\right)^{\dagger} \\
&= p\left(\mathbf{1}\otimes\mathbf{1}\right)\rho\left(\mathbf{1}\otimes\mathbf{1}\right)^{\dagger} + (1-p)\left(Z\otimes Z\right)\rho\left(Z\otimes Z\right)^{\dagger} \\
&= p\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}\rho\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}^{\dagger} + (1-p)\begin{pmatrix} Z & \\ & -Z \end{pmatrix}\rho\begin{pmatrix} Z & \\ & -Z \end{pmatrix}^{\dagger}, \qquad (159)
\end{aligned}
$$

for Hadamard

$$
\begin{aligned}
\mathcal{E}_{H}^{(2)}(\rho) &= \sum_{k=0}^{1} \lambda_k A_k^{\otimes 2} \rho \left(A_k^{\otimes 2}\right)^{\dagger} \\
&= p\left(\mathbf{1}\otimes\mathbf{1}\right)\rho\left(\mathbf{1}\otimes\mathbf{1}\right)^{\dagger} + (1-p)\left(H\otimes H\right)\rho\left(H\otimes H\right)^{\dagger} \\
&= p\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}\rho\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}^{\dagger} + (1-p)\frac{1}{\sqrt{2}}\begin{pmatrix} H & H \\ H & -H \end{pmatrix}\rho\frac{1}{\sqrt{2}}\begin{pmatrix} H & H \\ H & -H \end{pmatrix}^{\dagger}, \qquad (160)
\end{aligned}
$$

for Amplitude-Damping

$$
\begin{aligned}
\mathcal{E}_{A-D}^{(2)}(\rho) &= \sum_{k=0}^{1} \lambda_k A_k^{\otimes 2} \rho \left(A_k^{\otimes 2}\right)^{\dagger} \\
&= p\left(A_0\otimes A_0\right)\rho\left(A_0\otimes A_0\right)^{\dagger} + (1-p)\left(A_1\otimes A_1\right)\rho\left(A_1\otimes A_1\right)^{\dagger} \\
&= p\begin{pmatrix} 0 & \sqrt{\gamma}A_0 \\ 0 & 0 \end{pmatrix}\rho\begin{pmatrix} 0 & \sqrt{\gamma}A_0 \\ 0 & 0 \end{pmatrix}^{\dagger} + (1-p)\begin{pmatrix} A_1 & 0 \\ 0 & \sqrt{1-\gamma}A_1 \end{pmatrix}\rho\begin{pmatrix} A_1 & 0 \\ 0 & \sqrt{1-\gamma}A_1 \end{pmatrix}^{\dagger}, \qquad (161)
\end{aligned}
$$

for Depolarizing

$$
\begin{aligned}
\mathcal{E}_{DP}^{(2)}(\rho) &= \sum_{k=0}^{3} \lambda_k A_k^{\otimes 2} \rho \left(A_k^{\otimes 2}\right)^{\dagger} \\
&= p\left(\mathbf{1}\otimes\mathbf{1}\right)\rho\left(\mathbf{1}\otimes\mathbf{1}\right)^{\dagger} + \frac{(1-p)}{3}\left(X\otimes X\right)\rho\left(X\otimes X\right)^{\dagger} \\
&\quad + \frac{(1-p)}{3}\left(Y\otimes Y\right)\rho\left(Y\otimes Y\right)^{\dagger} + \frac{(1-p)}{3}\left(Z\otimes Z\right)\rho\left(Z\otimes Z\right)^{\dagger} \qquad (162)
\end{aligned}
$$

The unitarity is valid for collective channels as well.

$$\begin{aligned}
\mathcal{E}^{(n)}\left(\mathbf{1}^{\otimes n}\right) &= \sum_k \lambda_k A_k^{\otimes n} \mathbf{1}^{\otimes n} \left(A_k^{\otimes n}\right)^\dagger \\
&= \sum_k \lambda_k A_k^{\otimes n} \left(A_k^{\otimes n}\right)^\dagger \\
&= \sum_k \lambda_k \left(A_k A_k^\dagger\right)^{\otimes n} \\
&= \sum_k \lambda_k \mathbf{1}^{\otimes n} \\
&= \mathbf{1}^{\otimes n}.
\end{aligned} \tag{163}$$

We notice that this only applies to channels with unitary Kraus operators. That is to say, the collective channels of X, Y, Z, Hadamard and Depolarizing channels are unitals. Once again, this tells us that Amplitude-Damping is non-unital.

6 Stochastic Matrices and CPTP Maps

6.1 Stochastic Matrix

A stochastic matrix Δ is a square matrix with real non-negative entries $p_{ij} \in \mathbb{R}$. There are several types of stochastic matrices. The *column-stochastic matrix where*

$$e^\intercal \Delta = e^\intercal, \tag{164}$$

$e^\intercal = (1, 1, \ldots, 1)^\intercal$. The *row-stochastic matrix where*

$$\Delta e = e, \tag{165}$$

$e = (1, 1, \ldots, 1)$. The *doubly-stochastic matrix* where both of the above hold simultaneously. A classic example to understand the definition better is that of a unitary matrix U $\left(UU^\dagger = \mathbf{1}\right)$. If $U = \begin{pmatrix} a & \beta \\ -\beta^* & a^* \end{pmatrix}$ then, the stochastic matrix

$$\Delta_U = U \circ U^* = \begin{pmatrix} |a|^2 & |\beta|^2 \\ |\beta|^2 & |a|^2 \end{pmatrix}, \tag{166}$$

where $|a|^2 + |\beta|^2 = 1$ so, Δ_U is row- and column-stochastic. The class of $n \times n$ doubly-stochastic matrices is a convex polytope known as the Birkhoff polytope $\mathcal{B}_n$.

Theorem 17 *The Birkhoff-von Neumann theorem states that this polytope $\mathcal{B}_n$ is the convex hull of the set of $n \times n$ permutation matrices, and furthermore that the vertices of $\mathcal{B}_n$ are precisely the permutation matrices.*

That is to say, (166) can be written,

$$\Delta_U = \begin{pmatrix} |a|^2 & 0 \\ 0 & |a|^2 \end{pmatrix} + \begin{pmatrix} 0 & |\beta|^2 \\ |\beta|^2 & 0 \end{pmatrix}, \tag{167}$$

which is,

$$\begin{aligned} \Delta_U &= |a|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + |\beta|^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= |a|^2 \, \mathbf{1} + |\beta|^2 \, \sigma_1. \end{aligned} \tag{168}$$

We notice that this is the convex hull $S_2 = \{\mathbf{1}, \sigma_1\}$.

Furthermore, there is the $N \times N$ *bi-stochastic matrix* B which must obey some rules. First of all, for all elements of the matrix $B_{ij} \geq 0$. This ensures that positive vectors transform into positive vectors. Moreover, for every column it must hold that $\sum_{i=0}^{N-1} B_{ij} = 1$, which tells us that the sum of the components of the vector remains invariant. A matrix that satisfies the first two conditions is a stochastic matrix, which means that if a discrete probability distribution is thought of as a vector $\overrightarrow{p}$ then the vector $\overrightarrow{q} = B\overrightarrow{p}$ is a probability distribution too. The third condition is $\sum_{j=0}^{N-1} B_{ij} = 1$ and it ensures that the uniform distribution, a vector all of whose entries are equal, is transformed into itself. Hence a bi-stochastic matrix causes a kind of contraction of the probability simplex with the uniform distribution as a fixed point. One way of obtaining a bi-stochastic matrix is to start with a unitary matrix U and take the absolute value squared of its matrix elements, $B_{ij} = |U_{ij}|^2$. If there exists such U then B is said to be *uni-stochastic[3]*. Likewise, an *ortho-stochastic* matrix is a doubly stochastic matrix whose entries are the square of the absolute value of some orthogonal matrix.

In general, we have already discussed that if we have a density matrix ρ there is a CPTP operator $\mathcal{E} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ that maps $\rho \xrightarrow{\mathcal{E}} \rho' = \mathcal{E}(\rho) = \sum_{i=0}^{n-1} A_i \rho A_i^\dagger$. At this point we will show how the elements of ρ' are related with the eigenvalues of ρ. To begin with, we will turn ρ into diagonal matrix through canonical decomposition. That is, $\rho = S\rho_D S^\dagger$, where $\rho_D = diag(p_0, p_1, \ldots, p_{n-1})$, where p_i are the eigenvalues of ρ. By substituting ρ we obtain,

$$\begin{aligned} \rho' &= \sum_{i=0}^{n-1} A_i S \rho_D S^\dagger A_i^\dagger \\ &= \sum_{i=0}^{n-1} T_i \rho_D T_i^\dagger. \end{aligned} \tag{169}$$

We have called $T_i = A_i S$.

$$\begin{aligned}
(\rho')_{aa} &= \left(\sum_{i=0}^{n-1} T_i \rho_D T_i^\dagger\right)_{aa} \\
&= \sum_{i=0}^{n-1} \left(T_i \rho_D T_i^\dagger\right)_{aa} \\
&= \sum_{i}^{n-1} \sum_{\beta,\gamma}^{n-1} (T_i)_{a\beta} (\rho_D)_{\beta\gamma} \left(T_i^\dagger\right)_{\gamma a}. \qquad (170)
\end{aligned}$$

but $(\rho_D)_{\beta\gamma} = \delta_{\beta\gamma} p_\beta$, where p_β are the eigenvalues of ρ, i.e. $0 \leq p_\beta \leq 1$ and $\sum_{\beta}^{n-1} p_\beta = 1$ so, it becomes,

$$\begin{aligned}
(\rho')_{aa} &= \sum_{i}^{n-1} \sum_{\beta}^{n-1} (T_i)_{a\beta} \, p_\beta \left(T_i^\dagger\right)_{\beta a} \\
&= \sum_{\beta}^{n-1} p_\beta \left[\sum_{i}^{n-1} (T_i)_{a\beta} \left(T_i^\dagger\right)_{\beta a}\right] \\
&= \sum_{\beta}^{n-1} p_\beta \left[\sum_{i}^{n-1} (T_i)_{a\beta} (T_i^*)_{a\beta}\right] \\
&= \sum_{\beta}^{n-1} p_\beta \left[\sum_{i}^{n-1} (T_i \circ T_i^*)_{a\beta}\right]. \qquad (171)
\end{aligned}$$

We define Δ to be $\Delta = \sum_{i}^{n-1} T_i \circ T_i^*$, then

$$(\rho')_{aa} = \sum_{\beta}^{n-1} p_\beta \Delta_{a\beta}. \qquad (172)$$

We have shown how the diagonal elements of ρ' are related with the eigenvalues of ρ. Our next step is to prove the relation of the eigenvalues of ρ' with the eigenvalues of ρ. As we did before we will transform ρ and ρ' into their respective diagonal matrices by applying canonical decomposition. This gives us $\rho = S\rho_D S^\dagger$ and $\rho' = L^\dagger \rho'_D L$, which imply that

$$\begin{aligned} \rho' &= \sum_{i=0}^{n-1} A_i \rho A_i^\dagger \\ L^\dagger \rho'_D L &= \sum_{i=0}^{n-1} A_i S \rho_D S^\dagger A_i^\dagger \\ \rho'_D &= \sum_{i=0}^{n-1} L A_i S \rho_D S^\dagger A_i^\dagger L^\dagger. \end{aligned} \tag{173}$$

We denote $M_i = LA_iS$ and the above turns into

$$\begin{aligned} \rho'_D &= \sum_{i=0}^{n-1} M_i \rho_D M_i^\dagger \Rightarrow \\ (\rho')_{aa} &= \left(\sum_{i=0}^{n-1} M_i \rho_D M_i^\dagger \right)_{aa} \\ &= \sum_{i=0}^{n-1} \left(M_i \rho_D M_i^\dagger \right)_{aa} \\ &= \sum_{i}^{n-1} \sum_{\beta,\gamma}^{n-1} (M_i)_{a\beta} (\rho_D)_{\beta\gamma} \left(M_i^\dagger \right)_{\gamma a}, \end{aligned} \tag{174}$$

but $(\rho_D)_{\beta\gamma} = \delta_{\beta\gamma} p_\beta$, where p_β are the eigenvalues of ρ, i.e. $0 \leq p_\beta \leq 1$ and $\sum_{\beta}^{n-1} p_\beta = 1$ so, this leads to

$$\begin{aligned} (\rho'_D)_{aa} &= \sum_{i}^{n-1} \sum_{\beta}^{n-1} (M_i)_{a\beta} \, p_\beta \left(M_i^\dagger \right)_{\beta a} \\ &= \sum_{\beta}^{n-1} p_\beta \left[\sum_{i}^{n-1} (M_i)_{a\beta} \left(M_i^\dagger \right)_{\beta a} \right] \\ &= \sum_{\beta}^{n-1} p_\beta \left[\sum_{i}^{n-1} (M_i)_{a\beta} (M_i^*)_{a\beta} \right] \\ &= \sum_{\beta}^{n-1} p_\beta \left[\sum_{i}^{n-1} (M_i \circ M_i^*)_{a\beta} \right]. \end{aligned} \tag{175}$$

We define Δ to be $\Delta = \sum_{i}^{n-1} M_i \circ M_i^*$ and $p_a = (\rho'_D)_{aa}$, then last equation reads

$$p_a = \sum_{\beta}^{n-1} p_\beta \Delta_{a\beta}, \tag{176}$$

which means that $p' = \Delta p$. Now, let $p \xrightarrow{\Delta} p'$, where $p = (p_1, p_2, \ldots, p_n)$ and $p' = (p'_1, p'_2, \ldots, p'_n)$ are the eigenvalues of ρ and $\mathcal{E}(\rho)$, respectively. For a probability distribution $p = (p_1, p_2, \ldots, p_n)$ the normalization of the probabilities $\sum_{i=1}^{n} p_i = 1$, can be expressed as $e^{\intercal} p = 1$ or $p^{\intercal} e = 1$, where $e = (1, 1, \ldots, 1)$. We want to show that for p' is valid also that $e^{\intercal} p' = 1$ and $p'^{\intercal} e = 1$, namely the map $p \xrightarrow{\Delta} p'$ is probability preserving. In order to show this we will use the column- and row-stochasticity, of bi-stochastic matrices i.e. $e^{\intercal} \Delta = e^{\intercal}$ and $\Delta e = e$, respectively. To begin with we will show $e^{\intercal} p' = 1$. That is,

$$\begin{aligned} e^{\intercal} p' &= 1 \Rightarrow \\ e^{\intercal} \Delta p &= 1 \overset{e^{\intercal} \Delta = e^{\intercal}}{\Rightarrow} \\ e^{\intercal} p &= 1. \end{aligned} \tag{177}$$

Now, we will show $p'^{\intercal} e = 1$, in the same way,

$$\begin{aligned} p'^{\intercal} e &= 1 \Rightarrow \\ (\Delta p)^{\intercal} e &= 1 \Rightarrow \\ p^{\intercal} \Delta^{\intercal} e &= 1 \underset{\Delta^{\intercal} e = e}{\overset{(e^{\intercal} \Delta)^{\intercal} = (e^{\intercal})^{\intercal}}{\Rightarrow}} \\ p^{\intercal} e &= 1. \end{aligned} \tag{178}$$

Next we prove the uni-stochasticity of the matrix $\Delta = U \circ U^{\dagger}$, where $\Delta = \left(\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n)}\right)$. It must hold that $\sum_{a=1}^{n} \Delta_a^{(i)} = 1$. Unitarity of a matrix U can be written in respect of columns as

$$\begin{aligned} U^{\dagger} U &= \mathbf{1} \Rightarrow \\ \begin{pmatrix} C_1^{\dagger} \\ C_2^{\dagger} \\ \vdots \\ C_n^{\dagger} \end{pmatrix} \begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \Rightarrow \\ \begin{pmatrix} C_1^{\dagger} C_1 & C_1^{\dagger} C_2 & \cdots & C_1^{\dagger} C_n \\ C_2^{\dagger} C_1 & C_2^{\dagger} C_2 & \cdots & C_2^{\dagger} C_n \\ \vdots & \vdots & \ddots & \vdots \\ C_n^{\dagger} C_1 & C_n^{\dagger} C_2 & \cdots & C_n^{\dagger} C_n \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \Rightarrow \\ \begin{pmatrix} \langle C_1, C_1 \rangle & \langle C_1, C_2 \rangle & \cdots & \langle C_1, C_n \rangle \\ \langle C_2, C_1 \rangle & \langle C_2, C_2 \rangle & \cdots & \langle C_2, C_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle C_n, C_1 \rangle & \langle C_n, C_2 \rangle & \cdots & \langle C_n, C_n \rangle \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \end{aligned} \tag{179}$$

From the above we obtain, $\langle C_i, C_j\rangle = \delta_{ij}$ which means $\|C_i\| = 1$. In respect of rows is written

$$
\begin{aligned}
UU^\dagger &= \mathbf{1} \Rightarrow \\
\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} \begin{pmatrix} R_1^\dagger & R_2^\dagger & \cdots & R_n^\dagger \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \Rightarrow \qquad (180) \\
\begin{pmatrix} R_1R_1^\dagger & R_1R_2^\dagger & \cdots & R_1R_n^\dagger \\ R_2R_1^\dagger & R_2R_2^\dagger & \cdots & R_2R_n^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ R_nR_1^\dagger & R_nR_2^\dagger & \cdots & R_nR_n^\dagger \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \Rightarrow \\
\begin{pmatrix} \langle R_1, R_1\rangle & \langle R_1, R_2\rangle & \cdots & \langle R_1, R_n\rangle \\ \langle R_2, R_1\rangle & \langle R_2, R_2\rangle & \cdots & \langle R_2, R_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle R_n, R_1\rangle & \langle R_n, R_2\rangle & \cdots & \langle R_n, R_n\rangle \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \qquad (181)
\end{aligned}
$$

As a result $\langle R_i, R_j\rangle = \delta_{ij}$ which is $\|R_i\| = 1$. This leads to some consequences on the matrix Δ.

$$
\begin{aligned}
\Delta &= U \circ U^* \\
&= \begin{pmatrix} C_\Delta^1 & C_\Delta^2 & \cdots & C_\Delta^n \end{pmatrix} \qquad (182) \\
&= \begin{pmatrix} R_\Delta^1 \\ R_\Delta^2 \\ \vdots \\ R_\Delta^n \end{pmatrix}, \qquad (183)
\end{aligned}
$$

where $C_\Delta^i = C_U^i \circ C_U^{i*}$ and $R_\Delta^i = R_U^i \circ R_U^{i*}$. For the elements of each column we have

$$
\begin{aligned}
\left(C_\Delta^i\right)_a &= \left(C_U^i \circ C_U^{i*}\right)_a \\
&= \left(C_U^i\right)_a \left(C_U^{i*}\right)_a \\
&= \left|\left(C_U^i\right)_a\right|^2 \Rightarrow \\
\sum_{a=0}^{n-1} \left(C_\Delta^i\right)_a &= \sum_{a=0}^{n-1} \left|\left(C_U^i\right)_a\right|^2 = \|C_U^i\| = 1. \qquad (184)
\end{aligned}
$$

In the same way, for the elements of each row we have

$$\begin{aligned}
\left(R^i_\Delta\right)_a &= \left(R^i_U \circ R^{i*}_U\right)_a \\
&= \left(R^i_U\right)_a \left(R^{i*}_U\right)_a \\
&= \left|\left(R^i_U\right)_a\right|^2 \Rightarrow \\
\sum_{a=0}^{n-1} \left(R^i_\Delta\right)_a &= \sum_{a=0}^{n-1} \left|\left(R^i_U\right)_a\right|^2 = \left\|R^i_U\right\| = 1.
\end{aligned} \tag{185}$$

This generalizes into $UU^\dagger = \mathbf{1} \Leftrightarrow \left\langle U^i, U^j \right\rangle = \delta_{ij} \Leftrightarrow \sum_{a=0}^{n-1} \left|U^i_a\right|^2 = \left\|U^i\right\| = 1$.

As a consequence, $\Delta^{(i)} = U^{(i)} \circ U^{(i)*}$

$$\begin{aligned}
\Delta^i_a &= \left(U^i \circ U^{i*}\right)_a \\
&= U^i_a U^{i*}_a \\
&= \left|U^i_a\right|^2 \\
\sum_a^{n-1} \Delta^i_a &= \sum_a^{n-1} \left|U^i_a\right|^2 = 1.
\end{aligned} \tag{186}$$

6.2 Stochastic Matrices of One Qubit Quantum Channels

We will continue by computing the stochastic matrices of the channels we have seen earlier. These are, for X channel

$$\begin{aligned}
\Delta_X &= \sum_{i=0}^{1} p_i A_i \circ A^*_i \\
&= p\mathbf{1} \circ \mathbf{1}^* + (1-p)\, X \circ X^* \\
&= p\mathbf{1} + (1-p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix},
\end{aligned} \tag{187}$$

for Y channel

$$\begin{aligned}
\Delta_Y &= \sum_{i=0}^{1} p_i A_i \circ A^*_i \\
&= p\mathbf{1} \circ \mathbf{1}^* + (1-p)\, Y \circ Y^* \\
&= p\mathbf{1} + (1-p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix},
\end{aligned} \tag{188}$$

for Z channel

$$\begin{aligned}
\Delta_Z &= \sum_{i=0}^{1} p_i A_i \circ A_i^* \\
&= p\mathbf{1} \circ \mathbf{1}^* + (1-p)\, Z \circ Z^* \\
&= p\mathbf{1} + (1-p) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned} \tag{189}$$

for Hadamard channel

$$\begin{aligned}
\Delta_H &= \sum_{i=0}^{1} p_i A_i \circ A_i^* \\
&= p\mathbf{1} \circ \mathbf{1}^* + (1-p)\, H \circ H^* \\
&= p\mathbf{1} + \frac{1-p}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} p + \frac{1-p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & p + \frac{1-p}{2} \end{pmatrix},
\end{aligned} \tag{190}$$

for Amplitude-Damping channel

$$\begin{aligned}
\Delta_{A-D} &= \sum_{i=0}^{1} p_i A_i \circ A_i^* \\
&= p \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} + (1-p) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\
&= p \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + (1-p) \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} \\
&= \begin{pmatrix} 1-p & p\gamma \\ 0 & (1-p)\,(1-\gamma) \end{pmatrix},
\end{aligned} \tag{191}$$

for Depolarizing channel

$$\begin{aligned}
\Delta_{DP} &= \sum_{i=0}^{3} p_i A_i \circ A_i^* \\
&= p\mathbf{1} \circ \mathbf{1}^* + \frac{(1-p)}{3} X \circ X^* + \frac{(1-p)}{3} Y \circ Y^* + \frac{(1-p)}{3} Z \circ Z^* \\
&= \begin{pmatrix} p + \frac{1-p}{3} & 2\frac{1-p}{3} \\ 2\frac{1-p}{3} & p + \frac{1-p}{3} \end{pmatrix}.
\end{aligned} \tag{192}$$

We observe that for channels X, Y, Z, Hadamard and Depolarizing, the associated stochastic matrices are doubly-stochastic, whereas Amplitude-Damping's associated matrix Δ_{A-D} is not neither row nor column stochastic.

6.3 Stochastic Matrices of Common Collective Channels

The stochastic matrix of a collective channel is obtained from

$$\Delta^{(n)} = \sum_k \lambda_k A_k^{\otimes n} \circ \left(A_k^{\otimes n}\right)^* . \tag{193}$$

For the common channels we obtain: for X channel

$$\begin{aligned} \Delta_X^{(n)} &= \sum_k \lambda_k A_k^{\otimes n} \circ \left(A_k^{\otimes n}\right)^* \\ &= p\mathbf{1}^{\otimes n} \circ \mathbf{1}^{\otimes n} + (1-p)\,\sigma_1^{\otimes n} \circ \left(\sigma_1^{\otimes n}\right)^* \\ &= \begin{pmatrix} p & 0 & \cdots & 0 & 1-p \\ 0 & p & \cdots & 1-p & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1-p & \cdots & p & 0 \\ 1-p & 0 & \cdots & 0 & p \end{pmatrix}, \end{aligned} \tag{194}$$

$$\begin{aligned} \Delta_X^{(n)} e^{\otimes n} &= \left(p\mathbf{1}^{\otimes n} + (1-p)\sigma_1^{\otimes n}\right) e^{\otimes n} && (195) \\ &= pe^{\otimes n} + (1-p)e^{\otimes n} && (196) \\ &= e^{\otimes n}. && (197) \end{aligned}$$

For Y channel

$$\begin{aligned} \Delta_Y^{(n)} &= \sum_k \lambda_k A_k^{\otimes n} \circ \left(A_k^{\otimes n}\right)^* \\ &= p\mathbf{1}^{\otimes n} \circ \mathbf{1}^{\otimes n} + (1-p)\,\sigma_2^{\otimes n} \circ \left(\sigma_2^{\otimes n}\right)^* \\ &= \begin{pmatrix} p & 0 & \cdots & 0 & 1-p \\ 0 & p & \cdots & 1-p & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1-p & \cdots & p & 0 \\ 1-p & 0 & \cdots & 0 & p \end{pmatrix}, \end{aligned} \tag{198}$$

we notice that the row and column elements add up to one, which proves the double-stochasticity of $\Delta_Y^{(n)}$. For Z channel

$$\begin{aligned}
\Delta_Z^{(n)} &= \sum_k \lambda_k A_k^{\otimes n} \circ \left(A_k^{\otimes n}\right)^* \\
&= p\mathbf{1}^{\otimes n} \circ \mathbf{1}^{\otimes n} + (1-p)\,\sigma_3^{\otimes n} \circ \left(\sigma_3^{\otimes n}\right)^* \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},
\end{aligned} \tag{199}$$

we notice that it forms the identity matrix which is as a fact doubly-stochastic matrix. For Amplitude-Damping channel

$$\begin{aligned}
\Delta_{A-D}^{(n)} &= \sum_k \lambda_k A_k^{\otimes n} \circ \left(A_k^{\otimes n}\right)^* = pA_1^{\otimes n} \circ \left(A_1^{\otimes n}\right)^* + (1-p)\,A_2^{\otimes n} \circ \left(A_2^{\otimes n}\right)^* \\
&= p\begin{pmatrix} 0 & 0 & \cdots & 0 & \gamma^{\frac{n}{2}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} + (1-p)\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{1-\gamma} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (1-\gamma)^{\frac{n-1}{2}} & 0 \\ 0 & 0 & \cdots & 0 & (1-\gamma)^{\frac{n}{2}} \end{pmatrix},
\end{aligned} \tag{200}$$

we do not get a doubly-stochastic matrix as the sum of rows or columns does not add up to one. For Depolarizing channel

$$\begin{aligned}
\Delta_{DP}^{(n)} &= \sum_k \lambda_k A_k^{\otimes n} \circ \left(A_k^{\otimes n}\right)^* \\
&= p\mathbf{1}^{\otimes n} \circ \mathbf{1}^{\otimes n} + \frac{1-p}{3}\sigma_1^{\otimes n} \circ \left(\sigma_1^{\otimes n}\right)^* + \frac{1-p}{3}\sigma_2^{\otimes n} \circ \left(\sigma_2^{\otimes n}\right)^* + \frac{1-p}{3}\sigma_3^{\otimes n} \circ \left(\sigma_3^{\otimes n}\right)^* \\
&= \begin{pmatrix} p+\frac{1-p}{3} & 0 & \cdots & 0 & 2\frac{1-p}{3} \\ 0 & p+\frac{1-p}{3} & \cdots & 2\frac{1-p}{3} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2\frac{1-p}{3} & \cdots & p+\frac{1-p}{3} & 0 \\ 2\frac{1-p}{3} & 0 & \cdots & 0 & p+\frac{1-p}{3} \end{pmatrix},
\end{aligned} \tag{201}$$

we observe that the rows and columns satisfy the criteria of double-stochasticity.

It is worth noticing that the collective channels share the same stochasticity properties with the corresponding 1 qubit channels, namely the associated delta matrix $\Delta_a^{(n)}$ and Δ_a are double-stochastic for the cases $\alpha = \{$channels X, Y, Z, Hadamard and Depolarizing$\}$, while Amplitude-Damping matrix $\Delta_{AD}^{(n)}$ and Δ_{AD} is neither a row- stochastic nor a column stochastic.

7 Circulant Matrix Theory

A special case of Toeplitz matrix is the circulant matrix. A circulant matrix is of the form $M = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}$. Given the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ and $\omega = e^{\frac{2\pi i}{n}}$, then the eigenvalues are $\lambda_k = f(\omega^k)$, the eigenvectors are $v_k = \left(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}\right)^\intercal$ and the determinant of M is $\det(M) = \prod_{k=0}^{n-1} f(\omega^k)$ for $k = 0, \ldots, n-1$.

Circulant matrices are important due to being diagonalizable by the discrete Fourier transformation. Conversely, the inverse Fourier transformation takes a diagonal matrix into a circulant matrix. An $n \times n$ circulant matrix C is of the form

$$C = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{pmatrix}. \tag{202}$$

As we see, a circulant matrix needs only one vector c to be formed as all the other columns are the cyclic permutations of the elements of this vector.

Some properties of the circulant matrices are:

We can write the matrix C as a polynomial

$$p_\gamma(h) = c_0 \mathbf{1} + c_1 h + c_2 h^2 + c_3 h^3 + \ldots + c_{n-2} h^{n-2} + c_{n-1} h^{n-1}, \tag{203}$$

where h is the permutation matrix

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{204}$$

The permutation matrix is a circulant matrix with entries $c = (0, 1, 0, \ldots, 0)$.

The orthogonal and complete bases of h and g, where $g = diag\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right)$, are diagonal and are related by a finite Fourier transform $\mathcal{F}$,

$$|\phi_k\rangle = \mathcal{F}\,|k\rangle = \frac{1}{\sqrt{2j+1}} \sum_{n=0}^{2j} \omega^{kn}\,|n\rangle\,, \tag{205}$$

which maps angular momentum states to phase states. Also $g\,(h)$ acts as step operator in the phase state basis, i.e.

$$h\,|n\rangle = |n+1\rangle\,, h\,|\phi_m\rangle = \omega^m\,|\phi_m\rangle\,,\ \mathrm{mod}(2j+1), \tag{206}$$

while

$$g^{2j+1} = h^{2j+1} = \mathbf{1},\ h^{-1} = h^\dagger, \tag{207}$$

and of course $\mathcal{F}\mathcal{F}^\dagger = \mathcal{F}^\dagger\mathcal{F} = \mathbf{1}$. The conjugation between the number and the phase operators becomes transparent by the relations, $\mathcal{F}g\mathcal{F}^\dagger = h$, $\mathcal{F}h\mathcal{F}^\dagger = g^{-1}$. The Fourier matrix $\mathcal{F}$ can be written as[4]

$$\mathcal{F} = \frac{1}{\sqrt{2j+1}}\sum_{m,n=0}^{2j}\omega^{mn}\,|m\rangle\,\langle n|\,.$$

Proposition 18 *It holds about the commutation relations that*

$$g^m h^n = \omega^{mn} h^n g^m, \tag{208}$$

for every $m, n \in \mathbb{R}$ where, $g = diag\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right)$ and $h = circ\,(0, 1, 0, \ldots, 0)$ [1].

Proof. We have that

$$gh = \omega hg. \tag{209}$$

We multiply both sides of the equation from the left by g^{m-1} and from the right by h^{n-1}. This gives us

$$g^{m-1}ghh^{n-1} = \omega g^{m-1}hgh^{n-1}, \tag{210}$$

which is $g^m h^n = \omega g^{m-1}hgh^{n-1}$. Now, we need to move every h to the left and every g to the right. We know that $gh = \omega hg$ which means that with every commutation of g and h we get an extra ω. We have $n-1$ and $n(m-1)$ commutations. This gives us, $\omega\omega^{n-1}\omega^{n(m-1)} = \omega^{1+n-1+nm-n} = \omega^{mn}$. Thus,

$$g^m h^n = \omega^{mn} h^n g^m. \tag{211}$$

■

Theorem 19 *Having two circulant matrices A and B we notice that the sum $A+B$, and the product AB are circulant and that it also holds that $AB = BA$ which leads us to circulant matrices being a form of commutative algebra. If $C =$*

$circ\,(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix and $\mathcal{F}$ is the Fourier transformation matrix then

$$\mathcal{F}^\dagger C\mathcal{F} = \sqrt{n}diag\left[\mathcal{F}^\dagger c\right], \tag{212}$$

where $c = (c_0, c_1, \ldots, c_{n-1})$

Proof. First of all we will start with finding the matrix $\mathcal{F}c$. So we have,

$$\begin{aligned}
\mathcal{F}^\dagger c &= \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \\
&= \frac{1}{\sqrt{n}} \begin{pmatrix} \sum_{k=0}^{n-1} c_k \\ \sum_{k=0}^{n-1} c_k^{-k} \omega^{-k} \\ \sum_{k=0}^{n-1} c_k^{-2k} \omega^{-2k} \\ \vdots \\ \sum_{k=0}^{n-1} c_k^{-k(n-1)} \omega^{-k(n-1)} \end{pmatrix}. \qquad (213)
\end{aligned}$$

$$\begin{aligned}
\left(\mathcal{F}^\dagger h \mathcal{F}\right)_{\alpha\beta} &= \sum_{\kappa,\lambda=0}^{n-1} \left(\mathcal{F}^\dagger\right)_{\alpha\kappa} h_{\kappa\lambda} \left(\mathcal{F}\right)_{\lambda\beta} \\
&= \frac{1}{n} \sum_{\kappa,\lambda=0}^{n-1} \omega^{*\kappa\alpha} \delta_{\kappa,\lambda+1} \omega^{\lambda\beta} \\
&= \frac{1}{n} \sum_{\lambda=0}^{n-1} \omega^{*(\lambda+1)\alpha} \omega^{\lambda\beta} \\
&= \frac{1}{n} \omega^{*\alpha} \sum_{\lambda=0}^{n-1} \omega^{*\lambda\alpha} \omega^{\lambda\beta} \\
&= \frac{1}{n} \omega^{*\alpha} \sum_{\lambda=0}^{n-1} \omega^{-\lambda\alpha} \omega^{\lambda\beta} \\
&= \frac{1}{n} \omega^{*\alpha} \sum_{\lambda=0}^{n-1} \omega^{\lambda(\beta-\alpha)} \\
&= \delta_{\alpha,\beta} \omega^{*\alpha} \\
&= \delta_{\alpha,\beta} \omega^{-\alpha} \qquad (214)
\end{aligned}$$

From this we obtain

$$\begin{aligned}
\mathcal{F}^{\dagger} h \mathcal{F} &= g^{-1} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega^{*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{*(n-1)} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{-(n-1)} \end{pmatrix}.
\end{aligned} \tag{215}$$

Now, we have that

$$\mathcal{F}^{\dagger} h \mathcal{F} = g \Rightarrow \ \mathcal{F}^{\dagger} h^{2} \mathcal{F} = g^{2} \Rightarrow \cdots \Rightarrow \ \mathcal{F}^{\dagger} h^{k} \mathcal{F} = g^{k}. \tag{216}$$

If $C = \sum_{\alpha=0}^{n-1} c_{\alpha} h^{\alpha}$ then $\mathcal{F}^{\dagger} C \mathcal{F} = \mathcal{F}^{\dagger} \left(\sum_{\alpha=0}^{n-1} c_{\alpha} h^{\alpha} \right) \mathcal{F} = \sum_{\alpha=0}^{n-1} c_{\alpha} \mathcal{F}^{\dagger} h^{\alpha} \mathcal{F} = \sum_{\alpha=0}^{n-1} c_{\alpha} g^{\alpha}$.
We need to show that $\mathcal{F}^{\dagger} C \mathcal{F} = diag\,(\mathcal{F} c)$. ■

$$\begin{aligned}
\mathcal{F}^{\dagger} C \mathcal{F} &= \sum_{\alpha=0}^{n-1} c_{\alpha} g^{\alpha} \\
&= c_0 g^0 + c_1 g + c_2 g^2 + \cdots + c_{n-1} g^{n-1} \\
&= c_0 \mathbf{1} + c_1 g + c_2 g^2 + \cdots + c_{n-1} g^{n-1}
\end{aligned}$$

$$= \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_0 & 0 & \cdots & 0 \\ 0 & 0 & c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix} + \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_1 \omega^{-1} & 0 & \cdots & 0 \\ 0 & 0 & c_1^{-2} \omega^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{-(n-1)} \omega^{-(n-1)} \end{pmatrix} +$$

$$+ \begin{pmatrix} c_2 & 0 & 0 & \cdots & 0 \\ 0 & c_2 \omega^{-2} & 0 & \cdots & 0 \\ 0 & 0 & c_2 \omega^{-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_2 \omega^{-2(n-1)} \end{pmatrix} + \begin{pmatrix} c_3 & 0 & 0 & \cdots & 0 \\ 0 & c_3 \omega^{-3} & 0 & \cdots & 0 \\ 0 & 0 & c_3 \omega^{-6} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_3 \omega^{-3(n-1)} \end{pmatrix} +$$

$$+\cdots+\begin{pmatrix} c_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & c_{n-1}\omega^{-(n-1)} & 0 & \cdots & 0 \\ 0 & 0 & c_{n-1}\omega^{-2(n-1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-1}\omega^{-(n-1)^2} \end{pmatrix}=$$

$$=\begin{pmatrix} \sum_{k=0}^{n-1} c_k & 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{n-1} c_k\omega^{-k} & 0 & \cdots & 0 \\ 0 & 0 & \sum_{k=0}^{n-1} c_k\omega^{-2k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{k=0}^{n-1} c_k\omega^{-k(n-1)} \end{pmatrix}. \tag{217}$$

So, we have proven that $Fc = \frac{1}{\sqrt{n}}\begin{pmatrix} \sum_{k=0}^{n-1} c_k \\ \sum_{k=0}^{n-1} c_k\omega^{-k} \\ \sum_{k=0}^{n-1} c_k\omega^{-2k} \\ \vdots \\ \sum_{k=0}^{n-1} c_k\omega^{-k(n-1)} \end{pmatrix}$ and that

$$\mathcal{F}^\dagger C\mathcal{F}=\begin{pmatrix} \sum_{k=0}^{n-1} c_k & 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{n-1} c_k\omega^{-k} & 0 & \cdots & 0 \\ 0 & 0 & \sum_{k=0}^{n-1} c_k\omega^{-2k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{k=0}^{n-1} c_k\omega^{-k(n-1)} \end{pmatrix} \tag{218}$$

.

Thus, $\mathcal{F}^{\dagger}C\mathcal{F}=\sqrt{n}diag\left(\mathcal{F}c\right)$

Lemma 20 *It holds that* $\sum_{n=0}^{k-1}\omega^{n}=0$ *and* $\sum_{n=0}^{k-1}(\omega^{b})^{n}=0$ *, for every* b*, where* $\omega=e^{i\frac{2\pi k}{n}}$.

Proof. This is a geometric progression which sums into $S_n=a_1\frac{\lambda^n-1}{\lambda-1}$, where a_1 is the first element of the sequence, $\lambda\neq 0$ is the common ratio. Then

$$\begin{aligned}\sum_{n=0}^{k-1}\omega^{n} &= \omega^{0}+\omega+\omega^{2}+\omega^{3}+\cdots+\omega^{(k-1)}\\ &= e^{i\frac{2\pi 0}{n}}+e^{i\frac{2\pi 1}{n}}+e^{i\frac{2\pi 2}{n}}+e^{i\frac{2\pi 3}{n}}+\cdots+e^{i\frac{2\pi(n-1)}{n}}\\ &= a_0\frac{\omega^{n}-1}{\omega-1}=1\frac{e^{i\frac{2\pi n}{n}}-1}{e^{i\frac{2\pi}{n}}-1}=\frac{1-1}{e^{i\frac{2\pi}{n}}-1}=0, \end{aligned}\tag{219}$$

also

$$\begin{aligned}\sum_{n=0}^{k-1}\left(\omega^{b}\right)^{n} &= \omega^{0}+\omega^{b}+\omega^{2b}+\omega^{3b}+\cdots+\omega^{(k-1)b}\\ &= e^{i\frac{2\pi 0}{n}b}+e^{i\frac{2\pi 1}{n}b}+e^{i\frac{2\pi 2}{n}b}+e^{i\frac{2\pi 3}{n}b}+\cdots+e^{i\frac{2\pi(n-1)}{n}b}\\ &= a_0\frac{\omega^{nb}-1}{\omega^{b}-1}=1\frac{e^{i\frac{2\pi n}{n}b-1}}{e^{i\frac{2\pi}{n}b-1}}=\frac{1-1}{e^{i\frac{2\pi}{n}b-1}}=0, \end{aligned}\tag{220}$$

because, $e^{i\frac{2\pi n}{n}b}=e^{i2\pi b}=\cos\left(2\pi b\right)+i\sin\left(2\pi b\right)=1+i0=1.$ ∎

To sum up, circulants are matrices of basic Fourier analysis. Circulant matrices have projectors which are also circulants, and every well-defined function of a circulant is another circulant. We have seen that the matrices C and C_{diag} contain the same amount of information as the circulant matrix C is diagonalizable only by performing a Fourier transformation.

8 Circulant CPTP Maps

8.1 Introduction

Following the notion of Muldoon[20] and Scwaiger[25] on generalized hyperbolic functions and their characterization by functional equations we will discuss how a function can be expressed as the sum of its components. To begin with, we will analyse the case for $n=2$. A function $f\colon\mathbb{C}\to\mathbb{C}$ can be written as the sum of even and odd components as $f\left(x\right)=f_0\left(x\right)+f_1\left(x\right)$, where the even component is $f_0(x) = \frac{1}{2}(f(x) + f(-x))$ and the odd component is $f_1(x) = \frac{1}{2}(f(x) - f(-x))$. If the function f is the exponential function then it satisfies

$$f(x+y) = f(x) f(y) \tag{221}$$

for every x and y and its components satisfy

$$\begin{aligned} f_0(x+y) &= f_0(x) f_0(y) + f_1(x) f_1(y) \\ f_1(x+y) &= f_1(x) f_0(y) + f_0(x) f_1(y) \end{aligned} \tag{222}$$

and

$$\begin{aligned} f_0(x-y) &= f_0(x) f_0(y) - f_1(x) f_1(y) \\ f_1(x-y) &= f_1(x) f_0(y) - f_0(x) f_1(y) \end{aligned} \tag{223}$$

In case f is the exponential function e^x, then $f_0(x) = \cosh x$ and $f_1(x) = \sinh x$ and the above are the familiar sum and difference relations for these functions.

Conversely, the general solution of (223) is expressible in terms of a single arbitrary exponential function, i.e., it is known that if f_0 and f_1 satisfy (223), then f_0 and f_1 are the even and odd components of a single exponential function: $f_0(x) = \frac{1}{2}(g(x) + g(-x))$, $f_1(x) = \frac{1}{2}(g(x) - g(-x))$. In fact, $g(x) = f_0(x) + f_1(x)$. Whereas, the general solution of (222) depends on two exponential functions: $f_0(x) = \frac{1}{2}(g_1(x) + g_2(x))$, $f_1(x) = \frac{1}{2}(g_1(x) - g_2(x))$. For example, $g_1(x) = e^x$ and $g_2(x) = 0$([20]).

For (223) having less solutions than (222) is that (223) implies (222) but not the opposite. To show that (223) implies (222), first we interchange x and y in (223) and we observe that f_0 must be even and f_2 must be odd. Afterwards we replace y by $-y$ in (223) which yields (222). While, $f_0(x) = f_1(x) = \frac{e^x}{2}$ satisfies (222) but not (223).

The (222) and (223) can also be written

$$\begin{pmatrix} f_0(x+y) & f_1(x+y) \\ f_1(x+y) & f_0(x+y) \end{pmatrix} = \begin{pmatrix} f_0(y) & f_1(y) \\ f_1(y) & f_0(y) \end{pmatrix} \begin{pmatrix} f_0(x) & f_1(x) \\ f_1(x) & f_0(x) \end{pmatrix}, \tag{224}$$

and

$$\begin{pmatrix} f_0(x-y) & f_1(x-y) \\ f_1(x-y) & f_0(x-y) \end{pmatrix} = \begin{pmatrix} f_0(y) & -f_1(y) \\ -f_1(y) & f_0(y) \end{pmatrix} \begin{pmatrix} f_0(x) & f_1(x) \\ f_1(x) & f_0(x) \end{pmatrix}. \tag{225}$$

Definition 21 *A function $f : \mathbb{C} \to \mathbb{C}$ is of type j for $j = 0, 1, \ldots, n-1$, if $f(\omega x) = \omega^{n-j} f(x)$ where $\omega = e^{\frac{2\pi i}{n}}$.*

Lemma 22 *Every function $f : \mathbb{C} \to \mathbb{C}$ can be expressed uniquely as a sum of functions f_j of type j, for $j = 0, 1, 2, \ldots, n-1$, called the components of f, where*

$$\begin{pmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{pmatrix} = \frac{1}{\sqrt{n}} \mathcal{F} \begin{pmatrix} f(x) \\ f(\omega x) \\ \vdots \\ f(\omega^{n-1} x) \end{pmatrix}. \tag{226}$$

In fact $f = \sum_{j=0}^{n-1} f_j$([20]).

8.2 Circulant Function Valued Matrix

Here, we will introduce the matrix with entries functions derived from a chosen function with certain properties as well as how the matrix is constructed and its properties.

Definition 23 *We will call the circulant matrix $F(x)$ corresponding to function f the circulant matrix*

$$F(x) = circ(f_0, f_1, \ldots, f_{n-1}) = \begin{pmatrix} f_0 & f_{n-1} & \cdots & f_1 \\ f_1 & f_0 & \cdots & f_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1} & f_{n-2} & \cdots & f_0 \end{pmatrix}, \tag{227}$$

whose first column is formed by the components of the function f in increasing order.

Lemma 24 *If $f : \mathbb{C} \to \mathbb{C}$ is any function, the corresponding circulant matrix function is given by $F(x) = \mathcal{F} diag\left[f(x), f(\omega x), \ldots, f\left(\omega^{n-1}x\right)\right]\mathcal{F}^\dagger$*

Lemma 25 *If f is an exponential function, the corresponding circulant matrix function F satisfies*

$$F(x+y) = F(y)F(x). \tag{228}$$

Proof. *This is fairly immediate consequence of previous lemma*

$$F(x) = f_0\mathbf{1} + f_1 h + f_2 h^2 + \cdots + f_{n-1}h^{n-1}. \tag{229}$$

Then by means of Fourier matrix $\mathcal{F}$ we obtain

$$\begin{aligned} \mathcal{F}F(x)\mathcal{F}^\dagger &= f_0(x)\mathcal{F}\mathbf{1}\mathcal{F}^\dagger + f_1(x)\mathcal{F}h\mathcal{F}^\dagger + f_2(x)\mathcal{F}h^2\mathcal{F}^\dagger + \cdots + f_{n-1}(x)\mathcal{F}h^{n-1}\mathcal{F}^\dagger \\ &= f_0(x)\mathbf{1} + f_1(x)\, diag\left[1, \omega, \omega^2, \ldots, \omega^{n-1}\right] + \cdots + f_{n-1}(x)\, diag\left[1, \omega^{n-1}, \ldots, \omega^{(n-1)^2}\right] \\ &= f_0(x)\mathbf{1} + f_1(x)g + \cdots + f_{n-1}(x)g^{n-1}. \end{aligned} \tag{230}$$

Likewise,

$$\mathcal{F}F(y)\mathcal{F}^\dagger = f_0(y)\mathbf{1} + f_1(y)g + \cdots + f_{n-1}(y)g^{n-1}, \tag{231}$$

and

$$\mathcal{F}F(x+y)\mathcal{F}^\dagger = f_0(x+y)\mathbf{1} + f_1(x+y)g + \cdots + f_{n-1}(x+y)g^{n-1}. \tag{232}$$

Therefore

$$F(x+y) = F(x)F(y) \Rightarrow \mathcal{F}F(x+y)\mathcal{F}^\dagger = \mathcal{F}F(x)\mathcal{F}^\dagger\mathcal{F}F(y)\mathcal{F}^\dagger. \tag{233}$$

■

Example 26 *We will use the exponential function $f(x) = e^x$ for $n = 2$. To begin with, the function can be written as the sum of two other functions of type j, $j = 0, 1, \ldots, n-1$, the f_0 and f_1.*

$$f(x) = \sum_{j=0}^{1} f_j(x). \tag{234}$$

These functions will be found by the equation below where $\mathcal{F}$ is the 2×2 Fourier transformation matrix

$$\begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \mathcal{F} \begin{pmatrix} f(x) \\ f(\omega x) \end{pmatrix}, \tag{235}$$

$$\begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f(x) \\ f(-x) \end{pmatrix}. \tag{236}$$

When the matrix is $2-$dimentional we notice that we get the even and odd functions

$$f_0(x) = \frac{1}{2} \left[f(x) + f(-x) \right], \tag{237}$$

$$f_1(x) = \frac{1}{2} \left[f(x) - f(-x) \right]. \tag{238}$$

By substituting $f(x) = e^x$ and $f(-x) = e^{-x}$, f_0 and f_1 are

$$f_0(x) = \frac{1}{2} \left[e^x + e^{-x} \right] = \cosh x, \tag{239}$$

$$f_1(x) = \frac{1}{2} \left[e^x - e^{-x} \right] = i \sinh x. \tag{240}$$

Our last step is to use the functions we found above as entries of each column

$$F(x) = \begin{pmatrix} f_0 & f_1 \\ f_1 & f_0 \end{pmatrix} = \begin{pmatrix} \cosh x & i \sinh x \\ i \sinh x & \cosh x \end{pmatrix}. \tag{241}$$

The final circulant matrix that is derived is also unitary([20]).

$$F(x) F^\dagger(x) = \begin{pmatrix} \cosh x & i \sinh x \\ i \sinh x & \cosh x \end{pmatrix} \begin{pmatrix} \cosh x & -i \sinh x \\ -i \sinh x & \cosh x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}. \tag{242}$$

8.3 The Choice $f(x) = e^{ix}$

At this point we will apply the procedure to find the function valued matrix with the function $f(x) = e^{ix}$. To begin with, we will compute the matrix $F(x)$ for $n = 2$ by finding the components of $f(x)$

$$\begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \mathcal{F} \begin{pmatrix} f(x) \\ f(\omega x) \end{pmatrix}, \tag{243}$$

$$\begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f(x) \\ f(-x) \end{pmatrix}. \tag{244}$$

By substituting $f(x) = e^{ix}$ and $f(-x) = e^{-ix}$, f_0 and f_1 become

$$f_0(x) = \frac{1}{2} \left[e^{ix} + e^{-ix} \right] = \cos x, \tag{245}$$

$$f_1(x) = \frac{1}{2} \left[e^{ix} - e^{-ix} \right] = i \sin x. \tag{246}$$

As a consequence, $F(x)$ is

$$F(x) = \begin{pmatrix} f_0 & f_1 \\ f_1 & f_0 \end{pmatrix} = \begin{pmatrix} \cos x & i \sin x \\ i \sin x & \cos x \end{pmatrix}. \tag{247}$$

For $n = 2$ the constructed matrix $F(x)$ is unitary

$$\begin{aligned} F(x) F^{\dagger}(x) &= \begin{pmatrix} \cos x & i \sin x \\ i \sin x & \cos x \end{pmatrix} \begin{pmatrix} \cos x & -i \sin x \\ -i \sin x & \cos x \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 x + \sin^2 x & 0 \\ 0 & \cos^2 x + \sin^2 x \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}. \end{aligned}$$

Now, we will apply all the above to our exponential complex function for $n = 3$

$$f(x) = e^{ix}. \tag{248}$$

To begin with, we will determine the components of f which are functions f_j of type j, where $j = 0, 1, 2$,

$$f(x) = \sum_{j=0}^{2} f_j(x). \tag{249}$$

So,

$$\begin{pmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{pmatrix} = \frac{1}{\sqrt{3}} \mathcal{F} \begin{pmatrix} f(x) \\ f(\omega x) \\ f(\omega^2 x) \end{pmatrix}, \tag{250}$$

where $\mathcal{F}$ is the Fourier transformation matrix and $\omega = e^{\frac{2\pi i}{3}}$.

That is to say,

$$\begin{pmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \begin{pmatrix} f(x) \\ f(\omega x) \\ f(\omega^2 x) \end{pmatrix}, \tag{251}$$

$$\begin{pmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} f(x) \\ f(\omega x) \\ f(\omega^2 x) \end{pmatrix}, \tag{252}$$

$$\begin{pmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} e^{ix} \\ e^{i\omega x} \\ e^{i\omega^2 x} \end{pmatrix}. \tag{253}$$

From this equation we obtain the following:

$$\begin{aligned} f_0(x) &= \frac{1}{3}\left(e^{ix} + e^{i\omega x} + e^{i\omega^2 x}\right) \\ &= \frac{1}{3}\left(e^{ix} + 2e^{-\frac{1}{2}ix}\cosh\frac{\sqrt{3}}{2}x\right), \end{aligned} \tag{254}$$

$$\begin{aligned} f_1(x) &= \frac{1}{3}\left(e^{ix} + \omega e^{i\omega x} + \omega^2 e^{i\omega^2 x}\right) \\ &= \frac{1}{3}\left(e^{ix} - e^{-\frac{1}{2}ix}\left(\cosh\frac{\sqrt{3}}{2}x + \sqrt{3}i\sinh\frac{\sqrt{3}}{2}x\right)\right), \end{aligned} \tag{255}$$

$$\begin{aligned} f_2(x) &= \frac{1}{3}\left(e^{ix} + \omega^2 e^{i\omega x} + \omega e^{i\omega^2 x}\right) \\ &= \frac{1}{3}\left(e^{ix} - e^{-\frac{1}{2}ix}\left(\cosh\frac{\sqrt{3}}{2}x - \sqrt{3}i\sinh\frac{\sqrt{3}}{2}x\right)\right). \end{aligned} \tag{256}$$

These components are of type j, $j = 0, .., n-1$ because it holds that $f_j(\omega x) = \omega^{n-j} f_j(x)$.

$$\begin{aligned} f_0(\omega x) &= \frac{1}{3}\left(e^{i\omega x} + e^{i\omega^2 x} + e^{i\omega^3 x}\right) = \\ &= \frac{1}{3}\left(e^{i\omega x} + e^{i\omega^2 x} + e^{ix}\right) = f_0(x), \end{aligned} \tag{257}$$

$$
\begin{aligned}
f_1(\omega x) &= \frac{1}{3}\left(e^{i\omega x} + \omega e^{i\omega^2 x} + \omega^2 e^{i\omega^3 x}\right) \\
&= \frac{1}{3}\left(e^{i\omega x} + \omega e^{i\omega^2 x} + \omega^2 e^{ix}\right) \\
&= \frac{1}{3}\omega^2\left(\omega e^{i\omega x} + \omega^2 e^{i\omega^2 x} + e^{ix}\right) = \omega^2 f_1(x),
\end{aligned} \tag{258}
$$

$$
\begin{aligned}
f_2(\omega x) &= \frac{1}{3}\left(e^{i\omega x} + \omega^2 e^{i\omega^2 x} + \omega e^{i\omega^3 x}\right) \\
&= \frac{1}{3}\left(e^{i\omega x} + \omega^2 e^{i\omega^2 x} + \omega e^{ix}\right) \\
&= \frac{1}{3}\omega\left(\omega^2 e^{i\omega x} + \omega e^{i\omega^2 x} + e^{ix}\right) = \omega f_2(x).
\end{aligned} \tag{259}
$$

Now that we know that the components of the function $f(x) = e^{ix}$ are of type j we can proceed in finding the circulant matrix $F(x) = circ(f_0, f_1, f_2)$

$$
F(x) = \begin{pmatrix} f_0 & f_2 & f_1 \\ f_1 & f_0 & f_2 \\ f_2 & f_1 & f_0 \end{pmatrix} = e^{ixh}. \tag{260}
$$

We have from the properties of the exponential matrix that if Y is invertible then $e^{YXY^{-1}} = Ye^{X}Y^{-1}$ and from the properties of the Fourier matrix we know that $\mathcal{F}^{-1} = \mathcal{F}^{*}$.

We also know for the exponential matrix that if $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ then $e^A = \begin{pmatrix} e^{a_1} & 0 & 0 \\ 0 & e^{a_2} & 0 \\ 0 & 0 & e^{a_3} \end{pmatrix}$.

And now, we need to write our circulant matrix $F(x) = e^{ixh}$

$$
F(x) = e^{ixh} = e^{ix\mathcal{F}g\mathcal{F}^{\dagger}}, \tag{261}
$$

ix are elements and $g = diag(1, \omega, \omega^2)$.

So,

$$\begin{aligned}
F(x) &= \mathcal{F} e^{ixg} \mathcal{F}^{\dagger} \\
&= \mathcal{F} \begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x} \end{pmatrix} \mathcal{F}^{\dagger} \\
&= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} \\ 1 & \omega^{-2} & \omega^{-1} \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} e^{ix\omega} + e^{ix\omega^2} + e^{ix} & e^{ix} + \omega^2 e^{ix\omega} + \omega e^{ix\omega^2} & e^{ix} + \omega e^{ix\omega} + \omega^2 e^{ix\omega^2} \\ e^{ix} + \omega^2 e^{ix\omega^2} + \omega e^{ix\omega} & e^{ix\omega} + e^{ix\omega^2} + e^{ix} & e^{ix} + \omega^2 e^{ix\omega} + \omega e^{ix\omega^2} \\ e^{ix} + \omega e^{ix\omega^2} + \omega^2 e^{ix\omega} & e^{ix} + \omega^2 e^{ix\omega^2} + \omega e^{ix\omega} & e^{ix\omega} + e^{ix\omega^2} + e^{ix} \end{pmatrix} \\
&= \begin{pmatrix} f_0 & f_2 & f_1 \\ f_1 & f_0 & f_2 \\ f_2 & f_1 & f_0 \end{pmatrix} \qquad (262) \\
&= circ(f_0, f_1, f_2).
\end{aligned}$$

From this decomposition it is easy to show that the matrix $F(x)$ is unitary for $n = 3$ and as a consequence for every $n \in \mathbb{N}$

$$
\begin{aligned}
F(x)\,F^{\dagger}(x) &= \mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\mathcal{F}^{\dagger}\left[\mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\mathcal{F}^{\dagger}\right]^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\mathcal{F}^{\dagger}\left(\mathcal{F}^{\dagger}\right)^{\dagger}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}^{\dagger}\mathcal{F}^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\mathcal{F}^{\dagger}\mathcal{F}\begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & e^{-i\omega^* x} & 0 \\ 0 & 0 & e^{-i\omega^{*2} x}\end{pmatrix}\mathcal{F}^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\mathbf{1}\begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & e^{-i\omega^* x} & 0 \\ 0 & 0 & e^{-i\omega^{*2} x}\end{pmatrix}\mathcal{F}^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & e^{-i\omega^{-1} x} & 0 \\ 0 & 0 & e^{-i\omega^{-2} x}\end{pmatrix}\mathcal{F}^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{i\omega x} & 0 \\ 0 & 0 & e^{i\omega^2 x}\end{pmatrix}\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{-i\omega^2 x} & 0 \\ 0 & 0 & e^{-i\omega x}\end{pmatrix}\mathcal{F}^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} e^{ix-ix} & 0 & 0 \\ 0 & e^{ix\left(\omega-\omega^2\right)} & 0 \\ 0 & 0 & e^{ix\left(\omega^2-\omega\right)}\end{pmatrix}\mathcal{F}^{\dagger} \\
&= \mathcal{F}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\sqrt{3}x} & 0 \\ 0 & 0 & e^{\sqrt{3}x}\end{pmatrix}\mathcal{F}^{\dagger} \neq \mathbf{1}. \qquad (263)
\end{aligned}
$$

This means that $F(x)$ is not unitary because g and h are not Hermitian. That is, $g^{\dagger} \neq g$ and $h^{\dagger} \neq h$. The previous matrix is only unitary for $x = 0$, but we are not interested in a fixed solution. We are interested in finding the quantum channel of a circulant matrix that holds for every x. We have to find its closest unitary to continue into finding the CPTP map.

8.4 The powers of $F(x)$

At this point we will compute the the matrix $F(x)$ in the general power k in order to determine how it affects the properties of the matrix. Later, we are supposed to use the powers of matrix $F(x)$ to compute the quantum channel. That is

$$
F(x) = \mathcal{F}diag\left(f(x), f(\omega x), \ldots, f\left(\omega^{n-1}x\right)\right)\mathcal{F}^{\dagger}, \qquad (264)
$$

$$F^k(x) = \mathcal{F}\left[diag\left(f(x), f(\omega x), \ldots, f\left(\omega^{n-1}x\right)\right)\right]^k \mathcal{F}^{\dagger}. \tag{265}$$

For the 2×2 matrix we get that

$$F^k(x) = \begin{pmatrix} \cos kx & i\sin kx \\ i\sin kx & \cos kx \end{pmatrix}. \tag{266}$$

For the 3×3 matrix we get that

$$\begin{aligned} F^k(x) &= \begin{pmatrix} f_0(kx) & f_2(kx) & f_1(kx) \\ f_1(kx) & f_0(kx) & f_2(kx) \\ f_2(kx) & f_1(kx) & f_0(kx) \end{pmatrix} = \\ &= \frac{1}{3}\begin{pmatrix} e^{ikx}+e^{i\omega kx}+e^{i\omega^2 kx} & e^{ikx}+\omega e^{i\omega kx}+\omega^2 e^{i\omega^2 kx} & e^{ikx}+\omega^2 e^{i\omega kx}+\omega e^{i\omega^2 kx} \\ e^{ikx}+\omega^2 e^{i\omega kx}+\omega e^{i\omega^2 kx} & e^{ikx}+e^{i\omega kx}+e^{i\omega^2 kx} & e^{ikx}+\omega e^{i\omega kx}+\omega^2 e^{i\omega^2 kx} \\ e^{ikx}+\omega e^{i\omega kx}+\omega^2 e^{i\omega^2 kx} & e^{ikx}+\omega^2 e^{i\omega kx}+\omega e^{i\omega^2 kx} & e^{ikx}+e^{i\omega kx}+e^{i\omega^2 kx} \end{pmatrix}. \end{aligned}$$

After carrying out the explicit calculations we obtain that,
for $k=0$

$$\begin{aligned} f_0(0) &= 1, & (267) \\ f_1(0) &= 0, & (268) \\ f_2(0) &= 0, & (269) \end{aligned}$$

and

$$\begin{aligned} f_0^*(0) &= 1, & (270) \\ f_1^*(0) &= 0, & (271) \\ f_2^*(0) &= 0, & (272) \end{aligned}$$

for $k=1$

$$\begin{aligned} f_0(x) &= \frac{1}{3}\left(e^{ix} + 2e^{-\frac{1}{2}ix}\cosh\frac{\sqrt{3}}{2}x\right), & (273) \\ f_1(x) &= \frac{1}{3}\left(e^{ix} - e^{-\frac{1}{2}ix}\left(\cosh\frac{\sqrt{3}}{2}x + i\sqrt{3}\sinh\frac{\sqrt{3}}{2}x\right)\right), & (274) \\ f_2(x) &= \frac{1}{3}\left(e^{ix} - e^{-\frac{1}{2}ix}\left(\cosh\frac{\sqrt{3}}{2}x - i\sqrt{3}\sinh\frac{\sqrt{3}}{2}x\right)\right), & (275) \end{aligned}$$

and

$$f_0^*(x) = \frac{1}{3}\left(e^{-ix} + 2e^{\frac{1}{2}ix}\cosh\frac{\sqrt{3}}{2}x\right), \tag{276}$$

$$f_1^*(x) = \frac{1}{3}\left(e^{-ix} - e^{\frac{1}{2}ix}\left(\cosh\frac{\sqrt{3}}{2}x - i\sqrt{3}\sinh\frac{\sqrt{3}}{2}x\right)\right), \tag{277}$$

$$f_2^*(x) = \frac{1}{3}\left(e^{-ix} - e^{\frac{1}{2}ix}\left(\cosh\frac{\sqrt{3}}{2}x + i\sqrt{3}\sinh\frac{\sqrt{3}}{2}x\right)\right), \tag{278}$$

for $k = 2$

$$f_0(2x) = \frac{1}{3}\left(e^{2ix} + 2e^{-ix}\cosh\sqrt{3}x\right), \tag{279}$$

$$f_1(2x) = \frac{1}{3}\left(e^{2ix} - e^{-ix}\left(\cosh\sqrt{3}x + \sqrt{3}i\sinh\sqrt{3}x\right)\right), \tag{280}$$

$$f_2(2x) = \frac{1}{3}\left(e^{2ix} - e^{-ix}\left(\cosh\sqrt{3}x - \sqrt{3}i\sinh\sqrt{3}x\right)\right), \tag{281}$$

and

$$f_0^*(2x) = \frac{1}{3}\left(e^{-2ix} + 2e^{ix}\cosh\sqrt{3}x\right), \tag{282}$$

$$f_1^*(2x) = \frac{1}{3}\left(e^{-2ix} - e^{ix}\left(\cosh\sqrt{3}x - \sqrt{3}i\sinh\sqrt{3}x\right)\right), \tag{283}$$

$$f_2^*(2x) = \frac{1}{3}\left(e^{-2ix} - e^{ix}\left(\cosh\sqrt{3}x + \sqrt{3}i\sinh\sqrt{3}x\right)\right). \tag{284}$$

Next we will prove the general form of the $n \times n$ $F(x)$ matrix. First of all, we will use the already proven fact that

$$F(x) = e^{ixh} = e^{ix\mathcal{F}g\mathcal{F}^\dagger} = \mathcal{F}e^{ixg}\mathcal{F}^\dagger, \tag{285}$$

where

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}, \tag{286}$$

and where $\mathcal{F}$ is the Fourrier transformation matrix

$$e^{ixg} = \sum_k e^{ix\omega^k} |k\rangle\langle k|. \tag{287}$$

Thus,

$$\begin{aligned} \mathcal{F}e^{ixg}\mathcal{F}^\dagger &= \mathcal{F}\sum_k e^{ix\omega^k}|k\rangle\langle k|\mathcal{F}^\dagger \\ &= \sum_k e^{ix\omega^k}\mathcal{F}|k\rangle\langle k|\mathcal{F}^\dagger, \end{aligned} \tag{288}$$

$$\mathcal{F}\left|k\right\rangle\left\langle k\right|\mathcal{F}^{\dagger}=\sum_{\alpha,\beta'}\omega^{k\left(\alpha-\beta'\right)}\left|\alpha\right\rangle\left\langle\beta'\right|\Rightarrow \tag{289}$$

$$\begin{aligned}F\left(x\right) &= \sum_{k,\alpha,\beta'}e^{ix\omega^{k}}\omega^{k\left(\alpha-\beta'\right)}\left|\alpha\right\rangle\left\langle\beta'\right| \\ &= \sum_{\alpha,\beta'}\left[\sum_{k}e^{ix\omega^{k}}\omega^{k\left(\alpha-\beta'\right)}\right]\left|\alpha\right\rangle\left\langle\beta'\right|.\end{aligned} \tag{290}$$

8.5 Optimal Unitary of the Circulant Matrix $F\left(x\right)$

Here, we are going to find the optimal unitary of matrix $F\left(x\right)$ in order to compute the quantum channel as well as the density matrix that derives, due to the fact that we have shown $F\left(x\right)$ is not unitary. We have shown that $F\left(x\right)=\mathcal{F}e^{ixg}\mathcal{F}^{\dagger}$. At this point we will compute the product

$$\begin{aligned}F\left(x\right)F^{\dagger}\left(x\right) &= \mathcal{F}e^{ixg}\mathcal{F}^{\dagger}\left(\mathcal{F}e^{ixg}\mathcal{F}^{\dagger}\right)^{\dagger} \\ &= \mathcal{F}e^{ixg}\mathcal{F}^{\dagger}\mathcal{F}e^{-ixg^{\dagger}}\mathcal{F}^{\dagger} \\ &= \mathcal{F}e^{ixg}\mathbf{1}e^{-ixg^{\dagger}}\mathcal{F}^{\dagger} \\ &= \mathcal{F}e^{ixg}e^{-ixg^{\dagger}}\mathcal{F}^{\dagger} \\ &= \mathcal{F}e^{ix\left(g-g^{\dagger}\right)}\mathcal{F}^{\dagger}\neq\mathbf{1},\end{aligned} \tag{291}$$

where

$$\begin{aligned}g-g^{\dagger} &= \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^{2}\end{pmatrix}-\begin{pmatrix}1&0&0\\0&\omega^{*}&0\\0&0&\omega^{*2}\end{pmatrix} \\ &= \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^{2}\end{pmatrix}-\begin{pmatrix}1&0&0\\0&\omega^{-1}&0\\0&0&\omega^{-2}\end{pmatrix} \\ &= \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^{2}\end{pmatrix}-\begin{pmatrix}1&0&0\\0&\omega^{2}&0\\0&0&\omega\end{pmatrix} \\ &= \begin{pmatrix}0&0&0\\0&\omega-\omega^{2}&0\\0&0&\omega^{2}-\omega\end{pmatrix}\neq\mathbf{0}.\end{aligned} \tag{292}$$

At this point we will proceed to compute the closest unitary matrix $V\left(x\right)$ of $F\left(x\right)$. For this purpose we are going to follow the process as we have seen before in the proof of the closest unitary matrix in (97).

The closest unitary matrix V of the circulant matrix F is

$$V=\left(FF^{\dagger}\right)^{-\frac{1}{2}}F. \tag{293}$$

If $FF^\dagger = \mathcal{F}e^{ix\left(g-g^\dagger\right)}\mathcal{F}^\dagger$ then, $\left(FF^\dagger\right)^{-\frac{1}{2}} = \mathcal{F}e^{-\frac{1}{2}ix\left(g-g^\dagger\right)}\mathcal{F}^\dagger$. So, next we determine explicitly matrix V,

$$
\begin{aligned}
V &= \mathcal{F}e^{-\frac{1}{2}ix\left(g-g^\dagger\right)}\mathcal{F}^\dagger\mathcal{F}e^{ixg}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{-\frac{1}{2}ix\left(g-g^\dagger\right)}\mathbf{1}e^{ixg}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{-\frac{1}{2}ix\left(g-g^\dagger\right)+ixg}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{-\frac{1}{2}ixg+\frac{1}{2}ixg^\dagger+ixg}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{\frac{1}{2}ixg+\frac{1}{2}ixg^\dagger}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{\frac{1}{2}ix\left(g+g^\dagger\right)}\mathcal{F}^\dagger, \qquad (294)
\end{aligned}
$$

where matrix V is now unitary by construction as the next calculation verifies,

$$
\begin{aligned}
VV^\dagger &= \mathcal{F}e^{\frac{1}{2}ix\left(g+g^\dagger\right)}\mathcal{F}^\dagger\mathcal{F}e^{-\frac{1}{2}ix\left(g+g^\dagger\right)}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{\frac{1}{2}ix\left(g+g^\dagger\right)}\mathbf{1}e^{-\frac{1}{2}ix\left(g+g^\dagger\right)}\mathcal{F}^\dagger \\
&= \mathcal{F}e^{\frac{1}{2}ix\left(g+g^\dagger\right)-\frac{1}{2}ix\left(g+g^\dagger\right)}\mathcal{F}^\dagger \\
&= \mathcal{F}\mathbf{1}\mathcal{F}^\dagger \\
&= \mathcal{F}\mathcal{F}^\dagger \qquad (295) \\
&= \mathbf{1}. \qquad (296)
\end{aligned}
$$

The matrix $g+g^\dagger$ occurring in the definition of V is diagonal and reads

$$
\begin{aligned}
g+g^\dagger &= \sum_{k=0}^{n-1}\omega^k\left|k\right\rangle\left\langle k\right| + \sum_{k=0}^{n-1}\omega^{-k}\left|k\right\rangle\left\langle k\right| \\
&= \sum_{k=0}^{n-1}\left(\omega^k+\omega^{-k}\right)\left|k\right\rangle\left\langle k\right| \\
&= \sum_{k=0}^{n-1}2\,\mathrm{Re}\,\omega^k\left|k\right\rangle\left\langle k\right| \\
&= 2\sum_{k=0}^{n-1}\cos\left(\frac{2\pi}{n}k\right)\left|k\right\rangle\left\langle k\right|. \qquad (297)
\end{aligned}
$$

This leads to the following form of unitary matrix V

$$
\begin{aligned}
V &= \mathcal{F}e^{\frac{1}{2}ix\left(g+g^\dagger\right)}\mathcal{F}^\dagger \\
&= \sum_{k=0}^{n-1}e^{ix\cos\left(\frac{2\pi}{n}k\right)}\mathcal{F}\left|k\right\rangle\left\langle k\right|\mathcal{F}^\dagger \\
&= \sum_{k=0}^{n-1}e^{ix\cos\left(\frac{2\pi}{n}k\right)}\left|k'\right\rangle\left\langle k'\right|, \qquad (298)
\end{aligned}
$$

where $\left|k'\right\rangle = \mathcal{F}\left|k\right\rangle$. Since $g+g^\dagger$ is a diagonal matrix, we can assume that

$$V^k = \mathcal{F}e^{\frac{1}{2}ikx\left(g+g^\dagger\right)}\mathcal{F}^\dagger. \qquad (299)$$

Our next step is address the main target of our work which is to construct new families of unitary channel maps $\mathcal{E} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ of the form

$$\mathcal{E}(\varrho) = \sum_{k=0}^{n-1} p_k V^k(x)\, \varrho V^{k\dagger}(x). \tag{300}$$

For example for $n = 2$ the matrix V should read

$$\begin{aligned} V &= \mathcal{F} e^{\frac{1}{2}ix\left(g+g^{\dagger}\right)} \mathcal{F}^{\dagger} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{ix}+e^{-ix} & e^{ix}-e^{-ix} \\ e^{ix}-e^{-ix} & e^{ix}+e^{-ix} \end{pmatrix} \\ &= \begin{pmatrix} \cos x & \sin x \\ \sin x & \cos x \end{pmatrix}, \end{aligned} \tag{301}$$

and its hermitean conjugate is

$$\begin{aligned} V^{\dagger} &= \mathcal{F} e^{-\frac{1}{2}ix\left(g+g^{\dagger}\right)} \mathcal{F}^{\dagger} \\ &= \frac{1}{2} \begin{pmatrix} e^{ix}+e^{-ix} & e^{-ix}-e^{ix} \\ e^{-ix}-e^{ix} & e^{ix}+e^{-ix} \end{pmatrix} \\ &= \begin{pmatrix} \cos x & -\sin x \\ -\sin x & \cos x \end{pmatrix}, \end{aligned}$$

where $g + g^{\dagger}$ is

$$\begin{aligned} g+g^{\dagger} &= 2\sum_{k=0}^{2} \cos \pi k \left|k\right\rangle \left\langle k\right| \\ &= 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{302}$$

We notice that $F(x)$ for $n = 2$ coincides with its closest unitary because we have seen that $F(x)$ was already unitary. We have already seen similar work when we calculated the one qubit quantum channels where the Kraus operators were unitary matrices X, Y, Z, *Hadamard* and *Depolarizing*, except for the Amplitude-Damping channel. The CPTP map is explicitly

$$\begin{aligned} \mathcal{E}(\varrho) &= \sum_{k=0}^{1} p_k V^k(x)\, \varrho V^{k\dagger}(x) \\ &= p_0 V^0(x)\, \varrho V^{0\dagger}(x) + p_1 V(x)\, \varrho V^{\dagger}(x) \\ &= p_0 \varrho + p_1 \begin{pmatrix} \cos x & \sin x \\ \sin x & \cos x \end{pmatrix} \varrho \begin{pmatrix} \cos x & \sin x \\ \sin x & \cos x \end{pmatrix}, \end{aligned} \tag{303}$$

and it is associated with a bi-stochastic matrix Δ via the element-wise multiplication of its generator matrices as follows,

$$
\begin{aligned}
\Delta_2 &= \sum_{k=0}^{1} p_k V^k(x) \circ V^{k*}(x) \\
&= p_0 \mathbf{1} + p_1 V(x) \circ V^*(x) && (304) \\
&= p_0 \mathbf{1} + p_1 \begin{pmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{pmatrix}. && (305)
\end{aligned}
$$

The above is also a process we have already witnessed when we tried to calculate the bi-stochastic matrix of the one qubit quantum channels (187), (188), (189), (190), (191) and (192).

The similar construction for dimension for $n = 3$ provides the unitary matrix V

$$
\begin{aligned}
V &= \mathcal{F} e^{\frac{1}{2}ix\left(g+g^{\dagger}\right)} \mathcal{F}^{\dagger} \\
&= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{-\frac{1}{2}ix} & 0 \\ 0 & 0 & e^{-\frac{1}{2}ix} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} \\ 1 & \omega^{-2} & \omega^{-1} \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{-\frac{1}{2}ix} & 0 \\ 0 & 0 & e^{-\frac{1}{2}ix} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega^1 \\ 1 & \omega^1 & \omega^2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} e^{ix} + 2e^{-\frac{1}{2}ix} & e^{ix} - e^{-\frac{1}{2}ix} & e^{ix} - e^{-\frac{1}{2}ix} \\ e^{ix} - e^{-\frac{1}{2}ix} & e^{ix} + 2e^{-\frac{1}{2}ix} & e^{ix} - e^{-\frac{1}{2}ix} \\ e^{ix} - e^{-\frac{1}{2}ix} & e^{ix} - e^{-\frac{1}{2}ix} & e^{ix} + 2e^{-\frac{1}{2}ix} \end{pmatrix}, && (306)
\end{aligned}
$$

and its hermitian conjugate is

$$
\begin{aligned}
V^{\dagger} &= \mathcal{F} e^{-\frac{1}{2}ix\left(g+g^{\dagger}\right)} \mathcal{F}^{\dagger} \\
&= \frac{1}{3} \begin{pmatrix} e^{-ix} + 2e^{\frac{1}{2}ix} & e^{-ix} - e^{\frac{1}{2}ix} & e^{-ix} - e^{\frac{1}{2}ix} \\ e^{-ix} - e^{\frac{1}{2}ix} & e^{-ix} + 2e^{\frac{1}{2}ix} & e^{-ix} - e^{\frac{1}{2}ix} \\ e^{-ix} - e^{\frac{1}{2}ix} & e^{-ix} - e^{\frac{1}{2}ix} & e^{-ix} + 2e^{\frac{1}{2}ix} \end{pmatrix}.
\end{aligned}
$$

The matrix V^2 is

$$
\begin{aligned}
V^2 &= \mathcal{F} e^{ix\left(g+g^{\dagger}\right)} \mathcal{F}^{\dagger} \\
&= \frac{1}{3} \begin{pmatrix} e^{2ix} + 2e^{-ix} & e^{2ix} - e^{-ix} & e^{2ix} - e^{-ix} \\ e^{2ix} - e^{-ix} & e^{2ix} + 2e^{-ix} & e^{2ix} - e^{-ix} \\ e^{2ix} - e^{-ix} & e^{2ix} - e^{-ix} & e^{2ix} + 2e^{-ix} \end{pmatrix}, && (307)
\end{aligned}
$$

and its hermitian conjugate

$$
\begin{aligned}
V^{2\dagger} &= \mathcal{F} e^{-ix\left(g+g^{\dagger}\right)} \mathcal{F}^{\dagger} \\
&= \frac{1}{3} \begin{pmatrix} e^{-2ix} + 2e^{ix} & e^{-2ix} - e^{ix} & e^{-2ix} - e^{ix} \\ e^{-2ix} - e^{ix} & e^{-2ix} + 2e^{ix} & e^{-2ix} - e^{ix} \\ e^{-2ix} - e^{ix} & e^{-2ix} - e^{ix} & e^{-2ix} + 2e^{ix} \end{pmatrix},
\end{aligned}
$$

where $g+g^{\dagger}$ is

$$\begin{aligned} g+g^{\dagger} &= 2\sum_{k=0}^{2}\cos\frac{2\pi}{3}k\left|k\right\rangle\left\langle k\right| &(308)\\ &= 2\begin{pmatrix}1 & 0 & 0\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2}\end{pmatrix}. &(309)\end{aligned}$$

Then CPTP map reads

$$\begin{aligned}\mathcal{E}\left(\varrho\right) &= \sum_{k=0}^{2}p_k V^k\left(x\right)\varrho V^{k\dagger}\left(x\right)\\ &= p_0 V^0\left(x\right)\varrho V^{0\dagger}\left(x\right)+p_1 V\left(x\right)\varrho V^{\dagger}\left(x\right)+p_2 V^2\left(x\right)\varrho V^{2\dagger}\left(x\right)\\ &= p_0 V\left(0\right)\varrho V^{\dagger}\left(0\right)+p_1 V\left(x\right)\varrho V^{\dagger}\left(x\right)+p_2 V\left(2x\right)\varrho V^{\dagger}\left(2x\right). \quad (310)\end{aligned}$$

Now that we have shown in (296) that V^k is unitary we can easily understand that using it as a Kraus generator forms a CPTP map. The CPTP map satisfies the properties of a density matrix as we have seen previously in (137),(135) and (138). That is for the map $\mathcal{E}:\mathcal{D}\left(\mathcal{H}\right)\rightarrow\mathcal{D}\left(\mathcal{H}\right)$ it holds that $\mathcal{E}^{\dagger}\left(\varrho\right)=\mathcal{E}\left(\varrho\right)$, $\mathcal{E}\left(\varrho\right)>0$ and $Tr\mathcal{E}\left(\varrho\right)=1$. Also, it is valid that for unitary Kraus generators map $\mathcal{E}$ is unital, i.e. $\mathcal{E}\left(\mathbf{1}\right)=\mathbf{1}$.

Next we deal with the bistochastic matrices associated with the $n=3$ new family of unitary channel map constructed above, and in particular with the associated bistochastic matrix Δ_3. Refering to equations (306,307) we have

$$\begin{aligned}\Delta_3 &= \sum_{k=0}^{2}p_k V^k\left(x\right)\circ V^{k*}\left(x\right)\\ &= p_0 V^0\left(x\right)\circ V^{0*}\left(x\right)+p_1 V\left(x\right)\circ V^{*}\left(x\right)+p_2 V^2\left(x\right)\circ V^{2*}\left(x\right)\\ &= p_0 V\left(0\right)\circ V^{*}\left(0\right)+p_1 V\left(x\right)\circ V^{*}\left(x\right)+p_2 V\left(2x\right)\circ V^{*}\left(2x\right), \quad (311)\end{aligned}$$

which explicitly is expressed as

$$\begin{aligned}\Delta_3 &= p_0\mathbf{1}+\frac{p_1}{9}\begin{pmatrix}5+4\cos\frac{3}{2}x & 2-2\cos\frac{3}{2}x & 2-2\cos\frac{3}{2}x\\ 2-2\cos\frac{3}{2}x & 5+4\cos\frac{3}{2}x & 2-2\cos\frac{3}{2}x\\ 2-2\cos\frac{3}{2}x & 2-2\cos\frac{3}{2}x & 5+4\cos\frac{3}{2}x\end{pmatrix}\\ &\quad+\frac{p_2}{9}\begin{pmatrix}5+4\cos 3x & 2-2\cos 3x & 2-2\cos 3x\\ 2-2\cos 3x & 5+4\cos 3x & 2-2\cos 3x\\ 2-2\cos 3x & 2-2\cos 3x & 5+4\cos 3x\end{pmatrix} &(312)\\ &= (p_0+\frac{p_1}{9}\left(5+4\cos\frac{3}{2}x\right)+\frac{p_2}{9}\left(5+4\cos 3x\right))\mathbf{1}+ &(313)\\ &\quad+(\frac{p_1}{9}(2-2\cos\frac{3}{2}x)+\frac{p_2}{9}(2-2\cos 3x))(h+h^2). &(314)\end{aligned}$$

We have already found the relation between the eigenvalues of the density matrix ρ and of the density matrix $\mathcal{E}(\rho)$ for every unitary matrix as Kraus generators. At this point we will show the same relation but for every Fourier matrix. We assume ρ to be a unitary matrix. If the density matrix ρ is not unitary we have already proven the relation of the eigenvalues in (175).

$$\begin{aligned} (\rho')_{aa} &= \sum_{k=0}^{n-1} \lambda_k \left(V^k \rho V^{k\dagger}\right)_{aa} \\ &= \sum_{k=0}^{n-1} \sum_{m,s=0}^{n-1} \lambda_k \left(V^k\right)_{am} \rho_{ms} \left(V^{k\dagger}\right)_{sa}, \end{aligned} \tag{315}$$

but due to the fact that $\rho_{ms} = \delta_{mn} p_m$, since matrix ρ is taken to be diagonal, i.e. $\rho = diag(p_0, p_1, ..., p_{n-1})$, where p_m are the eigenvalues of ρ, the above turns into

$$\begin{aligned} & \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} \lambda_k \left(V^k\right)_{nm} \left(V^{k\dagger}\right)_{mn} p_m \\ = & \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} \lambda_k \left(V^k\right)_{nm} \left(V^{k*}\right)_{nm} p_m \\ = & \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} \lambda_k \left(V^k \circ V^{k*}\right)_{nm} p_m \\ = & \sum_{m=0}^{n-1} \left[\sum_{k=0}^{n-1} \lambda_k \left(V^k \circ V^{k*}\right)_{nm}\right] p_m \\ = & \sum_{m=0}^{n-1} \left[\sum_{k=0}^{n-1} \lambda_k \Delta^k\right]_{nm} p_m \\ = & \sum_{m=0}^{n-1} \Delta_{nm} p_m. \end{aligned} \tag{316}$$

Next we compute the change of stochastic vector q induced by the action of Δ_3 on it. We denote as $\Delta_3(p_0, p_1, p_2, x)$ the matrix Δ_3 that we found in (312). We choose for p_i the uniform probability distribution $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as well as the non uniform distribution $\left(\frac{1}{2}, \frac{1}{4} . \frac{1}{4}\right)$. Recalling that the action of stochastic matrices on stochastic vectors could have "fixed points" i.e. points that are left invariant under the action of the matrix. For the case of matrix $\Delta_3(p_0, p_1, p_2, x)$ vector $q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ i.e. the uniform 3 dimensional distribution, belongs to the invariant space of Δ_3 for any value of its arguments, i.e.

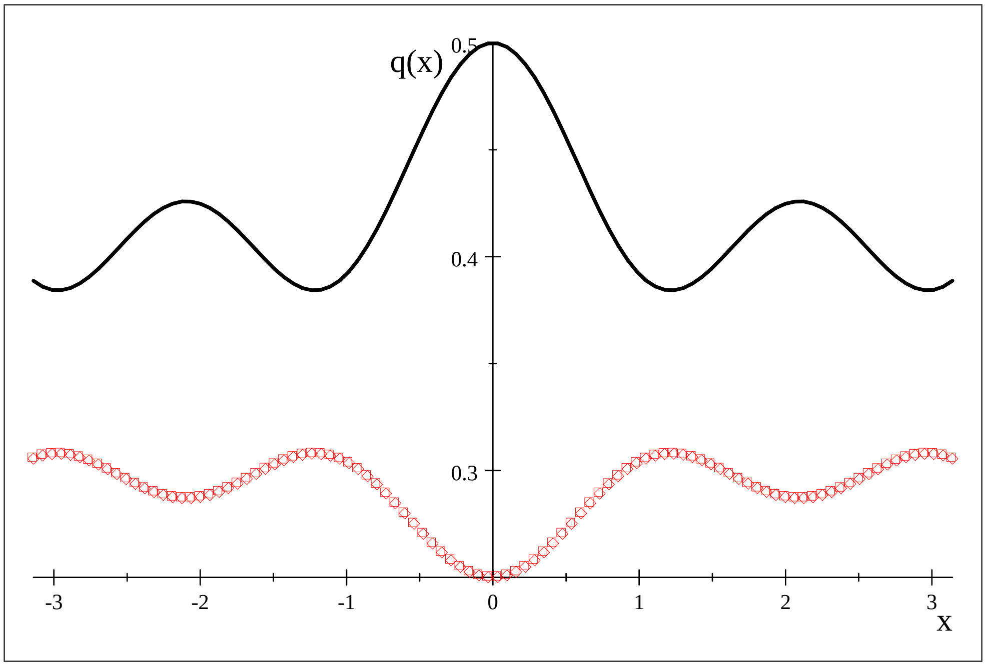
$$\Delta_3(p_0, p_1, p_2, x) \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}. \tag{317}$$

We choose though two other stochastic vectors $q_1 = \left(\frac{1}{2}, \frac{1}{4}.\frac{1}{4}\right)$ and $q_2 = \left(\frac{1}{2}, \frac{1}{7}, 1 - \frac{1}{2} - \frac{1}{7}\right)$ and notice their transformation.

For the combination of uniform probability for Δ_3 and q_1 we get

$$\Delta_3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, x) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{27}\cos 3x + \frac{1}{27}\cos\frac{3}{2}x + \frac{23}{54} \\ \frac{31}{108} - \frac{1}{54}\cos\frac{3}{2}x - \frac{1}{54}\cos 3x \\ \frac{31}{108} - \frac{1}{54}\cos\frac{3}{2}x - \frac{1}{54}\cos 3x \end{pmatrix}$$

where due to $\frac{1}{27}\cos 3x + \frac{1}{27}\cos\frac{3}{2}x + \frac{23}{54} + \frac{31}{108} - \frac{1}{54}\cos\frac{3}{2}x - \frac{1}{54}\cos 3x + \frac{31}{108} - \frac{1}{54}\cos\frac{3}{2}x - \frac{1}{54}\cos 3x = 1$ the resulting vector $q(x) \equiv \Delta_3(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, x)$ is stochastic. Its componentwise variation with parameter x is

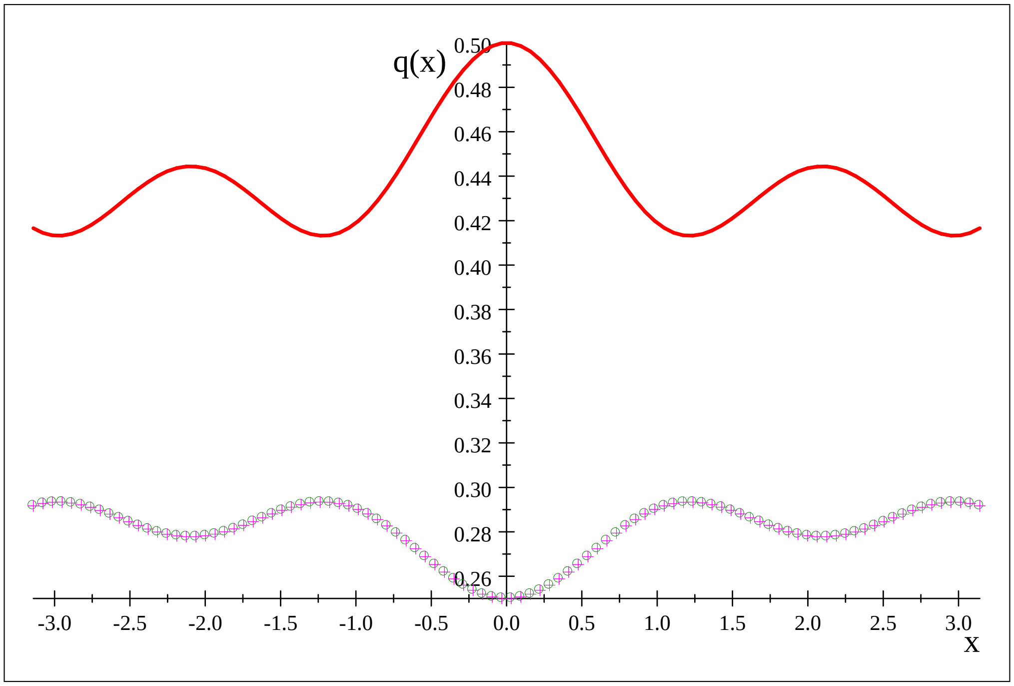


The solid thick line represents the first componet of $q(x)$, the curve with diamond marks $\Diamond$, refers to the second one and that marked with boxes $\Box$ to the third component, the two last curves overlap.

For the combination of $\left(\frac{1}{2}, \frac{1}{4}.\frac{1}{4}\right)$ for Δ_3 and q_1 we obtain

$$\Delta_3(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, x) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{36}\cos 3x + \frac{1}{36}\cos\frac{3}{2}x + \frac{4}{9} \\ \frac{5}{18} - \frac{1}{72}\cos\frac{3}{2}x - \frac{1}{72}\cos 3x \\ \frac{5}{18} - \frac{1}{72}\cos\frac{3}{2}x - \frac{1}{72}\cos 3x \end{pmatrix}$$

again due to $\frac{1}{36}\cos 3x + \frac{1}{36}\cos\frac{3}{2}x + \frac{4}{9} + \frac{5}{18} - \frac{1}{72}\cos\frac{3}{2}x - \frac{1}{72}\cos 3x + \frac{5}{18} - \frac{1}{72}\cos\frac{3}{2}x - \frac{1}{72}\cos 3x = 1$ the resulting vector is stochastic with variation wrt x

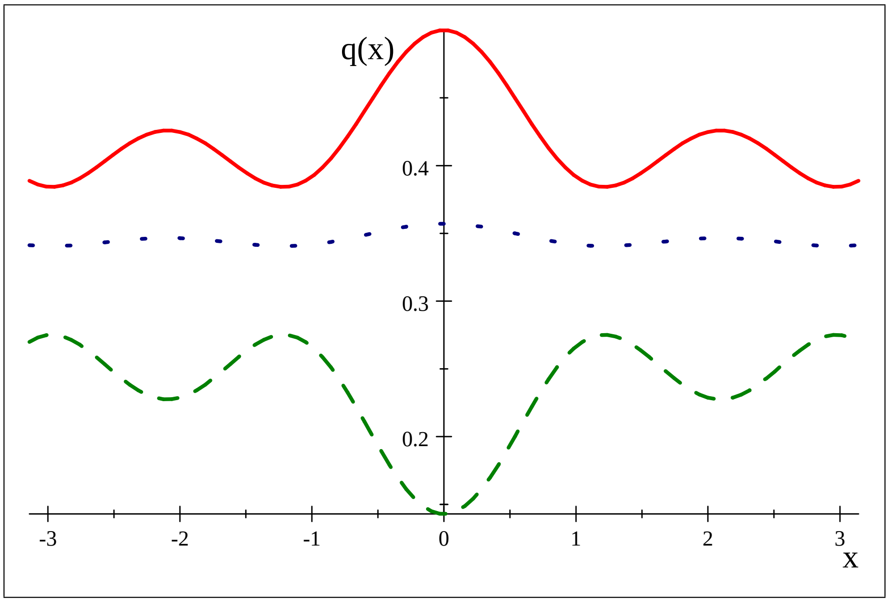


Here, the solid thick line represents the first component, and the overlapping curves with marks ◯ and + the rest two components

The combination of uniform distribution for Δ_3 and q_2 gives

$$\Delta_3(\frac{1}{3},\frac{1}{3},\frac{1}{3},x)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{7} \\ 1-\frac{1}{2}-\frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{27}\cos 3x + \frac{1}{27}\cos\frac{3}{2}x + \frac{23}{54} \\ \frac{43}{189} - \frac{8}{189}\cos\frac{3}{2}x - \frac{8}{189}\cos 3x \\ \frac{1}{189}\cos 3x + \frac{1}{189}\cos\frac{3}{2}x + \frac{131}{378} \end{pmatrix}$$

where relation $\frac{1}{27}\cos 3x+\frac{1}{27}\cos\frac{3}{2}x+\frac{23}{54}+\frac{43}{189}-\frac{8}{189}\cos\frac{3}{2}x-\frac{8}{189}\cos 3x+\frac{1}{189}\cos 3x+\frac{1}{189}\cos\frac{3}{2}x+\frac{131}{378}=1$ verifies the stochastic charater of the new vector, which varies as

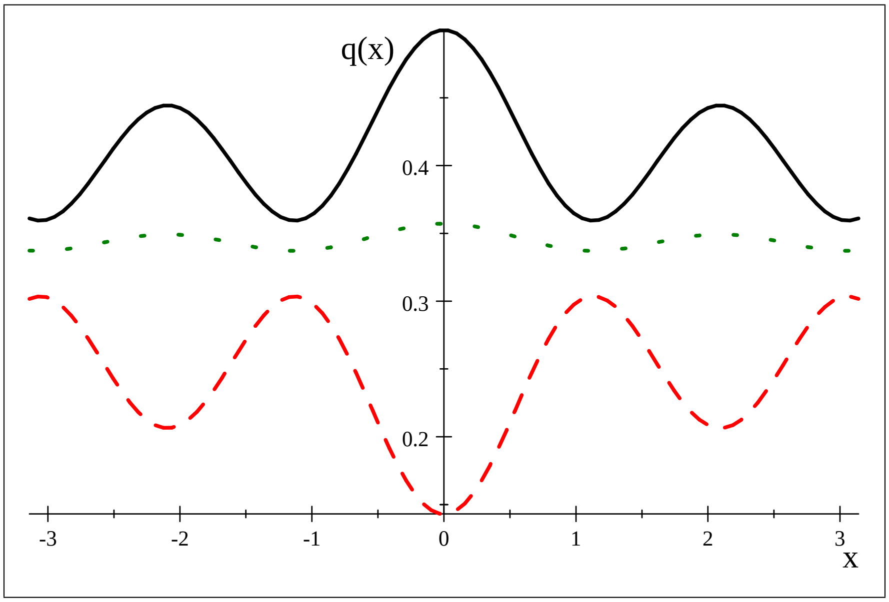


Here the solid line represents the first component and the dash and dot lines the second and third components respectively.

Finally for the combination $\left(\frac{1}{2}, \frac{1}{4}.\frac{1}{4}\right)$ for Δ_3 and q_2 we obtain

$$\Delta_3(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, x) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{7} \\ 1 - \frac{1}{2} - \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{18}\cos 3x + \frac{1}{36}\cos\frac{3}{2}x + \frac{5}{12} \\ \frac{5}{21} - \frac{2}{63}\cos\frac{3}{2}x - \frac{4}{63}\cos 3x \\ \frac{1}{126}\cos 3x + \frac{1}{252}\cos\frac{3}{2}x + \frac{29}{84} \end{pmatrix}$$

where the resulting stochastic vector has three components varying wrt x as the following curves displace from up to down respectively



9 Conclusions

This Thesis has put forward a construction technique for some new families of particular channels of the type of random and optimally unitary channels on finite dimensional Hilbert spaces. The effect of these channels on quantum signals has been given a elementary investigation. Basic algebraic and convex geometric properties of bi-stochastic matrices have been utilized for studying the effects of the new channels on signals. The utility and further development of the theory developed here remains to be addressed in future works in the field of Quantum Information. In outline the construction technique is implemented via the following steps:

- introduction of generalized hyperbolic function decomposition $f=f_0+f_1+...+f_{n-1}$ for a function f(x) of exponential type (V. Riccati 1757);
- assignment to each f(x) of a circulant matrix $F(x)=circ(f_0,f_1,...,f_{n-1})$;
- adoption of simple choice f(x)=exp(ix) for f;
- counterexample: f(x)=exp(ix), n=3, F(x) not unitary for non zero x;
- determination of optimally unitary V close to F wrt trace norm;
- construction of channel map En via family of unitary generators $\{V_k(x)\}$ k=0,...,n;
- construction of family of associated bi-stochastic matrices $\{D_k(x)\}$ via entry-wise matrix product;
- study of stochastic flows (quantum probability) of state matrix in the exemplary case of n=3 channel.

References

[1] R. Aldrovandi, Special Matrices of Mathematical Physics, World Scientific,

[2] Androulidakis Ioannis, *Dimensional Reduction Algorithm of Bipartite Entabgled Quantum Systems,* Diploma Thesis M. Sc., Technical University of Crete, General Department, Section of Mathematic.

[3] Ingemar Bengtsson, The Importance of Being Unistochastic, Stockholm University, Alba Nova, Physic um,

[4] Demosthenes Ellinas, Studies in Lie and Quantum Algebras, Commentationes Physico-Mathiticae, 120/1990, Dissertationes No 39,

[5] Demosthenes Ellinas and Ioannis Smyrnakis, *Quantum Rules and Quantum Simulation of Asymptotics*, Physical Review A 76, 022333, 2007.

[6] Marshall A W and Olkin I 1979, *Inequalities: Theory of Majorization and Its Applications (New York: Academic)*

[7] Aurelian Gheondea, The three equivalent forms of completely positive maps on matrices,

[8] David J. Griffiths and Carl A. Steinke, *Waves in Locally Periodic Media,* American Association of Physics Teachers, 2001.

[9] P.Hoel, S.Port, C. Stone, *Introduction to Probability Theory,* Performance in Greek: Apostolos Giannopoulos, University Press of Crete, Heraklion, 2004.

[10] Jozef Gruska, Quantum Computing, October 11, 2010,

[11] Roger A. Horn and Charles R. Johnson, *Matrix Analysis,* Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1985.

[12] Stephan Hoyer, Classification of Completely Positive Maps,

[13] Phillip Kaye, Raymond Laflamme and Michelle Mosca, *An introduction to Quantum Computing,* Oxford University Press, New York, 2007.

[14] Johannes Kofler, Vlatko Vedral, Myungshik S. Kim and Caslav Brukner, Entanglement between Collective Operators in the Linear Harmonic Chain, February 1, 2008,

[15] Christos Konstandakis, Quantum Search Algorithms in Unstructured Data Bases,

[16] Krontiris Ioannis, Quantum Information Theory.

[17] David McMahon, Quantum Computing Explained,

[18] J.C. Mason and D.C. Handscomb, *Chebyshev Polynomials,* A CRC Press Company, Boca Raton, London, New York, Washington, D.C., 2003.

[19] N. David Mermin, Quantum Computer Science : An Introduction, September 2007

[20] Martin E. Muldoon, Generalized hyperbolic functions, circulant matrices and functional equations,

[21] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information,* 10TH Anniversary Edition, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, Sao Paolo, Delhi, Dubai, Tokyo, Mexico City, 2010.

[22] Riley T. Perry, Temple of Quantum Computing, Version 1.0 - December 19, 2004,

[23] Wojciech Roga, Entropy of quantum channel in the theory of quantum information,

[24] Mary Beth Ruskai, Stanislaw Szarek, Elizabeth Werner, An Analysis of Completely-Positive Trace Preserving Maps on M_2,

[25] J. Schwaiger, Remark 15, Report on 28th International Symposium on Functional Equations, Aequations Math. 41 (1991) 289-290,

[26] http://en.wikipedia.org/wiki/Generalizations_of_Pauli_matrices

[27] Willi-Hans Steeb and Yorick Hardy, *Problems and Solutions in Quantum Computing and Quantum Information,* Rand Afrikaans University, South Africa, World Scientific, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai, 2004.

[28] George Moutzianou, Quantum Walk on Integers and Maximun Likelihood Parametric Estimation, Chania 2013