Technical University of Crete
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Ebits at a Distance: Quantum Algorithms and Circuits for Teleportation of Quantum Resources

Ebits εξ αποστάσεως : Κβαντικοί αλγόριθμοι και κυκλώματα τηλεμεταφοράς κβαντικών πόρων

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Abstract

An ebit is a pair of quantum mechanically entangled qubits i.e. locally un-factorizable state vectors of bipartite quantum states. The ebits constitute building blocks of the theoretical analyses and the technological tasks which are put forward in the area of quantum computation and information. Functioning as computational and communicational resource ebits are used in various quantum algorithms, most notably the state teleportation and quantum gate teleportation algorithms. This work addresses the problem of creating ebits at a distance by various low complexity elementary teleportation algorithms. The construction of quantum circuits follows the LOCC-SE protocol, according to which the allowed state transformations are restricted to local operations (LO), classical communication (CC) and shared quantum entanglement (SE). Explicitly the work investigates two types of quantum resource teleportation: in the first type single qubit state and gate teleportation is achieved, while in the second one the achievement concerns the state and gate teleportation of ebits, namely non-local quantum resources. The work investigates a unifying formalism for the respective quantum algorithms and provides relevant analytic proofs, quantum circuits and quantification of the classical and quantum resources required for their implementation.

Περίληψη

Τα εναγκαλισμένα qubit (ebit) είναι ζεύγη συζευγμένων qubit, δηλαδή τοπικά μη παραγοντοποιημένων διανυσμάτων κατάστασης διμερών κβαντικών συστημάτων. Τα ebit αποτελούν δομικά στοιχεία της θεωρίας και των τεχνολογικών εφαρμογών στην επιστήμη της Κβαντικής Πληροφορίας και Υπολογισμού. Λειτουργώντας ως υπολογιστικοί και επικοινωνιακοί πόροι τα ebit χρησιμοποιούνται σε διάφορους κβαντικούς αλγορίθμους και ειδικότερα σε αλγόριθμους τηλεμεταφοράς κβαντικών καταστάσεων και κβαντικών πυλών. Η παρούσα εργασία αντιμετωπίζει το πρόβλημα δημιουργίας εναγκαλισμένων qubit εξ αποστάσεως με χρήση γενικευμένων κυκλωμάτων τηλεμεταφοράς χαμηλής υπολογιστικής πολυπλοκότητας. Η κατασκευή των κυκλωμάτων ακολουθεί το πρωτόκολλο LOCC-SE, κατά το οποίο οι επιτρεπτοί μετασχηματισμοί των καταστασιακών διανυσμάτων περιορίζονται στις λεγόμενες τοπικές δράσεις (local operation), την ανταλλαγή κλασσικής πληροφορίας (classical communication), και την χρήση διαμοιρασμένου εναγκαλισμού (shared entanglement). Συγκεκριμένα η εργασία διερευνά δύο είδη αλγορίθμων τηλεμεταφοράς πόρων: το πρώτο είδος επιτυγχάνει την τηλεμεταφορά καταστάσεων και πυλών ενός qubit, και το δεύτερο επιτυγχάνει την τηλεμεταφορά καταστάσεων και πυλών για ebit δηλ. απεντοπισμένων κβαντικών πόρων. Η εργασία επεξεργάζεται ένα ενοποιημένο φορμαλισμό για τους αντίστοιχους αλγόριθμους και παρέχει σχετικές αναλυτικές αποδείξεις, κβαντικά κυκλώματα και ποσοτικοποίηση των απαιτούμενων κλασσικών και κβαντικών πόρων για την επίτευξή τους.

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Contents

1 Introduction

In this thesis, we try to take advantage of Entanglement of two quantum particles which is a weird phenomenon in quantum mechanics to teleport various kinds of quantum resources depending on Quantum Teleportation protocol which first introduced by Charles H. Bennett et. al in 1993(c.f [3]).Specifically we investigate low-complexity algorithms and circuits: for teleporting one arbitrary quantum state reformulating the one proposed by Bennett in a more compact form. Also we explain how we can succeed remotely computation, teleporting the action of One and Two Qubits gates using some pairs of prepared and shared entangled particles. Furthermore, we describe how we can remotely send one and two Qubits with purpose the receiver to reconstruct the gates for his own reasons. Last but not least we develop a protocol with which we can teleport one arbitrary pair of entangled qubits. All of the above protocols are examined from the perspective of resources that required to achieved and all of them belong to LOCC-SE protocols category which stands for Local Operator Classical Communication and Shared-Entanglement (c.f. [4]).

1.1 Qubit

In classical information the basic unit to represent information is bit which is a discrete value "0" or "1" $\in \mathbb{N}$ which is commonly the level of voltage ("0"=0Volts "1"=5Volts in TTL Logic). In quantum information things are a little different the basic information is not bit anymore but qubit. Which takes values not in the discrete $\mathbb{N}$ but in the continuous C^2 which in quantum mathematics is called Hilbert space. The value of a qubit is not the level of voltage but is represented by many physical ways, commonly Photons ("0"=Horizontal Polarization "1"=Vertical Polarization), Electrons ("0"=Spin Up "1"=Spin Down). The fact that qubit is not one discrete value but two complex numbers means that it is not only in one state but simultaneously in both states. This is a powerful advantage of quantum information which is based on the principles of quantum mechanics called superposition. In order to represent qubit in a mathematical form we will introduce the Dirac notation which is nothing more than a compact way to represent $\mathbb{C}^2$[2].

In Dirac notation (named for English physicist Paul Dirac) the symbol $|\rangle$ is called "ket" and it is a column vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and the symbol $\langle|$ is called "bra" and it is a row vector $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$.

There is an equality between them $\langle| = |\rangle^\dagger$ we refer to it is at "Hilbert Duality" where $\dagger$: is the conjucate and transpose.

Using Dirac notation we can represent a qubit

$$|\psi\rangle = a|0\rangle + \beta|1\rangle$$

where $|0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$, and $|1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$ they are called computational basis states and is one of the infinite basis that exist and we are going to use mainly. α, β are called probability amplitudes of the qubit. So the qubit can represented as a column vector

$$|\psi\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ \beta \end{pmatrix}.$$

The quantity $|a|^2$ declares the possibility our qubit is in the computational base $|0\rangle$ and the quantity $|\beta|^2$ declares the possibility our qubit is in the computational base $|1\rangle$ respectively.

Because of the fact that they represent possibilities we know that

$$0 \leq |a|^2, |\beta|^2 \geq 1 \ and \ |a|^2 + |\beta|^2 = 1.$$

The Axiom 1 of quantum information claims: Every quantum state is described by a normalized vector.

Which means that for every $|\psi\rangle$ represents a quantum state

$$\||\psi\rangle\| = \sqrt{\langle\psi||\psi\rangle} = 1.$$

Actually

$$\langle\psi||\psi\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} a \\ \beta \end{pmatrix} = a\alpha^* + \beta\beta^* = |a|^2 + |\beta|^2 = 1$$

In general every quantum state $|\psi\rangle$ is in the form:

$$|\psi\rangle = \sum_{i=1}^{\infty} c_i |u_i\rangle$$

where c_i are the probability amplitudes of the quantum state $|u_i\rangle$. $\sum_{i=1}^{\infty} |c_i|^2 = 1$ and $|u_i\rangle$ are the computational basis states.

Also a more Geometrical (spatial) representation of a qubit state is Bloch Sphere named after the physicist Felix Bloch as shown below:

1.2 Single Qubit Gate

As in classical computation we use gates made of transistors to manipulate bits so in quantum computation we use quantum gates to manipulate qubits. Unlike many classical logic gates, quantum logic gates are reversible.

In quantum computations, gates are unitary matrices ($A^\dagger A = AA^\dagger = I$) so that they respect the normality $\langle\psi||\psi\rangle = 1$ of a quantum state (we know that $|A| = |\det(A)| = 1$ for Unitary matrices). If we want to perform a gate on a

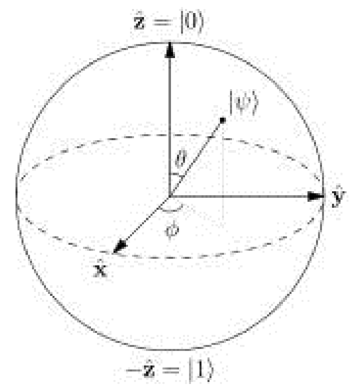


Figure 1: Gemetrical representation of one qubit on a Bloch Sphere

quantum state we only have to multiply the state with the matrix representation of our gate.

$$|\psi'\rangle = U|\psi\rangle$$

where $|\psi'\rangle$ is our new quantum state after the effect of the U gate on our quantum state $|\psi\rangle$.

Some common quantum gates are the Pauli matrices after the physicist Wolfgang Pauli.

$$\sigma_1 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Interestingly properties of these gates is that they all have $\det \sigma_i = -1$ so $|\det(\sigma_i)| = 1$, they are unitarity $\sigma_i \sigma_i^\dagger = I, Tr\sigma_i = 0, \sigma_i^2 = I$ their eigenvalues are: ± 1 and their normalized eigenvectors are spanning the Hilbert space and commonly used as computational basis for quantum states

Matrix	Eigenvectors
$X = \sigma_1$	$\|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$Y = \sigma_2$	$\|\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \|\downarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$
$Z = \sigma_3$	$\|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For the needs of this thesis we will use only the $\{|0\rangle, |1\rangle\}$ basis (Eigenvectors of σ_3) which is called canonical base.

X is the quantum analogue of classical "NOT" gate the action leads to reverse of the quantum state.

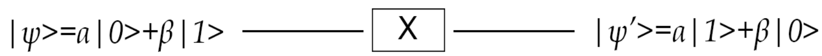


Figure 2: Action of X (NOT) gate on an arbitrary qubit

Before Action:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} a \\ \beta \end{pmatrix}$$

After Action

$$|\psi'\rangle = X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ a \end{pmatrix}$$

As we can observe $|\psi'\rangle = \alpha|1\rangle + \beta|0\rangle$ which is the reverse of the state before the action of the gate.

One other very important one qubit gate is Hadamard named for the French mathematician Jacques Hadamard:

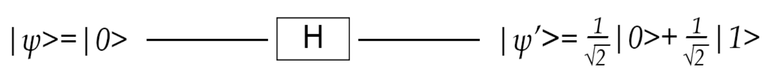


Figure 3: Action of H (Hadamard) gate on an qubit initialized at |0>

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The special property of this gate is that it transforms an initial base state to a maximaly superposition state (probability state $|0\rangle = |1\rangle = \frac{1}{2}$)

Suppose that our qubit is in the base state

$$|\psi\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

after the action of Hadamard gate our qubit is in the state:

$$|\psi'\rangle = H|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

This means that our qubit is in the state $|0\rangle$ with probability $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and in the state $|1\rangle$ with probability $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

Similar if our qubit was in the base state $|\psi\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ after the action of Hadamard gate our qubit is in the state

$$|\psi'\rangle = H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

This means that our qubit is in the state $|0\rangle$ with probability $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ and in the state $|1\rangle$ with probability $|-\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

Many quantum algorithms in our cases,will use the Hadamard transform as an initial step.

1.2.1 DIY one qubit unitary gate

In this subsection we are going to see how we can construct our own one qubit gate. The only thing we need to know is the outcome of the action on the computational basis vectors $|0\rangle, |1\rangle$ we call them $|u_0\rangle = U|0\rangle$ and $|u_1\rangle = U|1\rangle$

If we want to construct an one qubit Gate (Operator) U then the matrix representation of our Gate is

$$\begin{pmatrix} \langle 0||u_0\rangle & \langle 0||u_1\rangle \\ \langle 1||u_0\rangle & \langle 1||u_1\rangle \end{pmatrix}$$

Calculating these inner products we find the four elements (complex scalars) which compose the matrix representation of our one-qubit gate

From the definition of $|u_0\rangle$ and $|u_1\rangle$ one more abstract form of our matrix is below:

$$\begin{pmatrix} \langle 0|\underbrace{U|0\rangle} & \langle 0|\underbrace{U|1\rangle} \\ \langle 1|\underbrace{U|0\rangle} & \langle 1|\underbrace{U|1\rangle} \end{pmatrix}$$

<u>Remember</u> that our quantum Gate U <u>must</u> always be Unitary ($UU^\dagger = I$) to

respect the normality of the quantum state:

$$|\psi'\rangle = \sqrt{\langle\psi'||\psi'\rangle} = \sqrt{\langle\psi|U^\dagger U|\psi\rangle}$$

It must be

$$UU^\dagger = I$$

So that

$$|\psi'\rangle = \sqrt{\langle\psi||\psi\rangle} = 1$$

<u>Example:</u>

Suppose that we want construct the quantum NOT gate matrix represtation. We know that the action of U_{NOT} on the computational basis vectors $|0\rangle, |1\rangle$ is to reverse them.

Meaning $|u_0\rangle = U_{NOT}|0\rangle = |1\rangle$, $|u_1\rangle = U_{NOT}|1\rangle = |0\rangle$ so the matrix representation

$$\begin{pmatrix} \langle 0||u_0\rangle & \langle 0||u_1\rangle \\ \langle 1||u_0\rangle & \langle 1||u_1\rangle \end{pmatrix}$$

can be rewritten replacing $|u_i\rangle$

$$\begin{pmatrix} \langle 0||1\rangle & \langle 0||0\rangle \\ \langle 1||1\rangle & \langle 1||0\rangle \end{pmatrix}$$

Due to to orthonormality of the canonical base $\langle i||j\rangle = \delta_{ij}$ the final matrix representation of our gate is:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Known as quantum NOT gate.

1.3 Multiple Qubits

Like classical computation one system (PC) is not manipulating only one bit because information that one bit can carry is not enough so in quantum computation (Quantum computers) must handle more than one qubit. The mathematical tool that we use to express multiple qubits state is called tensor product and it is denoted " $\otimes$ ".

If we have two qubits $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, |\phi\rangle = \gamma|0\rangle + \delta|1\rangle$ which are the "components" of a two qubit quantum state then the total state is

$$\begin{aligned} |w\rangle &= |\psi\rangle \otimes |\phi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) \\ &= \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle \\ &= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \end{aligned}$$

For abbreviation we will denote $|i\rangle \otimes |j\rangle = |ij\rangle$

In vector formalism

$$|w\rangle = |\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

As we can observe a bipartite quantum state (2-qubits state) exists in the C^4 set.

It is also valid that the tensor product is not commutative

$$|\psi\rangle \otimes |\phi\rangle \neq |\phi\rangle \otimes |\psi\rangle$$

Proof

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \\ |\psi\rangle \otimes |\phi\rangle &= \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma\alpha \\ \gamma\beta \\ \delta\alpha \\ \delta\beta \end{pmatrix} \end{aligned}$$

To represent this vector a quantum state it must respect the axiom 1 of quantum mechanics and be normalized i.e

$$\langle w||w\rangle = 1$$

Proof

$$\begin{aligned} \langle w||w\rangle &= \begin{pmatrix} \alpha^*\gamma^* & \alpha^*\delta^* & \beta^*\gamma^* & \beta^*\delta^* \end{pmatrix} \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \\ \langle w||w\rangle &= \alpha\gamma\alpha^*\gamma^* + \alpha\delta\alpha^*\delta^* + \beta\gamma\beta^*\gamma^* + \beta\delta\beta^*\delta^* \\ \langle w||w\rangle &= |\alpha\gamma|^2 + |\alpha\delta|^2 + |\beta\gamma|^2 + |\beta\delta|^2 = 1 \end{aligned}$$

Which applies because we know that $\{|\alpha\gamma|^2, |\alpha\delta|^2, |\beta\gamma|^2, |\beta\delta|^2\}$ are the probabilities finding state $|w\rangle$ along the basis vectors $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ respectively.

Using this mathematical tool-tensor product we can composite N-Qubit quantum states but for the needs of this thesis we will use only bipartite quantum systems.

1.4 Two Qubit Gates

Similarly with classical computation we need gates to manipulate two qubits. The different between them is that the quantum gates are reversible it means that if we know the outputs of the gate we can find the input qubits applying the inverse ($U^{-1} = U^{\dagger}$) gate on the output qubits.

Also unlike with classical gates the number of inputs is equal to the number of outputs something that does not apply in classical gates (for example AND gate has two inputs and one output).

The gates act on bipartite C^4 quantum states must be represented as $4x4$ unitary (to repsect the normality of the quantum state) matrices.

Some common two qubits quantum gates are:

Control Not (CNOT)

It is the most used quantum gate which checks if the first qubit is in state $|1\rangle$ (control qubit) then it flips the second one (target qubit).

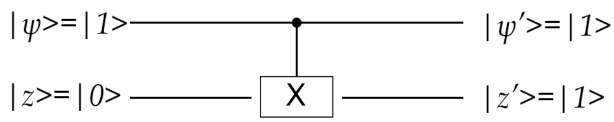


Figure 4: Action of CNOT gate on two qubits initialized at |10>

Truth table:

$$U_{CN}|x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle (\oplus : Exclusiveorsymbol)$$

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Matrix Representation of CNOT Operator:

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Swap

It swaps the two qubits.

Truth table:

$$U_{Swap}|x\rangle \otimes |y\rangle = |y\rangle \otimes |x\rangle$$

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|10\rangle$
$\|10\rangle$	$\|01\rangle$
$\|11\rangle$	$\|11\rangle$

Matrix Representation of SWAP Operator:

$$U_{Swap} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Control U (CU)

This is a general set of two qubits gates which checks if the first qubit is in state $|1\rangle$ then it acts with the U gate on the second one. CNOT belongs in this category for which $U = X$.

Truth table:

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|1\rangle \otimes U\|0\rangle$
$\|11\rangle$	$\|1\rangle \otimes U\|1\rangle$

Matrix Representation of CU Operator:

$$U_{CU} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix}$$

Example: Lets confirm the action of CNOT gate on the four possible inputs

$$\begin{aligned} U_{CNOT}|00\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle, \\ U_{CNOT}|01\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle \end{aligned}$$

$$
\begin{aligned}
U_{CNOT}|10\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle, \\
U_{CNOT}|11\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle
\end{aligned}
$$

Something that confirms the truth table of the gate

1.4.1 DIY two qubit unitary gate

In this subsection we are going to see how we can construct our own two qubits gate. The only thing we need to know is the outcome of the action on the computational basis vectors $|00\rangle, |01\rangle$. $|10\rangle, |11\rangle$. we call them $|u_{00}\rangle = U|00\rangle$, $|u_{01}\rangle = U|01\rangle$,$|u_{10}\rangle = U|10\rangle$, $|u_{11}\rangle = U|11\rangle$

If we want to construct a two qubit gate (Operator) U then the matrix representation of our gate is

$$
\begin{pmatrix}
\langle 00||u_{00}\rangle & \langle 00||u_{01}\rangle & \langle 00||u_{10}\rangle & \langle 00||u_{11}\rangle \\
\langle 01||u_{00}\rangle & \langle 01||u_{01}\rangle & \langle 01||u_{10}\rangle & \langle 01||u_{11}\rangle \\
\langle 10||u_{00}\rangle & \langle 10||u_{01}\rangle & \langle 10||u_{10}\rangle & \langle 10||u_{11}\rangle \\
\langle 11||u_{00}\rangle & \langle 11||u_{01}\rangle & \langle 11||u_{10}\rangle & \langle 11||u_{11}\rangle
\end{pmatrix}
$$

Calculating these inner products we find the sixteen elements (complex scalars) which compose the matrix representation of our two-qubit gate.

From the definition of $|u_{00}\rangle$,$|u_{01}\rangle$,$|u_{10}\rangle$,$|u_{11}\rangle$ one more abstract form of our matrix is below

$$
\begin{pmatrix}
\langle 00|\underbrace{U|00\rangle} & \langle 00|\underbrace{U|01\rangle} & \langle 00|\underbrace{U|10\rangle} & \langle 00|\underbrace{U|11\rangle} \\
\langle 01|\underbrace{U|00\rangle} & \langle 01|\underbrace{U|01\rangle} & \langle 01|\underbrace{U|10\rangle} & \langle 01|\underbrace{U|11\rangle} \\
\langle 10|\underbrace{U|00\rangle} & \langle 10|\underbrace{U|01\rangle} & \langle 10|\underbrace{U|10\rangle} & \langle 10|\underbrace{U|11\rangle} \\
\langle 11|\underbrace{U|00\rangle} & \langle 11|\underbrace{U|01\rangle} & \langle 11|\underbrace{U|10\rangle} & \langle 11|\underbrace{U|11\rangle}
\end{pmatrix}
$$

Remember that our quantum Gate U must always be Unitary ($UU^\dagger = I$) to respect the normality of the quantum state

Example:

Suppose that we want construct the quantum CNOT gate matrix represtation. We know that the action of U_{CNOT} on the computational basis vectors

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$ is to check if the first qubit is in state $|1\rangle$ then reverse the second.

Meaning $|u_{00}\rangle = U_{CNOT}|00\rangle = |00\rangle$, $|u_{01}\rangle = U_{CNOT}|01\rangle = |01\rangle, |u_{10}\rangle = U_{CNOT}|10\rangle = |11\rangle, |u_{11}\rangle = U_{CNOT}|11\rangle = |10\rangle$ so the matrix representation of our two qubit gate is:

$$\begin{pmatrix} \langle 00||u_{00}\rangle & \langle 00||u_{01}\rangle & \langle 00||u_{10}\rangle & \langle 00||u_{11}\rangle \\ \langle 01||u_{00}\rangle & \langle 01||u_{01}\rangle & \langle 01||u_{10}\rangle & \langle 01||u_{11}\rangle \\ \langle 10||u_{00}\rangle & \langle 10||u_{01}\rangle & \langle 10||u_{10}\rangle & \langle 10||u_{11}\rangle \\ \langle 11||u_{00}\rangle & \langle 11||u_{01}\rangle & \langle 11||u_{10}\rangle & \langle 11||u_{11}\rangle \end{pmatrix}$$

can be rewritten replacing $|u_i\rangle$

$$\begin{pmatrix} \langle 00||00\rangle & \langle 00||01\rangle & \langle 00||11\rangle & \langle 00||10\rangle \\ \langle 01||00\rangle & \langle 01||01\rangle & \langle 01||11\rangle & \langle 01||10\rangle \\ \langle 10||00\rangle & \langle 10||01\rangle & \langle 10||11\rangle & \langle 10||10\rangle \\ \langle 11||00\rangle & \langle 11||01\rangle & \langle 11||11\rangle & \langle 11||10\rangle \end{pmatrix}$$

Due to to orthonormality of the canonical basis $\langle xy||zw\rangle = \delta_{xz,yw}$ $\{x, y, z, w \in [0, 1]\}$ the final matrix representation of our gate is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Known as quantum CNOT gate.

1.5 "Double-Wedge" Notation

For the needs of this thesis we introduce a new notation called "double wedge" [7] it is a way to represent bipartite quatum systems in a form of a 2x2 matrices.

Lets suppose we have a two qubit quantum state

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} \text{ so that } |a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$$

then the double-wedge "ket" notation of this genaral bipartite quantum state is

$$|A\rangle\rangle = |\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\rangle\rangle$$

similar to vector representation the "bra" notation is

$$\langle\langle A| = (|A\rangle\rangle)^\dagger = \langle\langle\begin{pmatrix} a_{00}^* & a_{10}^* \\ a_{01}^* & a_{11}^* \end{pmatrix}|$$

We know that if $|A\rangle\rangle$ represents a quantum state it must be normalized which means $\langle\langle A||A\rangle\rangle = 1$ or $|||A\rangle\rangle|| = 1$

It is valid that:

$$\langle\langle A||B\rangle\rangle = Tr[A^\dagger B]$$

Proof:

$$\langle\langle A| = a_{00}^*\langle 00| + a_{01}^*\langle 01| + a_{10}^*\langle 10| + a_{11}^*\langle 11|$$

$$|B\rangle\rangle = |\beta_{00}|00\rangle + \beta_{01}|01\rangle + \beta_{10}|10\rangle + \beta_{11}|11\rangle$$

Calculating the product

$$\langle\langle A||B\rangle\rangle = (a_{00}^*\langle 00| + a_{01}^*\langle 01| + a_{10}^*\langle 10| + a_{11}^*\langle 11|)(\beta_{00}|00\rangle + \beta_{01}|01\rangle + \beta_{10}|10\rangle + \beta_{11}|11\rangle)$$

$$\begin{aligned} = \quad & a_{00}^*\beta_{00}\langle 00||00\rangle + a_{00}^*\beta_{01}\langle 00||01\rangle + a_{00}^*\beta_{10}\langle 00||10\rangle + a_{00}^*\beta_{11}\langle 00||11\rangle \\ & + a_{01}^*\beta_{00}\langle 01||00\rangle + a_{01}^*\beta_{01}\langle 01||01\rangle + a_{01}^*\beta_{10}\langle 01||10\rangle + a_{01}^*\beta_{11}\langle 01||11\rangle \\ & + a_{10}^*\beta_{00}\langle 10||00\rangle + a_{10}^*\beta_{01}\langle 10||01\rangle + a_{10}^*\beta_{10}\langle 10||10\rangle + a_{10}^*\beta_{11}\langle 10||11\rangle \\ & + a_{11}^*\beta_{00}\langle 11||00\rangle + a_{11}^*\beta_{01}\langle 11||01\rangle + a_{11}^*\beta_{10}\langle 11||10\rangle + a_{11}^*\beta_{11}\langle 11||11\rangle \end{aligned}$$

due orthonormality $\langle xy||x'y'\rangle = \delta_{xx',yy'}$ leads to

$$\langle\langle A||B\rangle\rangle = a_{00}^*\beta_{00} + a_{01}^*\beta_{01} + a_{10}^*\beta_{10} + a_{11}^*\beta_{11}$$

Next, we compute:

$$A^\dagger = \begin{pmatrix} a_{00}^* & a_{10}^* \\ a_{01}^* & a_{11}^* \end{pmatrix}, B = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix}$$

and

$$A^\dagger B = \begin{pmatrix} a_{00}^* & a_{10}^* \\ a_{01}^* & a_{11}^* \end{pmatrix}\begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} = \begin{pmatrix} a_{00}^*\beta_{00} + a_{10}^*\beta_{10} & a_{00}^*\beta_{01} + a_{10}^*\beta_{11} \\ a_{01}^*\beta_{00} + a_{11}^*\beta_{10} & a_{01}^*\beta_{01} + a_{11}^*\beta_{11} \end{pmatrix}$$

Taking the trace of this matrix:

$$Tr[A^\dagger B] = a_{00}^*\beta_{00} + a_{10}^*\beta_{10} + a_{01}^*\beta_{01} + a_{11}^*\beta_{11}$$

So

$$\langle\langle A||B\rangle\rangle = Tr[A^\dagger B]$$

The way of representing bipartite states via double-wedge notation is also very handy when we want to perform gates on the quantum state either one-qubit gates (X,Y,Z) or one two-qubit gate (SWAP,CNOT) due to the identity:

$$A\otimes B|C\rangle\rangle = |ACB^T\rangle\rangle$$

Proof:

$$|C\rangle\rangle = \sum_{ij} c_{ij}|i\rangle\otimes|j\rangle$$

$$A = \sum_{kl} A_{kl}|k\rangle\langle l|$$

$$B = \sum_{mn} A_{mn}|m\rangle\langle n|$$

$$\begin{aligned}
A\otimes B|C\rangle\rangle &= (A\otimes B)\sum_{ij} c_{ij}|i\rangle\otimes|j\rangle \\
A\otimes B|C\rangle\rangle &= \sum_{ij} c_{ij}A|i\rangle\otimes B|j\rangle \\
A\otimes B|C\rangle\rangle &= \sum_{ij} c_{ij}\left(\sum_{kl} A_{kl}|k\rangle\underbrace{\langle l||i\rangle}\right)\otimes\left(\sum_{mn} B_{mn}|m\rangle\underbrace{\langle n||j\rangle}\right)
\end{aligned}$$

due to orthonormality $\langle x||y\rangle = \delta_{xy}$ We have that $\langle l||i\rangle = \delta_{li}$ and $\langle n||j\rangle = \delta_{nj}$

$$A\otimes B|C\rangle\rangle = \sum_{ij} c_{ij}\left(\sum_{k} A_{ki}|k\rangle\right)\otimes\left(\sum_{m} B_{mj}|m\rangle\right)$$

$$\begin{aligned}
A\otimes B|C\rangle\rangle &= \sum_{ijkm} A_{ki}c_{ij}B_{mj}|k\rangle\otimes|m\rangle \\
A\otimes B|C\rangle\rangle &= \sum_{jkm}(AC)_{kj}B_{mj}|k\rangle\otimes|m\rangle
\end{aligned}$$

$$\begin{aligned}
A \otimes B|C\rangle\rangle &= \sum_{jkm} (AC)_{kj} (B_{jm})^T |k\rangle \otimes |m\rangle \\
A \otimes B|C\rangle\rangle &= \sum_{km} (ACB^T)_{km} |k\rangle \otimes |m\rangle \\
A \otimes B|C\rangle\rangle &= |ACB^T\rangle\rangle
\end{aligned}$$

Examples:

Suppose that we have a bipartite system

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

In double wedge notation

$$|A\rangle\rangle = |\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\rangle\rangle$$

If we act with X gate on the first qubit and with Z gate on the second one the unitary matrix U representation of this gate is

$$U = X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

The quantum state after the action of U using dirac notation

$$\begin{aligned}
U|\psi\rangle &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} = \begin{pmatrix} a_{10} \\ -a_{11} \\ a_{00} \\ -a_{01} \end{pmatrix} \\
&= a_{10}|00\rangle - a_{11}|01\rangle + a_{00}|10\rangle - a_{01}|11\rangle
\end{aligned}$$

The quantum state after the action of U using double wedge notation and the identity $A \otimes B|C\rangle\rangle = |ACB^T\rangle\rangle$

$$\begin{aligned}
X \otimes Z|A\rangle\rangle &= |XAZ^T\rangle\rangle = |\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\rangle\rangle \\
X \otimes Z|A\rangle\rangle &= |\begin{pmatrix} a_{10} & -a_{11} \\ a_{00} & -a_{01} \end{pmatrix}\rangle\rangle \\
X \otimes Z|A\rangle\rangle &= a_{10}|00\rangle - a_{11}|01\rangle + a_{00}|10\rangle - a_{01}|11\rangle.
\end{aligned}$$

We can easily observe that both ways lead to the same outcome and the second one (double wedge) does it in a very compact way.

Similarly if we act with a two-qubit gate specifically with a CNOT gate, whose unitary matrix U_{CNOT} representation is:

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The quantum state after the action of U_{CNOT} using dirac notation reads,

$$\begin{aligned} U|\psi\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{11} \\ a_{10} \end{pmatrix} \\ U|\psi\rangle &= a_{00}|00\rangle + a_{01}|01\rangle + a_{11}|10\rangle + a_{10}|11\rangle. \end{aligned}$$

Trying to express $CNOT$ as a tensor product of two independent one qubit gates is impossible.

Definition: We say that CNOT is a non-local gate because it can't be written as a tensor product of two one-qubit gates ($A \otimes B$) but only as a sum of tensor products because is not acting on each qubit independently but check the first and then acts on the second.

So decomposing CNOT as a sum of two matrices we have:

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The quantum state after the action of CNOT using double wedge notation and the identity $A \otimes B|C\rangle\rangle = |ACB^T\rangle\rangle$ reads

$$\begin{aligned} U_{CNOT}|A\rangle\rangle &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] | \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\rangle\rangle \\ U_{CNOT}|A\rangle\rangle &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] | \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\rangle\rangle + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} | \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\rangle\rangle \\ U_{CNOT}|A\rangle\rangle &= | \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\rangle\rangle + | \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\rangle\rangle \\ U_{CNOT}|A\rangle\rangle &= | \begin{pmatrix} a_{00} & a_{01} \\ 0 & 0 \end{pmatrix}\rangle\rangle + | \begin{pmatrix} 0 & 0 \\ a_{11} & a_{10} \end{pmatrix}\rangle\rangle \\ U_{CNOT}|A\rangle\rangle &= | \begin{pmatrix} a_{00} & a_{01} \\ a_{11} & a_{10} \end{pmatrix}\rangle\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{11}|10\rangle + a_{10}|11\rangle \end{aligned}$$

We can easily observe again that both ways lead to the same outcome.

1.6 Entanglement

One weird phenomenon we observe in quantum world is entanglement. Suppose that we have a bipartite system (two quantum particles(photons,electrons etc)) we call them entangled if there is an "invisible" connection between them. Changing the polarization of our own qubit (photon particle) performing one qubit gate for example then the polarization of the remote particle will transofm analogous. One of the first researchers which observed these phenomena was Albert Einstein in 1935 who called it "spooky action at a distance". In quantum computation and communication entanglement is a very important resource (ebit) because it allows us to use it as quantum channel to transfer quantum information and to make computations at distance.The main algorithms of this thesis are using quantum entanglement to teleport a quantum state from one location to other or to teleport the action of quantum gates on quantum states at a distance.

Mathematically, we say that two qubits are entangled if we can not express their quantum state as a tensor product of two independent one qubit states (un-factorized).

i.e $|w\rangle$ is an entangled bipartite system if there are not two qubits $|x\rangle, |\psi\rangle$ such that $|w\rangle = |x\rangle \otimes |\psi\rangle$

Suppose that we have the quantum state:

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

we say that this bipartite system is entangled because does not exist two one qubit states whose the composition will produce $|\phi_+\rangle$.

Proof

We suppose that exists two one qubit states $|x\rangle, |y\rangle$ whose the composition will produce $|\phi_+\rangle$

$$|x\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, |y\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$$

Taking the tensor product of them

$$|x\rangle \otimes |y\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

It must be:

$$\begin{aligned} |x\rangle \otimes |y\rangle &= |\phi_+\rangle \\ \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

which means

$$\begin{aligned} ac &= \frac{1}{\sqrt{2}} && (1)\\ ad &= 0 && (2)\\ bc &= 0 && (3)\\ bd &= \frac{1}{\sqrt{2}} && (4) \end{aligned}$$

from equation (1) $\alpha \neq 0$ and c $\neq 0$ which leads to $d = 0$ and $b = 0$. So that equations (2),(3) can be verified respectively but if $d = 0$ and $b = 0$ then equation (4) is invalid.

So our system has no solution, which means that there are no $|x\rangle, |y\rangle$ such that $|x\rangle \otimes |y\rangle = |\phi_+\rangle$ is valid.

In general, every bipartite system of the form $\alpha|00\rangle + \beta|11\rangle$ is entangled.

We call the state vector:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

a maximaly entangled state because the probability of two states $|00\rangle, |11\rangle$ are the same and equal to $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$. If we know the state of one particle we can guess the state of the other with confidence. In this specific bipartite quantum state if we know that the first qubit is in state $|0\rangle$ then and the second qubit is in state $|0\rangle$ similarly if the first qubit is in state $|1\rangle$ then and the second qubit is in state $|1\rangle$ because the fact that all the possible states of our qubits are $|00\rangle, |11\rangle$.

One other maximally entagled bipartite quantum state is

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Similarly with before if we know the state of the first qubit then we can quess deterministic the state of the second one. If the first qubit is in state $|0\rangle$ then the second is in state $|1\rangle$ similarly if the first is in state $|1\rangle$ then the second is in state $|0\rangle$.

John Bell promoted a set of four bipartite quantum states which are maximally entangled[1]:

$$\begin{aligned} |\phi_+\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\\ |\psi_+\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\\ |\phi_-\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle\\ |\psi_-\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle \end{aligned}$$

These four states define a new basis, the so called Bell basis which (as it is shown in Appendix A) is orthonormal.

1.6.1 Entanglement Circuit : Bell Operator

It's time to present our first quantum circuit (combination of quantum gates) which has as inputs two initialized qubits in a computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ (canonical-factorized) and produces at the output one of the four Bell States (un-factorized).

The action of the circuit evolves from left to right as we can see in the diagram.

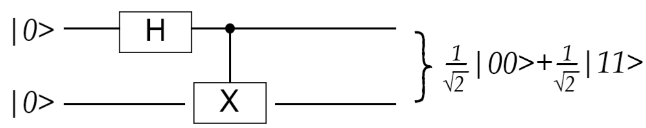


Figure 5: Action of Bell Circuit on two qubits initialized at |00>

In first step we apply a Hadamard gate on the first qubit and then a CNOT gate on the two qubits.

Example

Suppose input is in the quantum state

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

after the action of Hadamard gate on the first qubit:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

the quantum state transforms to

$$|\psi_1\rangle = (H \otimes I)\,|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Then the action of CNOT on both qubits transforms the quantum state to:

$$|\psi_2\rangle = U_{CNOT}|\psi_1\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

which is the final output for input $|00\rangle$.As we observe that it is equal to the first of the Bell states $|\phi_+\rangle$.

Using the notation we can prove that the action of the circuit is exactly the same.

Suppose the input is in the quantum state

$$|A_{00}\rangle\rangle = |\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\rangle\rangle$$

which is equal to $|00\rangle$.

After the action of Hadamard gate on the first qubit:

$$\begin{aligned} |\psi_1\rangle\rangle &= H \otimes I|A_{00}\rangle\rangle = H \otimes I|A_{00}\rangle\rangle = |HA_{00}\rangle\rangle \\ |\psi_1\rangle\rangle &= |\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\rangle\rangle \\ |\psi_1\rangle\rangle &= |\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\rangle\rangle \end{aligned}$$

Then the action of CNOT on both qubits transforms the quantum state to

$$|\psi_2\rangle\rangle = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] |\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\rangle\rangle$$

$$|\psi_2\rangle\rangle = |\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\rangle\rangle + |\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\rangle\rangle$$

$$|\psi_2\rangle\rangle = |\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}\rangle\rangle + |\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\rangle\rangle$$

$$|\psi_2\rangle\rangle = |\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\rangle\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

which is the final output for double wedge input $|A_{00}\rangle\rangle$. As we observe it is equal to the first of Bell states $|\phi_+\rangle$.

Similarly we can easily verify either by using Dirac notation or double wedge notation that this circuit has outputs $|\psi_+\rangle, |\phi_-\rangle, |\psi_-\rangle$ for $|01\rangle, |10\rangle, |11\rangle$ respectively.

Also in quantum world every circuit can be represented as a unitary matrix multiplying the gates from right to left of the diagram. The circuit which defines U_{Bell} gate is:

$$U_{Bell} = U_{CNOT}(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

which is unitary i.e ($U_{Bell}U_{Bell}^{\dagger} = I_4$) a requirement in order to respect the

normality of quantum state,indeed

$$U_{Bell}^{\dagger}U_{Bell} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can easily confirm that U_{Bell} acting on two qubits in the canonical basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ transforms it directly to a Bell state $\{|\phi_+\rangle, |\psi_+\rangle, |\phi_-\rangle, |\psi_-\rangle\}$ respectively.

Example

$$U_{Bell}|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |\phi_+\rangle$$

$$U_{Bell}|01\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = |\psi_+\rangle \; etc.$$

A significant property is that we can transform from one Bell State to another by applying one qubit gate only at the first particle which means that Bell states are locally interconnected. This is very useful in practical implementations such as supedense coding protocol which exceeds the range of interests of this thesis.

From\To	$\|\phi_+\rangle$	$\|\psi_+\rangle$	$\|\phi_-\rangle$	$\|\psi_-\rangle$
$\|\phi_+\rangle$	$I \otimes I$	$X \otimes I$	$Z \otimes I$	$iY \otimes I$
$\|\psi_+\rangle$	$X \otimes I$	$I \otimes I$	$iY \otimes I$	$Z \otimes I$
$\|\phi_-\rangle$	$Z \otimes I$	$-iY \otimes I$	$I \otimes I$	$-X \otimes I$
$\|\psi_-\rangle$	$-iY \otimes I$	$Z \otimes I$	$-X \otimes I$	$I \otimes I$

Example:

If we have a Bell state (From):

$$|\phi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and we want to transform it to an other Bell state (To) for example:

$$|\psi_+\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Then we have to act as follows,

$$\begin{aligned} |\psi_+\rangle &= (-iY \otimes I)|\phi_-\rangle = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ |\psi_+\rangle &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \end{aligned}$$

1.6.2 Un-Entaglement Circuit : Inverse of Bell Operator

As we said every quantum gate is reversible i.e if we know the outputs of the gate we can find the inputs applying the $U^\dagger$ gate at the output qubits. This can be generalized for quantum circuits since every quantum circuit can be written in a form of a unitary matrix who represents the whole circuit.

As we have shown the Matrix representation of Bell circuit is:

$$U_{Bell} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

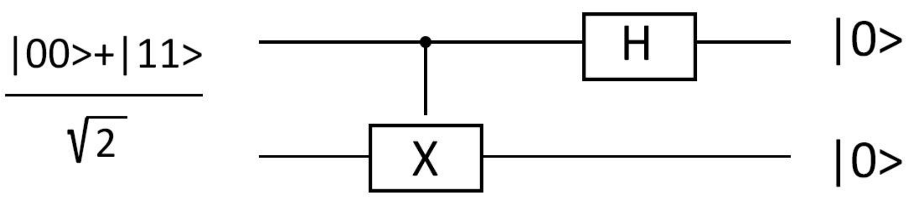
which takes an input from the canonical basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and transforms it to a Bell state.

Due to reversibility the reverse Bell gate reads

$$U_{un-Bell} \equiv U_{Bell}^{\dagger} = (U_{CNOT}(H \otimes I))^{\dagger} = (H \otimes I)^{\dagger} U_{CNOT}^{\dagger}$$

U_{CNOT} and H are Hermitians ($U_{CNOT}^{\dagger} = U_{CNOT}$ and $H^{\dagger} = H$),which reads

$$U_{un-Bell} = (H \otimes I)U_{CNOT} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$



Action of un-Bell Circuit on two qubits initialized at $|\phi_+ >$

<u>Example:</u>

If we "insert" to un-Bell circuit the Bell state

$$|\phi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

then the outcome is:

$$|\psi_{out}\rangle = U_{un-Bell}|\phi_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

which is the input (on canonical basis) which produces the Bell state $|\phi_-\rangle$

1.7 Density Operator

A more complete way of describing quantum systems is the so called density matrix/operator. This refers to the general case where we do not know the exact state in which our system finds itself. E.g a system is 30% in the state $|0\rangle$ and 70% is in state $|+\rangle$ it can be described by the following density operator $\rho = 0.3|0\rangle\langle 0| + 0.7|+\rangle\langle +|$. But this kind of systems are out of the needs of this thesis. In our cases we know that the state is $|\psi\rangle$ with certainty so an alternative way of saying that is by means of the density operator which is:

$$\rho_\psi = |\psi\rangle\langle\psi|$$

Suppose we have a general one qubit state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

let's compute the density operator of one qubit state using vector notation

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} a^* & \beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}$$

Three conditions which a density operator satisfies are:

a) It is Hermitian:

$$\rho^\dagger = \rho$$

b) The sum of the diagonal elements is equal to 1:

$$|a|^2 + |\beta|^2 = 1$$

because $|a|^2, |\beta|^2$ are possibilities

c) ϱ is a positive operator which means that $\langle\psi|\rho|\psi\rangle \geq 0$ for an arbitrary quantum state $|\psi\rangle$

Suppose we have one general two qubit-Bipartite state:

$$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) = \begin{pmatrix} \alpha\gamma \\ a\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

The general density operator of this arbitrary bipartite system is:

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha\gamma \\ a\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \begin{pmatrix} \alpha^*\gamma^* & \alpha^*\delta^* & \beta^*\gamma^* & \beta^*\delta^* \end{pmatrix}$$

$$\rho_\psi = \begin{pmatrix} |\alpha|^2|\gamma|^2 & |\alpha|^2\gamma\delta^* & |\gamma|^2\alpha\beta^* & \alpha\gamma\beta^*\delta^* \\ |\alpha|^2\delta\gamma^* & |\alpha|^2|\delta|^2 & a\delta\beta^*\gamma^* & |\delta|^2 a\beta^* \\ |\gamma|^2\beta\alpha^* & \beta\gamma\alpha^*\delta^* & |\beta|^2|\gamma|^2 & |\beta|^2\gamma\delta^* \\ \beta\delta\alpha^*\gamma^* & |\delta|^2\beta\alpha^* & |\beta|^2\delta\gamma^* & |\beta|^2|\delta|^2 \end{pmatrix}$$

Also as we used tensor product to describe multiple qubits quantum states. So we will use tensor product of density operators to describe the general density operator of two or more qubits quantum states

$$\rho = \rho_\alpha \otimes \rho_\beta \otimes \rho_\gamma ....$$

Especially for a general bipartite system we may have that

$$\rho_\psi = \rho_\phi \otimes \rho_x$$

<u>Proof</u>

$$\begin{aligned} |\phi\rangle &= \alpha|0\rangle + \beta|1\rangle \text{ then } \rho_\phi = |\phi\rangle\langle\phi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix} \\ |x\rangle &= \gamma|0\rangle + \delta|1\rangle \text{ then } \rho_x = |x\rangle\langle x| = \begin{pmatrix} |\gamma|^2 & \gamma\delta^* \\ \delta\gamma^* & |\delta|^2 \end{pmatrix} \end{aligned}$$

Which leads to:

$$\begin{aligned} \rho_\psi &= \rho_\phi \otimes \rho_x = |\phi\rangle\langle\phi| \otimes |x\rangle\langle x| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix} \otimes \begin{pmatrix} |\gamma|^2 & \gamma\delta^* \\ \delta\gamma^* & |\delta|^2 \end{pmatrix} \\ \rho_\psi &= \begin{pmatrix} |\alpha|^2|\gamma|^2 & |\alpha|^2\gamma\delta^* & |\gamma|^2\alpha\beta^* & \alpha\gamma\beta^*\delta^* \\ |\alpha|^2\delta\gamma^* & |\alpha|^2|\delta|^2 & a\delta\beta^*\gamma^* & |\delta|^2 a\beta^* \\ |\gamma|^2\beta\alpha^* & \beta\gamma\alpha^*\delta^* & |\beta|^2|\gamma|^2 & |\beta|^2\gamma\delta^* \\ \beta\delta\alpha^*\gamma^* & |\delta|^2\beta\alpha^* & |\beta|^2\delta\gamma^* & |\beta|^2|\delta|^2 \end{pmatrix} \end{aligned}$$

which applies because it is equal to the direct computation of the density operator for a general bipartite systems.

1.7.1 Evolution of Density Operator

We know that the evolution of a quantum state $|\psi\rangle$ after the action of a gate U is given by: $|\psi'\rangle = U|\psi\rangle$.

Similarly the evolution of the ρ after the action of U gate ρ' reads:

$$\rho' = U\rho U^\dagger$$

<u>Proof</u>

We know that

$$\begin{aligned} \rho' &= |\psi'\rangle\langle\psi'| \\ \text{where } |\psi'\rangle &= U|\psi\rangle \text{ and } \langle\psi'| = |\psi'\rangle^\dagger = (U|\psi\rangle)^\dagger = \langle\psi|U^\dagger \\ \text{So } \rho' &= U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger \end{aligned}$$

Example

We have a single qubit quantum state

$$|\psi\rangle = \begin{pmatrix} a \\ \beta \end{pmatrix}$$

which has as we have seen before Density Operator

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}$$

After the action of $U_{NOT} = X$ gate the new quantum state is:

$$|\psi'\rangle = X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

So

$$\rho' = |\psi'\rangle\langle\psi'| = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \begin{pmatrix} \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} |\beta|^2 & \beta\alpha^* \\ \alpha\beta^* & |\alpha|^2 \end{pmatrix}$$

Using the equation we just proved

$$\rho' = U\rho U^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} |\beta|^2 & \beta\alpha^* \\ \alpha\beta^* & |\alpha|^2 \end{pmatrix}$$

As we can observe both ways to compute the density operator after the action of a gate U are equivalent.

Respectivly we can generalize the evolution of density operator for bipartite quantum systems or N-partite quantum systems.

1.7.2 Reduced Density Operator of Bipartite Systems

In this subsection we will describe how we can calculate a very important object the reduced density operator (2x2) which describes one of two qubits given the general density operator (4×4) of a bipartite system using the mathematical tool known as "Partial Trace" over one or the other of the two quantum systems

Suppose that we have a bipartite density operator

$$\rho_{AB} = \sum_{ijkl} \rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|$$

$$\rho_{AB} = \begin{pmatrix} \rho_{0000} & \rho_{0001} & \rho_{0100} & \rho_{0101} \\ \rho_{0010} & \rho_{0011} & \rho_{0110} & \rho_{0111} \\ \rho_{1000} & \rho_{1001} & \rho_{1100} & \rho_{1101} \\ \rho_{1010} & \rho_{1011} & \rho_{1110} & \rho_{1111} \end{pmatrix}$$

Grouping the above 4×4 matrix in four 2×2 matrices we rewrite:

$$\rho_{AB} = \begin{pmatrix} \boldsymbol{\rho}_{00} & \boldsymbol{\rho}_{01} \\ \boldsymbol{\rho}_{10} & \boldsymbol{\rho}_{11} \end{pmatrix}$$

We can find that the reduced density operator ρ_A describing the first quantum system tracing out the second density operator ρ_B

$$\rho_{AB} = \sum_{ijkl} \rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|$$

$$\begin{aligned} \rho_A &= \sum_{ijkl} Tr_B(\rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|) \\ \rho_A &= \sum_{ijkl} \rho_{ijkl} |i\rangle\langle j| \otimes Tr(|k\rangle\langle l|) \end{aligned}$$

We know that

$$Tr(|k\rangle\langle l|) = \delta_{kl}$$

So,

$$\rho_A = \sum_{ijk} \rho_{ijkk} |i\rangle\langle j|$$

leads to:

$$\rho_A = \begin{pmatrix} \rho_{0000} + \rho_{0011} & \rho_{0100} + \rho_{0111} \\ \rho_{1000} + \rho_{1011} & \rho_{1100} + \rho_{1111} \end{pmatrix}$$

Which is equal to

$$\rho_A = \begin{pmatrix} Tr(\boldsymbol{\rho}_{00}) & Tr(\boldsymbol{\rho}_{01}) \\ Tr(\boldsymbol{\rho}_{10}) & Tr(\boldsymbol{\rho}_{11}) \end{pmatrix}$$

Similarly

$$\begin{aligned} \rho_B &= \sum_{ijkl} Tr_A(\rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|) \\ \rho_B &= \rho_{ijkl} Tr\left(|i\rangle\langle j|\right) \otimes |k\rangle\langle l| \end{aligned}$$

We know that

$$Tr(|i\rangle\langle j|) = \delta_{ij}$$

So,

$$\rho_B = \sum_{ikl} \rho_{iikl} |k\rangle\langle l|$$

$$\rho_B = \begin{pmatrix} \rho_{0000} + \rho_{1100} & \rho_{0001} + \rho_{1101} \\ \rho_{0010} + \rho_{1101} & \rho_{0011} + \rho_{1111} \end{pmatrix}$$

Which is equal to

$$\rho_B = (\boldsymbol{\rho}_{00} + \boldsymbol{\rho}_{11})$$

So using the above two relations we can calculate any of both reduced density operators (ρ_A, ρ_B) given the general Density operator ϱ of a bi-partite system.

<u>Example</u>:

Suppose that we have one of the Bell quantum states

$$|\phi_+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

then the density operator of this bipartite quantum state is

$$\rho_{\phi+} = |\phi_+\rangle\langle\phi_+| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Then

$$\rho_A = \begin{pmatrix} Tr\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}I_2$$

and

$$\rho_B = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}I_2$$

In general any of Bell States $(|\phi_+\rangle, |\phi_-\rangle, |\psi_+\rangle, |\psi_-\rangle,)$ have $\rho_A = \rho_B = \frac{1}{2}I_2$. (See Apendix B)

1.8 Entropy as a measure of Entanglement

As we have already seen we say that two particles are entangled when we can not write them as a form of two qubits using tensor product $(|\phi\rangle \otimes |\psi\rangle)$ but how we can measure the quantify of entanglement for a bipartite system?

One way to quantify the entanglement of one bipartite system is to find the reduced density operator according to the previous section and then calculate the Von Neumann Entropy introduced by John von Neumann, using the definition

$$S(\rho) = -Tr(\rho \log_2 \rho)$$

The outcome of this computation is a scalar which quantifies the entanglement.

It is very important to prove that the Bell States have the maximum quantity of entanglement.

Suppose we have one of the Bell State e.g

$$|\phi_+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

according to the previous section the reduced density operator of this bipartite system is

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Applying the definition of Von Neumann Entropy

$$S(\rho) = -Tr\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \log_2 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right)$$

The matrix is diagonal so we can take the logarithm of it's elements

$$S(\rho) = -Tr\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \log_2 \frac{1}{2} & 0 \\ 0 & \log_2 \frac{1}{2} \end{pmatrix}\right)$$

Due to the identity

$$\log(\frac{x}{y}) = \log x - \log y$$

Our equation takes the form:

$$S(\rho) = -Tr\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \log_2 1 - \log_2 2 & 0 \\ 0 & \log_2 1 - \log_2 2 \end{pmatrix}\right)$$

We know that

$$\log_2 1 = 0$$

So

$$S(\rho) = -Tr\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \log_2 2 & 0 \\ 0 & \log_2 2 \end{pmatrix}\right)$$

$$S(\rho) = -Tr\begin{pmatrix} \frac{1}{2}\log_2 2 & 0 \\ 0 & \frac{1}{2}\log_2 2 \end{pmatrix}$$

Finally,

$$S(\rho) = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2 = \log_2 2 = 1$$

Which is the maximum possible as we expected and it indicates that we have maximum entanglement.

Exactly the same applies for the rest of Bell states since all of them have the same reduced density operator($\rho = \frac{I}{2}$). (According to Appendix B)

1.9 Quantum Measurement

As in the classical case where we have to measure the level of voltage using a multimeter so in quantum computation we have to estimate the quantum state of qubit (for example Spin-up or Spin-down state for the case of electron states) using a mathematical tool called projection.Which is a set of matrices P_w that constructed with the help of a vector $|w\rangle$ on which we want to project our state. Explicitly

$$P_w = |w\rangle\langle w|$$

Assuming that $|||w\rangle||$=1 we can easily verify that $P_w = P_w^2$

<u>Proof</u>

$$P_w^2 = |w\rangle \underbrace{\langle w||w\rangle}\langle w|$$

Do to the fact that $|||w\rangle||$=1

$$P_w^2 = |w\rangle\langle w| = P_w$$

1.9.1 Canonical Basis Measurement

One Qubit Measurement Suppose that we have a qubit in a general form of superposition on the computational basis $|0\rangle$,$|1\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where

$$|a|^2 + |\beta|^2 = 1$$

The two projection operators we can construct to observe if the qubit is in state $|0\rangle$ or $|1\rangle$ at the moment of measurement are $P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|$ respectively.

Suppose that we want to act with the P_0 on the qubit.

$$|\psi'\rangle = P_0|\psi\rangle = |0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle \underbrace{\langle 0||0\rangle} + \beta|0\rangle \underbrace{\langle 0||1\rangle}$$

due to Orthonormality

$$\langle x||y\rangle = \delta_{xy}$$

Our system takes the form:

$$|\psi'\rangle = a|0\rangle$$

As we can easily observe this is not representing a quantum state because it is not normalized

$$\langle\psi'||\psi'\rangle = \alpha^*\langle 0|a|0\rangle = aa^*\langle 0||0\rangle = |\alpha|^2 \neq 1$$

this happens because projection operators are not unitary so they do not respect the norm of the quantum state

So we have to normalize it dividing with

$$\begin{aligned}
\sqrt{\langle\psi|P_0|\psi\rangle} &= \sqrt{\langle\psi||\psi'\rangle} = \sqrt{(\alpha^*\langle 0| + \beta^*\langle 1|)\, a|0\rangle} \\
\sqrt{\langle\psi|P_0|\psi\rangle} &= \sqrt{\alpha^*\alpha\langle 0|0\rangle + \beta^*\langle 1|0\rangle|} \\
\sqrt{\langle\psi|P_0|\psi\rangle} &= \sqrt{|\alpha|^2} = |\alpha|
\end{aligned}$$

where $\langle\psi|P_0|\psi\rangle = |\alpha|^2$ is the probability of finding our state $|\psi\rangle$ along $|0\rangle$

The state after P_0 measurement is:

$$|\psi_{P_0}\rangle = \frac{P_0|\psi\rangle}{\sqrt{\langle\psi|P_0|\psi\rangle}} = \frac{a|0\rangle}{|\alpha|}$$

which means that

$$|\psi_{P_0}\rangle = |0\rangle$$

or

$$|\psi_{P_0}\rangle = -|0\rangle$$

which are equal "up to a global phase". Which means that

$$|\psi_{P_0}\rangle = |0\rangle$$

Similarly under the measurment acting with P_1 on the qubit the post-measurement state is:

$$|\psi_{P_1}\rangle = \frac{P_1|\psi\rangle}{\sqrt{\langle\psi|P_1|\psi\rangle}} = \frac{\beta|1\rangle}{|\beta|}$$

which is equal to:

$$|\psi_{P_1}\rangle = |1\rangle$$

As we observe either of two cases the state after measurment collapses on $|0\rangle$ or $|1\rangle$ depending on the projection operator that we will use. This means that quantum measurement destroy our quantum state.

Two Qubits Measurement Suppose that we have a bipartite (two qubits)

quantum state in a general form on the computational base $|0\rangle$,$|1\rangle$

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

where

$$|a|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

We have four possible states in which a qubit may be $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ so we construct the respective projection operators

$$P_{00} = |00\rangle\langle 00|$$

$$P_{01} = |01\rangle\langle 01|$$

$$P_{10} = |10\rangle\langle 10|$$

$$P_{11} = |11\rangle\langle 11|$$

Suppose that we want to act with the P_{01} on the bipartite quantum system

$$|\psi'\rangle = P_{01}|\psi\rangle = |01\rangle\langle 01|(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle)$$

due to orthonormality

$$\langle x_1x_2||y_1y_2\rangle = \delta_{x_1y_1,x_2y_2}$$

Our system will be transformed:

$$|\psi'\rangle = \beta|01\rangle$$

As we can easily observe this is not representing a quantum state because it is not normalized

$$||\psi'\rangle| = \langle\psi'||\psi'\rangle = \beta^*\langle 01|\beta|01\rangle = \beta\beta^*\langle 01||01\rangle = |\beta|^2 \neq 1$$

So we have to normalize it dividing with

$$\sqrt{\langle\psi|P_{01}|\psi\rangle} = \sqrt{\langle\psi||\psi'\rangle} = \sqrt{(\alpha^*\langle 00| + \beta^*\langle 01| + \gamma^*\langle 10| + \delta^*|\langle 11|)(\beta|01\rangle)}$$

due to orthonormality

$$\langle x_1x_2||y_1y_2\rangle = \delta_{x_1x_2,y_1y_2}$$

$$\begin{aligned} \sqrt{\langle\psi|P_0|\psi\rangle} &= \sqrt{\beta^*\beta\langle 01|01\rangle} \\ \sqrt{\langle\psi|P_0|\psi\rangle} &= \sqrt{|\beta|^2} = |\beta| \end{aligned}$$

where $\langle\psi|P_{01}|\psi\rangle = |\beta|^2$ is the probability of finding our state $|\psi\rangle$ along $|0\rangle$

The state after P_{01} measurement is:

$$|\psi_{P_{01}}\rangle = \frac{P_{01}|\psi\rangle}{\sqrt{\langle\psi|P_{01}|\psi\rangle}} = \frac{\beta|01\rangle}{|\beta|}$$

Which is equal to

$$|\psi_{P_{01}}\rangle = |01\rangle$$

"up to a global phase"

Similarly we can calculate the post measurment states under the action of P_{00}, P_{10}, P_{11} on the bipartite quantum system which are $|00\rangle, |10\rangle.|11\rangle$ respectively

On bipartite quantum systems we have the ability to measure only one of the two qubits (particles).

For example we can construct projection operators so that the first qubit be in state $|0\rangle$ and leave alone the second one

$$P_{0,I} = |0\rangle\langle 0| \otimes I$$

or the second qubit be in state $|1\rangle$ and leave alone the first one.

$$P_{I,1} = I \otimes |1\rangle\langle 1|$$

etc.

Suppose that we have a general bipartite system

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

and we want to calculate the post measurement state of $P_{0,I}$ projection operator (first qubit be in state $|0\rangle$)

$$\begin{aligned}|\psi'\rangle &= P_{0,I}|\psi\rangle = |0\rangle\langle 0| \otimes I(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle)\\ |\psi'\rangle &= \alpha|0\rangle \underbrace{\langle 0||0\rangle} \otimes I|0 + \beta|0\rangle \underbrace{\langle 0||0\rangle} \otimes I|1\rangle + \gamma|0\rangle \underbrace{\langle 0||1\rangle} \otimes I|0\rangle + \delta|0\rangle \underbrace{\langle 0||1\rangle} \otimes I|1\rangle\end{aligned}$$

due to orthonormality

$$\langle x||y\rangle = \delta_{xy}$$

$$|\psi'\rangle = a|00\rangle + \beta|01\rangle$$

which is not representing a quantum state

$$||\psi'\rangle| = \langle\psi'||\psi'\rangle = (\alpha^*\langle 00| + \beta\langle 01)\,(|\alpha|00\rangle + \beta|01\rangle) = \alpha\alpha^* + \beta\beta^* = |\alpha|^2 + |\beta|^2 \neq 1$$

So we have to normalize it dividing with

$$\sqrt{\langle\psi|P_{0,I}|\psi\rangle} = \sqrt{\langle\psi||\psi'\rangle} = \sqrt{(\alpha^*\langle 00| + \beta^*\langle 01| + \gamma^*\langle 10| + \delta^*|\langle 11|)(a|00\rangle + \beta|01\rangle)}$$

due to Orthonormality

$$\langle x_1x_2||y_1y_2\rangle = \delta_{x_1y_1,x_2y_2}$$

$$\begin{aligned}\sqrt{\langle\psi|P_{0,I}|\psi\rangle} &= \sqrt{\alpha^*\alpha\langle 00|00\rangle + \beta^*\beta\langle 01|01\rangle}\\ \sqrt{\langle\psi|P_{0,I}|\psi\rangle} &= \sqrt{|\alpha|^2 + |\beta|^2}\end{aligned}$$

where $\langle\psi|P_{0,I}|\psi\rangle = |\alpha|^2 + |\beta|^2$ is the probability of finding the first qubit of our state $|\psi\rangle$ along $|0\rangle$

The state after $P_{0,I}$ measurement is:

$$|\psi_{P_{0,I}}\rangle = \frac{P_{0,I}|\psi\rangle}{\sqrt{\langle\psi|P_{0,I}|\psi\rangle}} = \frac{a|00\rangle + \beta|01\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}}$$

1.9.2 Bell Basis Measurement

One other basis as we have already described is Bell basis with base vectors $\{|\phi_+\rangle, |\phi_-\rangle, |\psi_+\rangle, |\psi_-\rangle\}$ which represent four maximally entagled states of two qubits.

The general form of a bipartite quantum system using Dirac notation of Bell States is

$$|\psi\rangle = \alpha|\phi_+\rangle + \beta|\phi_-\rangle + \gamma|\psi_+\rangle + \delta|\psi_-\rangle$$

We can construct four Projection Operators to measure in which Bell state is our system.

$$P_{\phi+} = |\phi_+\rangle\langle\phi_+|$$

$$P_{\phi-} = |\phi_-\rangle\langle\phi_-|$$
$$P_{\psi+} = |\psi_+\rangle\langle\psi_+|$$
$$P_{\psi-} = |\psi_-\rangle\langle\psi_-|$$

for respective Bell states. When we measure on this basis we call it Bell Measurement which is usefull for this thesis.

Suppose that we want to measure our bipartite system under the $P_{\phi+}$ projection operator

$$\begin{aligned} |\psi'\rangle &= P_{\phi+}|\psi\rangle = |\phi_+\rangle\langle\phi_+|(\alpha|\phi_+\rangle + \beta|\phi_-\rangle + \gamma|\psi_+\rangle + \delta|\psi_-\rangle) \\ &= \alpha \underbrace{\langle\phi_+||\phi_+\rangle}|\phi_+\rangle + \beta \underbrace{\langle\phi_+||\phi_-\rangle}|\phi_+\rangle + \gamma \underbrace{\langle\phi_+||\psi_+\rangle}|\phi_+\rangle + \delta \underbrace{\langle\phi_+||\psi_-\rangle}|\phi_+\rangle \end{aligned}$$

due to Orthonormality of Bell Basis

$$|\psi'\rangle = \alpha|\phi_+\rangle$$

So we have to normalize it dividing with

$$\begin{aligned} \sqrt{\langle\psi|P_{\phi+}|\psi\rangle} &= \sqrt{\langle\psi||\psi'\rangle} = \sqrt{(\alpha^*\langle\phi_+| + \beta^*\langle\phi_-| + \gamma^*\langle\psi_+| + \delta\langle\psi_-|)(\alpha|\phi_+\rangle)} \\ \sqrt{\langle\psi|P_{\phi+}|\psi\rangle} &= \sqrt{\alpha^*\alpha\langle\phi_+|\phi_+\rangle} \\ \sqrt{\langle\psi|P_{\phi+}|\psi\rangle} &= \sqrt{|\alpha|^2} = |\alpha| \end{aligned}$$

where $\langle\psi|P_{\phi+}|\psi\rangle = |\alpha|^2$ is the probability of finding our system $|\psi\rangle$ along $|\phi_+\rangle$ Bell state.

The final state after the measurement is

$$|\psi_{P_{\phi+}}\rangle = \frac{P_{\phi+}|\psi\rangle}{\sqrt{\langle\psi|P_{\phi+}|\psi\rangle}} = \frac{\alpha|\phi_+\rangle}{|\alpha|}$$

"up to a global phase"

$$|\psi_{P_{\phi+}}\rangle = |\phi_+\rangle$$

In this case it is evident that we can not construct projection operators on one of two qubits because our base is not factorized.

2 Quantum Teleportation Algorithms

Quantum teleportation is a protocol introduced by six scientists (C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters) in 1993 which allows us to teleport quantum information (quantum states) from one space to another taking advantage of quantum entanglement. In this thesis we are going to reformulate the protocol in a more compact form and describe some other cases for the teleportation of quantum resources further the one you teleport a quantum state. Specifically, on the first section we will describe the teleportation of a quantum state of one qubit according to the paper of Bennett something that requires 1shared pair of entangled qubits and two classical bits of information. On the second section we will describe the 1Qubit Gate Teleportation which will have as a scope Bob to receive the outcome of the action of one qubit gate on the arbitraty qubit that Alice had on her possession, as we will see from resources perspective it also requires 1shared pair of entangled qubits and two classical bits of information plus the action of a quantum gate by Alice.One other protocol which uses this kind of teleportation is the 1-Qubit Gate only Teleportation on which we will prove that Bob can reconstruct a gate which Alice teleported just sending the action of this gate on the basis of $|0\rangle, |1\rangle$ from resources perspective it requires 2*(1shared pair of entangled qubits and two classical bits of information)= 2 shared pair of entangled qubits and four classical bits of information.Also, we will demonstrate a protocol with which Alice can teleport an arbitrary (not necessarily maximally) entangled state of the form $a|00\rangle + \beta|11\rangle$ using 1shared-pair of entangled qubits and 3 classical bits of information. On the fifth section we are going to demonstrate how we can teleport the action of one two qubits gate (such as CNOT) on two arbitrary qubits, from resources persepctive it requires 2 shared-pairs of entangled qubits plus 4 bits of classsical information. Similarly, we will prove that Bob can reconstruct the two-qubit gate if he receive the action on the basis $[|00\rangle, |01\rangle, |10\rangle, |11\rangle]$ from resources perspective it requires 4*(2shared-pairs of entangled qubits and four classical bits of information)= 8 shared pairs of entangled qubits and sixteen classical bits of information.

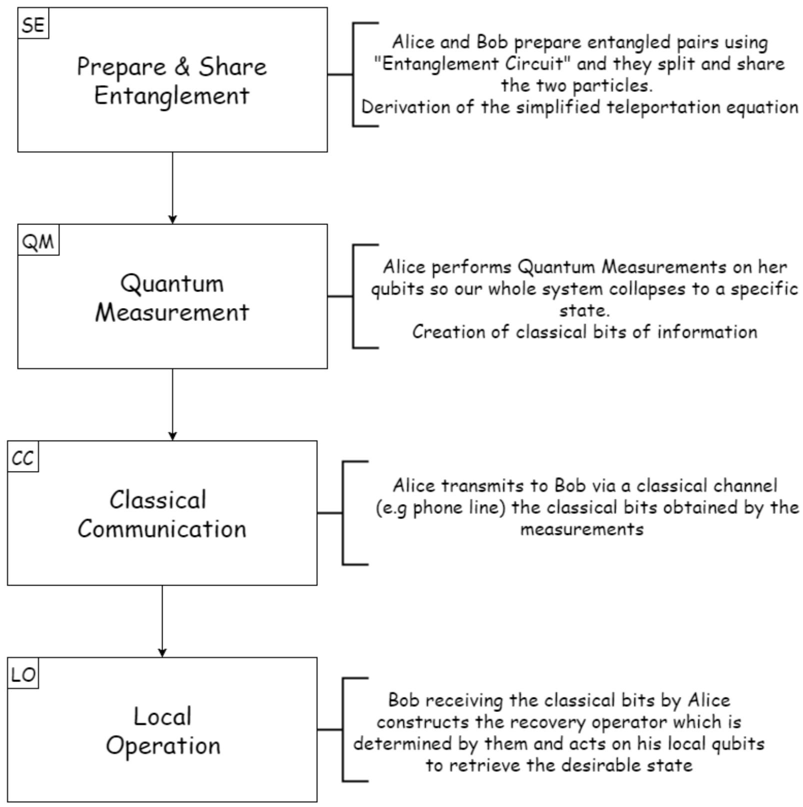
Summing up in a more visual form from resources perspective we have:

	1-Qubit	2-Qubits
State	1ebit 2cbits	1ebit 3cbits
Action	1ebit 2cbits	2ebits 4cbits
Gate	2ebits 4cbits	8ebits 16cbits

ebits: shared pairs of entangled qubits
cbits: classical bits of information

It is notable that all of the above protocols belong to a specific category of protocols called LOCC-SE (Local Operator Classical Communication Shared Entanglement) which means that the sender and the receiver have prepare and share a pair of entanglement and that the receiver performs some local operations determined by the bits of the classical communication between the sender and the receiver to retrieve the desirable quantum state.

One visual way to represent the steps of the protocols separated in modules is given below:



The whole description of the protocols in a form of steps shows that they belong to LOCC-SE protocol

2.1 1-Qubit State Teleportation

Alice wants to teleport an arbitrary unknown quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. To achieve this they have prepared and shared (taking one qubit each one) using the Entanglement Circuit a maximally entangled bipartite state

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\frac{I}{\sqrt{2}}\rangle\rangle$$

So the initial state of our system is:

$$
\begin{aligned}
|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\frac{|00\rangle + |11\rangle}{\sqrt{2}}) \\
|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle
\end{aligned}
$$

From which the first two qubits belongs to Alice and the third one to Bob.

So we can rewrite the equation separating their local qubits

$$|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{|00\rangle}{\sqrt{2}} \otimes \alpha|0\rangle + \frac{|01\rangle}{\sqrt{2}} \otimes \alpha|1\rangle + \frac{|10\rangle}{\sqrt{2}} \otimes \beta|0\rangle + \frac{|11\rangle}{\sqrt{2}} \otimes \beta|1\rangle$$

We can easily prove that

$$
\begin{aligned}
\frac{|00\rangle}{\sqrt{2}} &= \frac{|\phi_+\rangle + |\phi_-\rangle}{2}, \\
\frac{|01\rangle}{\sqrt{2}} &= \frac{|\psi_+\rangle + |\psi_-\rangle}{2}, \\
\\
\frac{|10\rangle}{\sqrt{2}} &= \frac{|\psi_+\rangle - |\psi_-\rangle}{2}, \\
\frac{|11\rangle}{\sqrt{2}} &= \frac{|\phi_+\rangle - |\phi_-\rangle}{2},
\end{aligned}
$$

Reformulating Alice's qubits on Bell Basis:

$$|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{|\phi_+\rangle + |\phi_-\rangle}{2} \otimes \alpha|0\rangle + \frac{|\psi_+\rangle + |\psi_-\rangle}{2} \otimes \alpha|1\rangle + \frac{|\psi_+\rangle - |\psi_-\rangle}{2} \otimes \beta|0\rangle + \frac{|\phi_+\rangle - |\phi_-\rangle}{2} \otimes \beta|1\rangle$$

Separating the Bell Basis factors:

$$|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2}\left[|\phi_+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |\phi_-\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + |\psi_+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |\psi_-\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\right].$$

Next we express the four Bell states belonging to Alice using "Double Wedge" notation and the Pauli matrices σ_1, σ_3 :

$$
\begin{aligned}
|\phi_+\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\rangle\rangle = |\frac{\sigma_1^0\sigma_3^0}{\sqrt{2}}\rangle\rangle = |\frac{(\sigma_1^0\sigma_3^0)^T}{\sqrt{2}}\rangle\rangle \\
|\phi_-\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\rangle\rangle = |\frac{\sigma_1^0\sigma_3^1}{\sqrt{2}}\rangle\rangle = |\frac{(\sigma_1^0\sigma_3^1)^T}{\sqrt{2}}\rangle\rangle \\
\\
|\psi_+\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\rangle\rangle = |\frac{\sigma_1^1\sigma_3^0}{\sqrt{2}}\rangle\rangle = |\frac{(\sigma_1^1\sigma_3^0)^T}{\sqrt{2}}\rangle\rangle \\
|\psi_-\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\rangle\rangle = |\frac{\sigma_3^1\sigma_1^1}{\sqrt{2}}\rangle\rangle = |\frac{(\sigma_1^1\sigma_3^1)^T}{\sqrt{2}}\rangle\rangle
\end{aligned}
$$

The four possible cases Bob's qubit that involve the unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ are expressed as:

$$\begin{aligned}
\alpha|0\rangle + \beta|1\rangle &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\sigma_1^0 \sigma_3^0\right) |\psi\rangle \\
\alpha|0\rangle - \beta|1\rangle &= \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\sigma_1^0 \sigma_3^1\right) |\psi\rangle
\end{aligned}$$

$$\begin{aligned}
\alpha|1\rangle + \beta|0\rangle &= \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\sigma_1^1 \sigma_3^0\right) |\psi\rangle \\
\alpha|1\rangle - \beta|0\rangle &= \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\sigma_1^1 \sigma_3^1\right) |\psi\rangle
\end{aligned}$$

Combining these expressions for Alice's and Bob's qubits the initial state can be written as:

$$\begin{aligned}
|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= \frac{1}{2}(|\frac{(\sigma_1^0\sigma_3^0)^T}{\sqrt{2}}\rangle\rangle \otimes \sigma_1^0\sigma_3^0|\psi\rangle + |\frac{(\sigma_1^0\sigma_3^1)^T}{\sqrt{2}}\rangle\rangle \otimes \sigma_1^0\sigma_3^1|\psi\rangle \\
&\quad + |\frac{(\sigma_1^1\sigma_3^0)^T}{\sqrt{2}}\rangle\rangle \otimes \sigma_1^1\sigma_3^0|\psi\rangle + |\frac{(\sigma_1^1\sigma_3^1)^T}{\sqrt{2}}\rangle\rangle \otimes \sigma_1^1\sigma_3^1|\psi\rangle)
\end{aligned}$$

To simplify the notation let:

$$V_{\alpha\beta} = \frac{(\sigma_1^\alpha \sigma_3^\beta)^T}{\sqrt{2}}.$$

Our equation then gets the following equivalent forms

$$\begin{aligned}
|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= \frac{1}{2}\left(|V_{00}\rangle\rangle \otimes \sqrt{2}V_{00}^T|\psi\rangle + |V_{01}\rangle\rangle \otimes \sqrt{2}V_{01}^T|\psi\rangle + |V_{10}\rangle\rangle \otimes \sqrt{2}V_{10}^T|\psi\rangle + |V_{11}\rangle\rangle \otimes \sqrt{2}V_{11}^T|\psi\rangle\right) \\
|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= \frac{1}{\sqrt{2}}(|V_{00}\rangle\rangle \otimes V_{00}^T|\psi\rangle + |V_{01}\rangle\rangle \otimes V_{01}^T|\psi\rangle + |V_{10}\rangle\rangle \otimes V_{10}^T|\psi\rangle + |V_{11}\rangle\rangle \otimes V_{11}^T|\psi\rangle)
\end{aligned}$$

And finally

$$|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{\sqrt{2}} \sum_{\alpha,\beta} |V_{\alpha\beta}\rangle\rangle \otimes V_{\alpha\beta}^T|\psi\rangle. \tag{1}$$

We call this the **simplified teleportation equation .**

Next Alice performs a joint Bell measurement on her two qubits so the above state which is described by a sum of quantum states collapses into one term. To achieve this she acts with the projection operator on her two qubits.

$$P_{V_{\alpha'\beta'}} = |V_{\alpha'\beta'}\rangle\rangle\langle\langle V_{\alpha'\beta'}|$$

where $\alpha', b' \in \{0, 1\}$ can be considered as two classical bits which exported from the Bell measurement and indicate uniquely with which of four Bell Projection operator Alice acted on her qubits.

So the teleportation equation after the measurement collapses into the state:

$$|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \rightarrow (P_{V_{\alpha'\beta'}} \otimes I)(|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle)$$

Using the simplified teleportation equation

$$(P_{V_{\alpha'\beta'}} \otimes I)(|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle) = (|V_{\alpha'\beta'}\rangle\rangle\langle\langle V_{\alpha'\beta'}| \otimes I)\frac{1}{\sqrt{2}}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes V_{\alpha\beta}^T|\psi\rangle$$

Acting on the respectively qubits:

$$(P_{V_{\alpha'\beta'}} \otimes I)(|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle) = \frac{1}{\sqrt{2}}\sum_{\alpha,\beta}|V_{\alpha'\beta'}\rangle\rangle\underbrace{\langle\langle V_{\alpha'\beta'}||V_{\alpha\beta}\rangle\rangle}_{\delta_{\alpha\alpha'}\delta_{\beta\beta'}} \otimes V_{\alpha\beta}^T|\psi\rangle$$

Due to orthonormality of the set of matricies $V_{\alpha\beta}\ \forall\ \alpha,\beta \in \{0,1\}$ which is proved at Appendix A

$$\langle\langle V_{\alpha'\beta'}||V_{\alpha\beta}\rangle\rangle = \begin{Bmatrix} 1 \text{ if } a' = a \text{ and } \beta' = \beta \\ 0 \text{ if } \alpha' \neq a' \text{ or } \beta' \neq \beta \end{Bmatrix} = \delta_{\alpha\alpha',\beta\beta'}$$

Only one term will "survive" after the action of projection operator this one where $a' = a$ and $\beta' = \beta$

$$|System_{post-measurement}\rangle = \frac{1}{\sqrt{2}}|V_{\alpha'\beta'}\rangle\rangle \otimes V_{\alpha'\beta'}^T|\psi\rangle$$

$$|System_{post-measurement}\rangle = \frac{1}{2}|V_{\alpha'\beta'}\rangle\rangle \otimes \sigma_1^{\alpha'}\sigma_3^{\beta'}|\psi\rangle$$

Normalizing the final state after measurement we obtain:

$$|System_{post-measurement}\rangle = |V_{\alpha'\beta'}\rangle\rangle \otimes \sigma_1^{\alpha'}\sigma_3^{\beta'}|\psi\rangle$$

We define:

$$W_{\alpha',\beta'} = \sigma_1^{\alpha'}\sigma_3^{\beta'}$$

Bob's state vector after measurement is $W_{\alpha',\beta'}|\psi\rangle$. To retrieve the initial state $|\psi\rangle$ he needs to undo the unitary operator $W_{\alpha',\beta'}$. To this end he needs to know exponents α' and β' which are identified with two classical bits obtained from Alice's measurement, these two classical bits known to Alice and needs to forward them to Bob by classical channel (e.g telephone line).

When Bob receives these two classical bits he acts with the appropriate recovery operator to his qubit as shown in the table below:

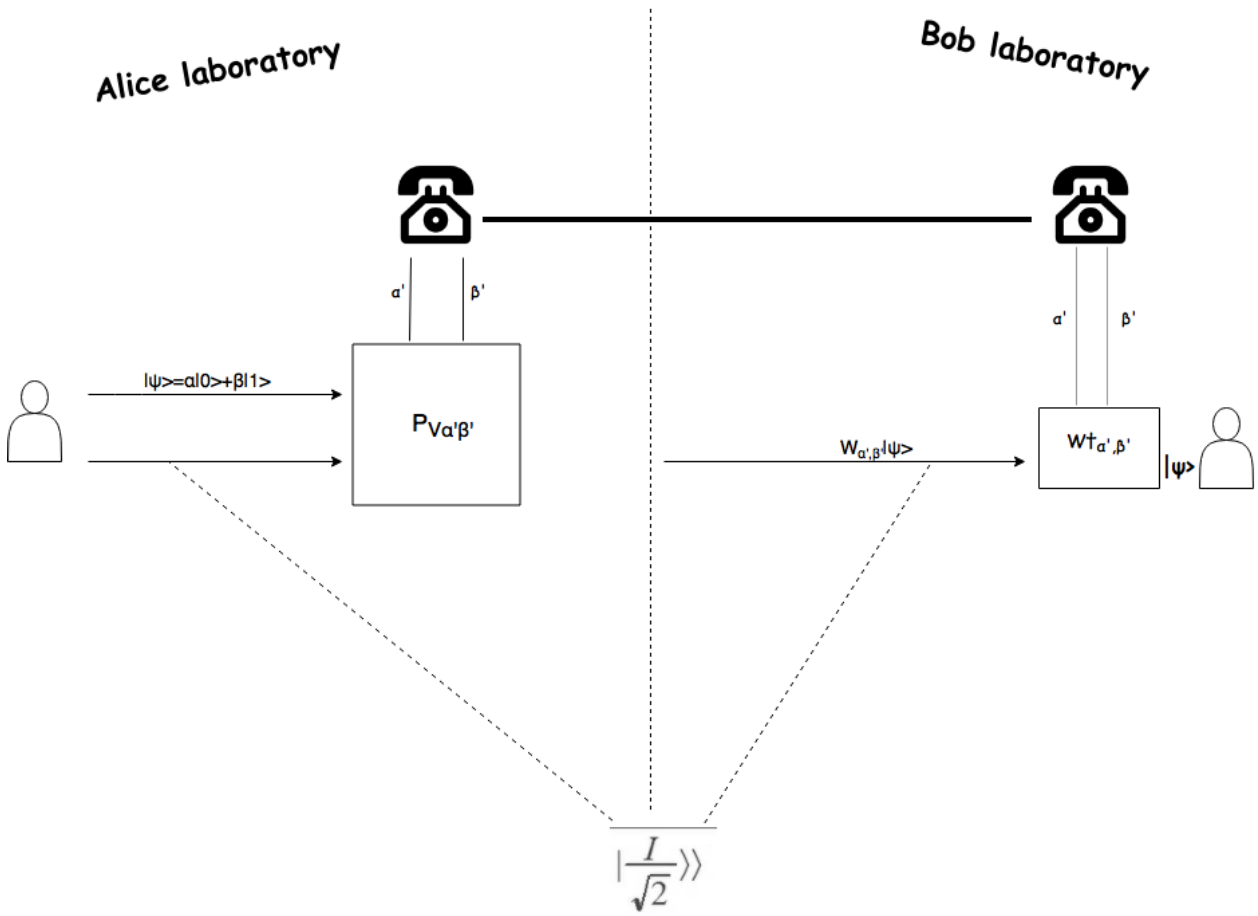
Cbit	Recovery Operator
$\alpha'\beta'$	$W^{\dagger}_{\alpha',\beta'} = \left(\sigma_1^{\alpha'}\sigma_3^{\beta'}\right)^{\dagger}$
00	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
01	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
10	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
11	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Finally Bob acting with the suitable operator he will obtain:

$$|\psi_{Bob}\rangle = W^{\dagger}_{\alpha',\beta'} W_{\alpha',\beta'}|\psi\rangle = I|\psi\rangle = |\psi\rangle.$$

where $|\psi\rangle$ is the initial quantum state which Alice wishes to teleport to Bob.

From resources perspective we achieved this state teleportation by using one pair of entangled qubits and classical communication of two bits.



The whole diagram of teleporting a quantum state. (Time evolves from left to right)

2.2 1-Qubit Gate Teleportation

Suppose that Alice wants to teleport the result of the action of one predefined qubit gate U_{2x2} on her qubit ($U|\psi\rangle$) to Bob. To achieve this she acts with U^T gate on her second qubit (the entangled one) [5] [6].

So the initial state of teleportation is reformulated as

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = |\psi\rangle \otimes \underbrace{\left(U^T \otimes \mathbf{1}_2\right)|\frac{I}{\sqrt{2}}\rangle\rangle}$$

due to identity:

$$(A \otimes B)|C\rangle\rangle = |ACB^T\rangle\rangle$$

It's a fact that:

$$\left(U^T \otimes \mathbf{1}_2\right)|\frac{I}{\sqrt{2}}\rangle\rangle = \left(\mathbf{1}_2 \otimes U\right)|\frac{I}{\sqrt{2}}\rangle\rangle = |\frac{U^T}{\sqrt{2}}\rangle\rangle$$

So our initial equation reads:

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes U)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle$$

Due to simplified teleportation equation introduced previously see equation (1)

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes U)\frac{1}{\sqrt{2}}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes V^T_{\alpha,\beta}|\psi\rangle$$

U gate acts on the third qubit

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{\sqrt{2}}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes UV^T_{\alpha,\beta}|\psi\rangle$$

Exploiting the fact that U is Unitary: $U^\dagger U = I$

$$\begin{aligned}(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= \frac{1}{\sqrt{2}}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes \underbrace{UV^T_{\alpha,\beta}U^\dagger}U|\psi\rangle \\ (\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle &= \frac{1}{\sqrt{2}}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes \frac{1}{\sqrt{2}}\underbrace{U\sigma_1^\alpha\sigma_3^\beta U^\dagger}U|\psi\rangle\end{aligned}$$

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes \underbrace{U\sigma_1^\alpha\sigma_3^\beta U^\dagger}U|\psi\rangle$$

So by defining the unitary operator:

$$W^U_{\alpha\beta} = U\sigma_1^\alpha\sigma_3^\beta U^\dagger$$

It leads to:

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^U_{\alpha\beta}U|\psi\rangle \tag{2}$$

This will be called the **simplified 1 qubit gate teleportation equation**

Next Alice performs a joint Bell measurement on her two qubits. Acting with the projection operator

$$P_{V_{\alpha'\beta'}} = |V_{\alpha'\beta'}\rangle\rangle\langle\langle V_{\alpha'\beta'}|$$

where $\alpha^{'}, b^{'} \in \{0,1\}$ can be considered as two classical bits which exported from measurement and indicate with which of four Projection Operator she acted

So the teleportation equation after measurement becomes

$$\frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^{U}_{\alpha\beta}U|\psi\rangle \rightarrow (P_{V_{\alpha'\beta'}} \otimes \mathbf{1_2})\frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^{U}_{\alpha\beta}U|\psi\rangle$$

$$(P_{V_{\alpha'\beta'}} \otimes \mathbf{1_2})\frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^{U}_{\alpha\beta}U|\psi\rangle = (|V_{\alpha'\beta'}\rangle\rangle\langle\langle V_{\alpha'\beta'}| \otimes \mathbf{1_2})\frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^{U}_{\alpha\beta}U|\psi\rangle$$

$$(P_{V_{\alpha'\beta'}} \otimes \mathbf{1_2})\frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^{U}_{\alpha\beta}U|\psi\rangle = \frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha'\beta'}\rangle\rangle\underbrace{\langle\langle V_{\alpha'\beta'}||V_{\alpha\beta}\rangle\rangle}_{\delta_{\alpha\alpha'}\delta_{\beta\beta'}} \otimes W^{U}_{\alpha\beta}U|\psi\rangle$$

Due to orthonormality of the set of matrices $V_{\alpha\beta}\forall\alpha,\beta \in\{0,1\}$

$$(P_{V_{\alpha'\beta'}} \otimes \mathbf{1_2})\frac{1}{2}\sum_{\alpha,\beta}|V_{\alpha\beta}\rangle\rangle \otimes W^{U}_{\alpha\beta}U|\psi\rangle = \frac{1}{2}|V_{\alpha'\beta'}\rangle\rangle \otimes W^{U}_{\alpha'\beta'}U|\psi\rangle$$

Normalizing our system after the measurement reads:

$$|System_{post-measurement}\rangle = |V_{\alpha'\beta'}\rangle\rangle \otimes W^{U}_{\alpha'\beta'}U|\psi\rangle$$

The qubit of Bob after measurement is in the state

$$|\psi_{Bob}\rangle = W^{U}_{\alpha'\beta'}U|\psi\rangle$$

To retrieve $U|\psi\rangle$ he has to undo the action of the unitary operator $W^{U}_{\alpha'\beta'}$ acting with the unitary recovery operator $W^{\dagger}_{\alpha'\beta'}$ on his qubit.

As a consequence:

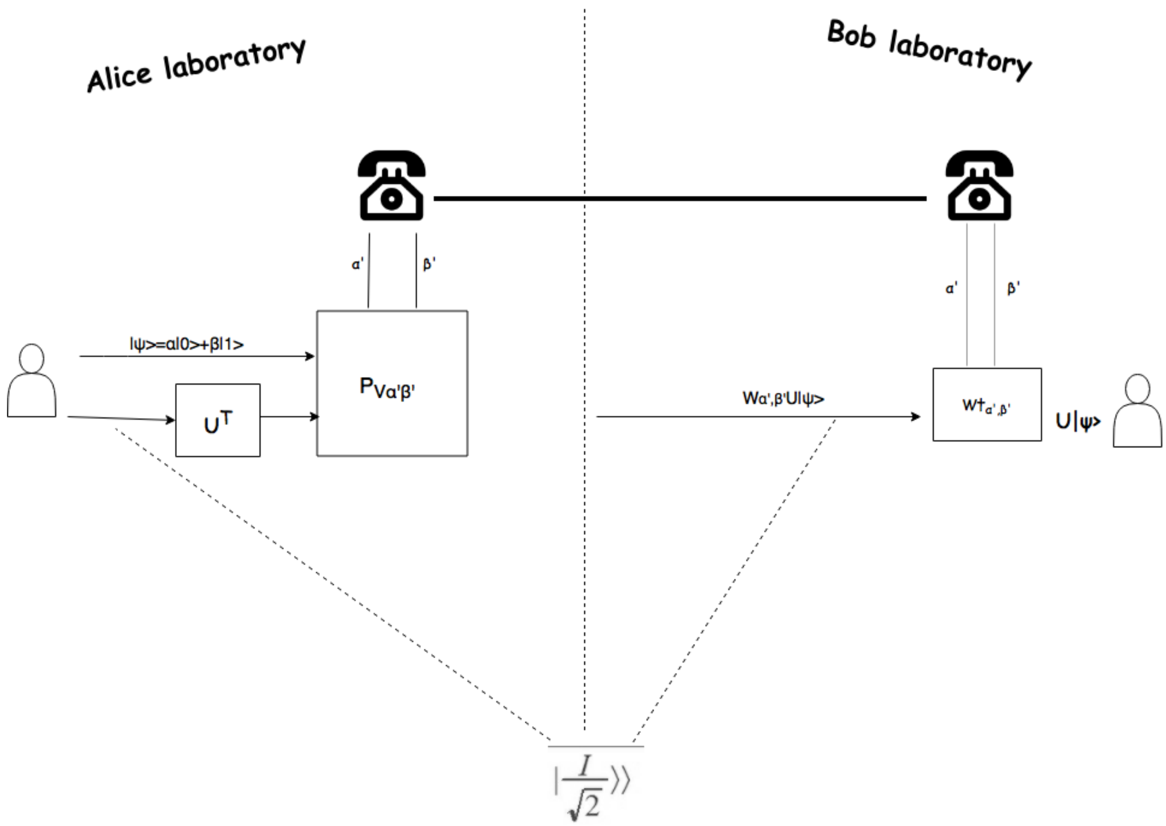
$$W^{\dagger}_{\alpha'\beta'}|\psi_{Bob}\rangle = W^{\dagger}_{\alpha'\beta'}W_{\alpha'\beta'}U|\psi\rangle = \mathbf{I}_2U|\psi\rangle = U|\psi\rangle.$$

$W_{a,\beta}$ is unitary as a product of unitary matrices

Below is a table with the recovery operators $W^{\dagger}_{\alpha'\beta'} = \left(U\sigma_1^{\alpha'}\sigma_3^{\beta'}U^{\dagger}\right)^{\dagger}$ that Bob has to act on his qubit for some common gates U (X, Y, Z, H) for the corresponding four possible results $(\alpha', \beta') \rightarrow \{00, 01, 10, 11\}$ obtained from the Bell measurement.

Cbit(α,β)	X	Y	Z	H
00	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
01	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
10	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
11	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

From resources perspective we achieved this using one pair of entangled qubits the action of a gate by Alice and classical communication of two bits.



The whole diagram of teleporting the action of one predetermined qubit gate on a quantum state. (Time evolves from left to right)

2.3 1-Qubit Gate only Teleportation

The gate only teleportation is a corollary of the gate teleportation. The purpose is to teleport a whole gate, not only it's action. To this end we will take advantage of the fact that the gate teleportation algorithm leaves unspecified the state $|\psi\rangle$.Suppose that we want to teleport a 2x2 unitary gate to Bob. We need a way to make possible to Bob to reconstruct the U gate local on his laboratory. To this end Alice has to send the action of the U gate to Bob on two predetermined basis vectors $|\psi_0\rangle, |\psi_1\rangle$ using the protocol we introduced above ("1-Qubit Gate Teleportation").

The initial qubits (basis vectors) that Alice has to send to Bob are e.g the canonical basis vectors $|\psi_0\rangle = |0\rangle$ and $|\psi_1\rangle = |1\rangle$.

We assume that Alice sends $U|\psi_0\rangle$ at the first one qubit gate teleportation and $U|\psi_1\rangle$ at the second one.

So the simplified teleportation equation in this case using the **1 qubit gate teleportation equation (2)** will take the form:

$$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)|i\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2}\sum_{\alpha,\beta} |V_{\alpha\beta}\rangle\rangle \otimes W^U_{\alpha\beta} U|i\rangle \tag{3}$$

where $i \in \{0,1\}$ for the first and second case respectively

This will be called the **simplified 1-qubit gate only teleportation equation.**

Therefore after the completion of the first one Qubit Gate Teleportation including recover operator by Bob.He will has in his possession the action of U on $|0\rangle$

$$|\psi_0'\rangle = U|\psi_0\rangle = U|0\rangle$$

Similarly after the second One Qubit Gate Teleportation Bob will has in his possession the action of U on $|1\rangle$

$$|\psi_1'\rangle = U|\psi_1\rangle = U|1\rangle$$

Now he is ready to reconstruct the U operator locally using these two vectors making some post processing calculations.

We have proved that the general Matrix representation of one qubit gate knowing the action of the gate on basis ($|\psi_0'\rangle = U|0\rangle, |\psi_1'\rangle = U|1\rangle$) is:

$$\begin{pmatrix} \langle 0|\psi_0'\rangle & \langle 0|\psi_1'\rangle \\ \langle 1|\psi_0'\rangle & \langle 1|\psi_1'\rangle \end{pmatrix}$$

where $|\psi_0'\rangle, |\psi_1'\rangle$ are the received vectors

The post-processing which Bob has to do is to calculate the inner products of the received vectors $|\psi_0'\rangle, |\psi_1'\rangle$ with the basis vectors $|0\rangle, |1\rangle$

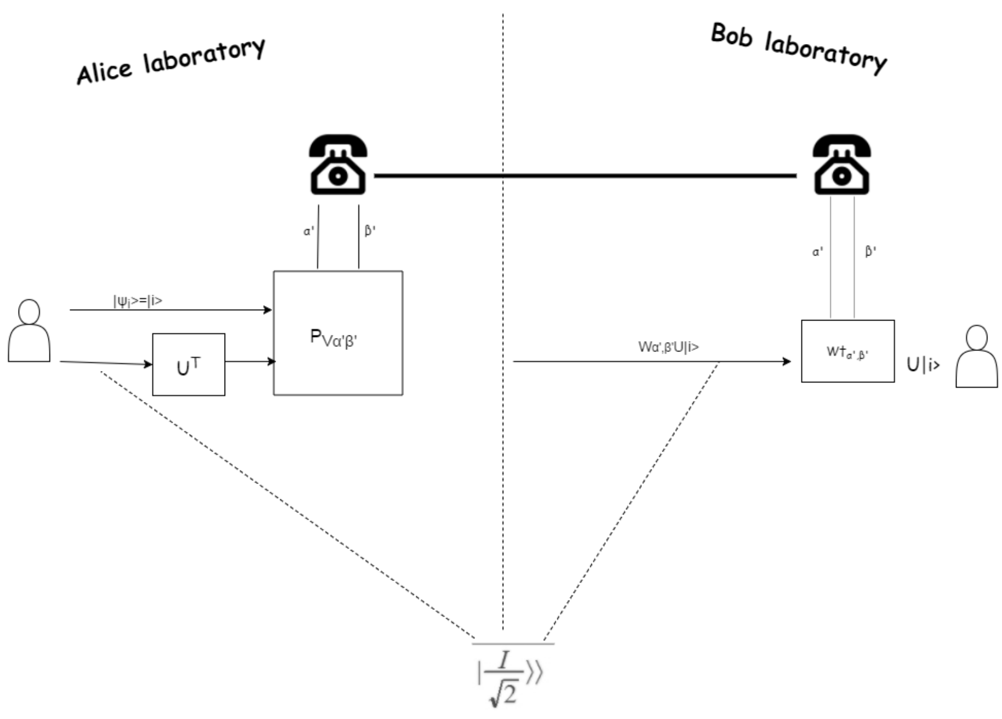
Explicitly:

$$
\begin{aligned}
a_{00} &= \langle 0|\psi'_0\rangle = \langle 0|U|0\rangle \\
a_{01} &= \langle 0|\psi'_1\rangle = \langle 0|U|1\rangle \\
a_{10} &= \langle 1|\psi'_0\rangle = \langle 1|U|0\rangle \\
a_{11} &= \langle 1|\psi'_1\rangle = \langle 1|U|1\rangle
\end{aligned}
$$

And construct the final matrix representation of the gate:

$$
\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}
$$

In conclusion, from resources perspective to teleport the whole 1-qubit gate you need to repeat 1-Qubit Gate teleportation twice which means: $2 * (1ebit + 2cbit) = 2ebits + 4cbits$.



The whole diagram of teleporting one predetermined qubit gate acting on a predetermined quantum state. (Time evolves from left to right)

Example:

Suppose that Alice wants to teleport to Bob the $U_{NOT} = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gate so he can act with it on his laboratory. To achieve this they have to follow the "1-Qubit Gate only Teleportation" algorithm.

According to the steps of protocol she initializes her qubit at state $|\psi_0\rangle = |0\rangle$ and acts on her second qubit with the U^T_{NOT}.

So the simplified 1-Qubit Gate only teleportation equation takes the form:

$$(\mathbf{1}_2 \otimes U^T_{NOT} \otimes \mathbf{1}_2)|0\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2}\sum_{\alpha,\beta} |V_{\alpha\beta}\rangle\rangle \otimes W^{U_{NOT}}_{\alpha\beta} U_{NOT}|0\rangle$$

Suppose that Alice performs a joint Bell measurement acting with:

$$P_{V_{01}} = |V_{01}\rangle\rangle\langle\langle V_{01}|$$

on her first two qubits. The outcome of this action is two classical bits specifically (0,1) and the state after measurement will be transformed to:

$$|system_{post-measurement}\rangle = |V_{01}\rangle\rangle \otimes W^{U_{NOT}}_{01} U_{NOT}|0\rangle$$

According to the table above with the recovery operators for the case that the gate is X and the two classical bits are (0,1) Bob has to act with

$$W^{\dagger}_{0,1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

to retrieve:

$$|\psi'_0\rangle = W^{\dagger}_{0,1} W_{0,1} U_{NOT}|0\rangle = U_{NOT}|0\rangle$$

which is equal to (according to truth table of NOT):

$$|\psi'_0\rangle = |1\rangle$$

Similarly, they repeat this protocol with initialized state by Alice $|\psi_1\rangle = |1\rangle$ and the action of U^T_{NOT} on her second qubit.

So after the recover by Bob his qubit will be in state

$$|\psi'_1\rangle = U_{NOT}|1\rangle = |0\rangle$$

Using these two vectors $(|\psi'_0\rangle, |\psi'_1\rangle)$ Bob has to do some post processing calculations to construct the matrix representation of the gate.

Specifically:

$$\begin{pmatrix} \langle 0|\psi'_0\rangle & \langle 0|\psi'_1\rangle \\ \langle 1|\psi'_0\rangle & \langle 1|\psi'_1\rangle \end{pmatrix}$$

substituting $|\psi'_i\rangle$ with the corresponding values:

$$\begin{pmatrix} \langle 0|1\rangle & \langle 0|0\rangle \\ \langle 1|1\rangle & \langle 1|0\rangle \end{pmatrix}$$

due to orthonormality of the canonical basis vectors, the final matrix representation is:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Which is the matrix representation of U_{NOT} gate tha Alice wanted to Teleport.

2.4 2-Qubit Entangled State Teleportation

In this subsection we will expand the "1-Qubit State Telportation" protocol trying to make possible to Alice to teleport an arbitrary entangled state of two qubits. A straightforward generalization of 1-Qubit state teleportation shows that any bipartite system can be teleported using two maximally entangled pairs and four bits of classical information, but our goal is to weaken the requirements on the resources needed to achieve the purpose (e.g the number of the prepared and shared entangled qubits).

Suppose that Alice wants to teleport the arbitrary entangled state [8]:

$$|\psi_{send}\rangle = a|00\rangle + \beta|11\rangle.$$

The whole quantum system consists of one shared Bell state between Alice & Bob in our case

$$|\phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

on the first and second Hillbert spaces.

An arbitrary entangled state that Alice wants to teleport

$$|\psi_{send}\rangle = a|00\rangle + \beta|11\rangle$$

located on the third and fourth qubit

Finally, an ancillary fifth qubit initialized at $|0\rangle$ belongs to Bob

Summing-up our systems consists of a five qubits space $|\rangle_1 \otimes |\rangle_2 \otimes |\rangle_3 \otimes |\rangle_4 \otimes |\rangle_5$

(where the subscript denotes the space that each qubit belongs) from which 1,3,4 belongs to Alice and 2,5 belongs to Bob.

Our system using Dirac notation can be written as:

$$\begin{aligned} |\psi_{sys}\rangle &= \frac{(|00\rangle_{12} + |11\rangle_{12})}{\sqrt{2}} \otimes (a|00\rangle_{34} + \beta|11\rangle_{34}) \otimes |0\rangle_5 \\ |\psi_{sys}\rangle &= \frac{\alpha|00000\rangle}{\sqrt{2}} + \frac{\beta|00110\rangle}{\sqrt{2}} + \frac{\alpha|11000\rangle}{\sqrt{2}} + \frac{\beta|11110\rangle}{\sqrt{2}} \end{aligned}$$

Separating two qubits which belongs to Alice (1,3) we can rewrite our system

$$|\psi_{sys}\rangle = \frac{|00\rangle_{13}}{\sqrt{2}} \otimes a|000\rangle_{245} + \frac{|01\rangle_{13}}{\sqrt{2}} \otimes \beta|010\rangle_{245} + \frac{|10\rangle_{13}}{\sqrt{2}} \otimes a|100\rangle_{245} + \frac{|11\rangle_{13}}{\sqrt{2}} \otimes \beta|110\rangle_{245}$$

Using the equations which connect the Bell states with the canonical basis we can reformulate our state to:

$$\begin{aligned}|\psi_{sys}\rangle &= \frac{|\phi_+\rangle_{13}+|\phi_-\rangle_{13}}{2}\otimes a|000\rangle_{245}+\frac{|\psi_+\rangle_{13}+|\psi_-\rangle_{13}}{2}\otimes\beta|010\rangle_{245}\\ &+\frac{|\psi_+\rangle_{13}-|\psi_-\rangle_{13}}{2}\otimes a|100\rangle_{245}+\frac{|\phi_+\rangle_{13}-|\phi_-\rangle_{13}}{2}\otimes\beta|110\rangle_{245}\end{aligned}$$

Taking the Bell states as common factor:

$$\begin{aligned}|\psi_{sys}\rangle &= \frac{1}{2}(|\phi_+\rangle_{13}\otimes(a|000\rangle_{245}+\beta|110\rangle_{245})+|\phi_-\rangle_{13}\otimes(a|000\rangle_{245}-\beta|110\rangle_{245})\\ &+|\psi_+\rangle_{13}\otimes(a|100\rangle_{245}+\beta|010\rangle_{245})+|\psi_-\rangle_{13}\otimes(-a|100\rangle_{245}+\beta|010\rangle_{245}))\end{aligned}$$

According to the protocol Alice performs a Hadamard gate on fourth qubit:

$$\begin{aligned}|\psi_{sys}\rangle &= (I_2\otimes I_2\otimes I_2\otimes H\otimes I_2)\frac{1}{2}(|\phi_+\rangle_{13}\otimes(a|000\rangle_{245}+\beta|110\rangle_{245})+|\phi_-\rangle_{13}\otimes(a|000\rangle_{245}-\beta|110\rangle_{245})\\ &+|\psi_+\rangle_{13}\otimes(a|100\rangle_{245}+\beta|010\rangle_{245})+|\psi_-\rangle_{13}\otimes(-a|100\rangle_{245}+\beta|010\rangle_{245}))\end{aligned}$$

so our system transforms to

$$|\psi_{sys}\rangle = \begin{array}{c}\frac{1}{2}\{|\phi_+\rangle_{13}\otimes(a|0\rangle_2(\frac{|0\rangle_4+|1\rangle_4}{\sqrt{2}})|0\rangle_5+\beta|1\rangle_2(\frac{|0\rangle_4-|1\rangle_4}{\sqrt{2}})|0\rangle_5)\\ +|\phi_-\rangle_{13}\otimes(a|0\rangle_2(\frac{|0\rangle_4+|1\rangle_4}{\sqrt{2}})|0\rangle_5-\beta|1\rangle_2(\frac{|0\rangle_4-|1\rangle_4}{\sqrt{2}})|0\rangle_5)\\ +|\psi_+\rangle_{13}\otimes(a|1\rangle_2(\frac{|0\rangle_4+|1\rangle_4}{\sqrt{2}})|0\rangle_5+\beta|0\rangle_2(\frac{|0\rangle_4-|1\rangle_4}{\sqrt{2}})|0\rangle_5)\\ +|\psi_-\rangle_{13}\otimes(-a|1\rangle_2(\frac{|0\rangle_4+|1\rangle_4}{\sqrt{2}})|0\rangle_5+\beta|0\rangle_2(\frac{|0\rangle_4-|1\rangle_4}{\sqrt{2}})|0\rangle_5\}\end{array}$$

Which is equal to:

$$|\psi_{sys}\rangle = \begin{array}{c}\frac{1}{2\sqrt{2}}\{|\phi_+\rangle_{13}\otimes(a|000\rangle_{245}+a|010\rangle_{245}+\beta|100\rangle_{245}-\beta|110\rangle_{245})\\ +|\phi_-\rangle_{13}\otimes(a|000\rangle_{245}+a|010\rangle_{245}-\beta|100\rangle_{245}+\beta|110\rangle_{245})\\ +|\psi_+\rangle_{13}\otimes(a|100\rangle_{245}+a|110\rangle_{245}+\beta|000\rangle_{245}-\beta|010\rangle_{245})\\ +|\psi_-\rangle_{13}\otimes(-a|100\rangle_{245}-a|110\rangle_{245}+\beta|000\rangle_{245}-\beta|010\rangle_{245})\}\end{array}$$

Bob as a receiver performs a CNOT gate on his two qubits the second and the fifth of our system having as a control qubit the second one and as a target qubit the fifth one. After the action our system takes the form:

$$|\psi_{sys}\rangle = \begin{array}{c}\frac{1}{2\sqrt{2}}\{|\phi_+\rangle_{13}\otimes(a|000\rangle_{245}+a|010\rangle_{245}+\beta|101\rangle_{245}-\beta|111\rangle_{245})\\ +|\phi_-\rangle_{13}\otimes(a|000\rangle_{245}+a|010\rangle_{245}-\beta|101\rangle_{245}+\beta|111\rangle_{245})\\ +|\psi_+\rangle_{13}\otimes(a|101\rangle_{245}+a|111\rangle_{245}+\beta|000\rangle_{245}-\beta|010\rangle_{245})\\ +|\psi_-\rangle_{13}\otimes(-a|101\rangle_{245}-a|111\rangle_{245}+\beta|000\rangle_{245}-\beta|010\rangle_{245})\}\end{array}$$

According to the protocol Alice is going to measure the fourth qubit, to make it explicitly we factor out the fourth qubit:

$$|\psi_{sys}\rangle = \begin{array}{c}\frac{1}{2\sqrt{2}}\{|\phi_+\rangle_{13}(|0\rangle_4(a|00\rangle_{25}+\beta|11\rangle_{25})+|1\rangle_4\,(a|00\rangle_{25}-\beta|11\rangle_{25}))\\ +|\phi_-\rangle_{13}(|0\rangle_4(a|00\rangle_{25}-\beta|11\rangle_{25})+|1\rangle_4\,(a|00\rangle_{25}+\beta|11\rangle_{25}))\\ +|\psi_+\rangle_{13}(|0\rangle_4(a|11\rangle_{25}+\beta|00\rangle_{25})+|1\rangle_4(a|11\rangle_{25}-\beta|00\rangle_{25}))\\ +|\psi_-\rangle_{13}\,(|0\rangle_4(-a|11\rangle_{25}+\beta|00\rangle_{25})+|1\rangle_4(-a|11\rangle_{25}-\beta|00\rangle_{25}))\}\end{array}$$

Anticipating the quantum measurement we express the state in the first and third qubit in the $|V_{\alpha\beta}\rangle\rangle$ notation and the state at Bob's side in terms of $|\psi_{send}\rangle = a|00\rangle + \beta|11\rangle$:

$$|\psi_{sys}\rangle = \begin{array}{l} \frac{1}{2\sqrt{2}}\{|V_{00}\rangle\rangle_{13} \otimes (|0\rangle_4(\sigma_3^0\sigma_1^0 \otimes \sigma_3^0\sigma_1^0)|\psi_{send}\rangle\rangle_{25} + |1\rangle_4(\sigma_3^1\sigma_1^0 \otimes \sigma_3^0\sigma_1^0)|\psi_{send}\rangle\rangle_{25} \\ +|V_{01}\rangle\rangle_{13} \otimes (|0\rangle_4(\sigma_3^1\sigma_1^0 \otimes \sigma_3^0\sigma_1^0)|\psi_{send}\rangle\rangle_{25} + |1\rangle_4(\sigma_3^0\sigma_1^0 \otimes \sigma_3^0\sigma_1^0)|\psi_{send}\rangle\rangle_{25} \\ +|V_{10}\rangle\rangle_{13} \otimes |0\rangle_4(\sigma_3^0\sigma_1^1 \otimes \sigma_3^0\sigma_1^1)|\psi_{send}\rangle\rangle_{25} + |1\rangle_4(\sigma_3^1\sigma_1^1 \otimes \sigma_3^0\sigma_1^1)|\psi_{send}\rangle\rangle_{25} \\ +|V_{11}\rangle\rangle_{13} \otimes |0\rangle_4(\sigma_3^1\sigma_1^1 \otimes \sigma_3^0\sigma_1^1)|\psi_{send}\rangle\rangle_{25} + |1\rangle_4(\sigma_3^0\sigma_1^1 \otimes \sigma_3^0\sigma_1^1)|\psi_{send}\rangle\rangle_{25}\} \end{array}$$

Obviously now the state $|y_{send}\rangle\rangle$ is obstructed to reach Bob by the unitaries.

For simplicity we define $W_{\alpha,\beta,\gamma}$

α,β,γ	$W_{\alpha,\beta,\gamma}$
000	$I \otimes I$
001	$\sigma_3 \otimes I$
010	$\sigma_3 \otimes I$
011	$I \otimes I$
100	$\sigma_1 \otimes \sigma_1$
101	$\sigma_3\sigma_1 \otimes \sigma_1$
110	$\sigma_3\sigma_1 \otimes \sigma_1$
111	$\sigma_1 \otimes \sigma_1$

So, we can rewrite our whole system in a very compact form:

$$|\psi_{sys}\rangle = \frac{1}{2\sqrt{2}} \sum_{\alpha,\beta} \sum_{\gamma} |V_{\alpha,\beta}\rangle\rangle_{13} \otimes |\gamma\rangle_4 \otimes W_{\alpha,\beta,\gamma}|\psi_{send}\rangle\rangle_{25} \quad (4)$$

where $\alpha,\beta,\gamma \in \{0,1\}$

This will be called the **simplified entangled state teleportation.**

Alice performs two measurements on her three qubits one Bell measurement on her first and third qubit acting with $P_{V_{\alpha',\beta'}} = |V_{\alpha',\beta'}\rangle\rangle\langle\langle V_{a',\beta'}|$ so she exports two classical bits (α',β') which identify uniquely the state of these two qubits and one canonical measurement on her fourth qubit acting with $P_{\gamma'} = |\gamma'\rangle\langle\gamma'|$ from which she exports one classical bit (γ')

The state after the mesaurement will collapse in

$$|\psi_{sys\_final}\rangle = |V_{\alpha',\beta'}\rangle\rangle_{13} \otimes |\gamma'\rangle_4 \otimes W_{\alpha',\beta',\gamma'}|\psi_{send}\rangle\rangle_{25}$$

So the only action that Bob has to do after receiving the three classical bits (α',β',γ') from Alice e.g via a telephone line is to retrieve the initial entangled state $|\psi_{send}\rangle\rangle$ undoing the action of $W_{\alpha',\beta',\gamma'}$. To this end he acts with the appropriate recovery operator $W^{\dagger}_{\alpha',\beta',\gamma'}$ according to the table below on his two qubits. Which leads his state in:

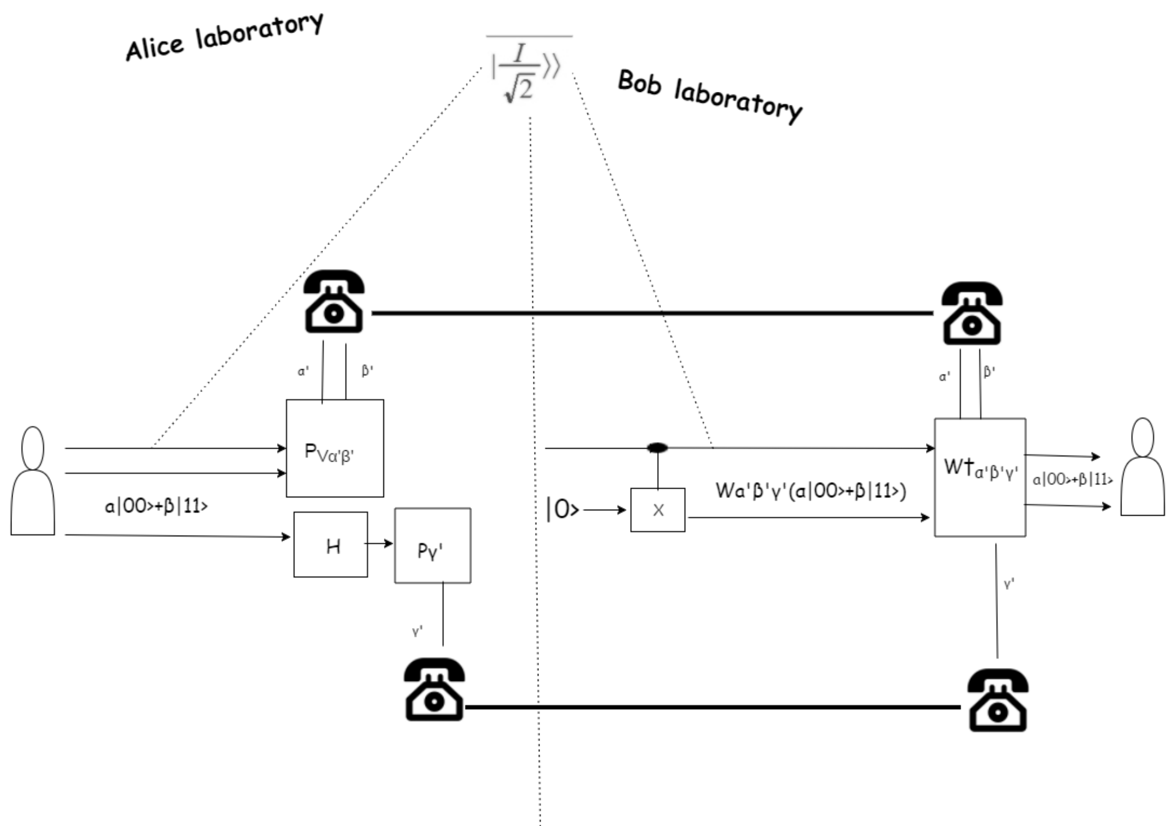
$$|\psi_{Bob}\rangle_{25} = W^{\dagger}_{\alpha',\beta',\gamma'} W_{\alpha',\beta',\gamma'}|\psi_{send}\rangle\rangle_{25} = I|\psi_{send}\rangle\rangle_{25} = |\psi_{send}\rangle\rangle_{25}$$

We note that $W_{\alpha,\beta,\gamma}$ is unitary for any triplet (a,β,γ) as a product of unitary matrices $(W^{\dagger}_{\alpha',\beta',\gamma'} W_{\alpha,\beta,\gamma} = I)$

Recovery operators depending on $(\alpha', \beta', \gamma')$:

α', β', γ'	$W^{\dagger}_{\alpha', \beta', \gamma'}$
000	$I \otimes I$
001	$\sigma_3 \otimes I$
010	$\sigma_3 \otimes I$
011	$I \otimes I$
100	$\sigma_1 \otimes \sigma_1$
101	$\sigma_1 \sigma_3 \otimes \sigma_1$
110	$\sigma_1 \sigma_3 \otimes \sigma_1$
111	$\sigma_1 \otimes \sigma_1$

From resources point of view this algorithm achieved its goal i.e. to teleport a single entangled pair by using as much resources as the original single state teleportation (1 maximally entangled pair a Bell measurment and 2 cbits), plus additional resources firstly at the sender's side, consisting of a Hadamard transformation, a single qubit measurement and the sending of a cbit, and secondly at the receiver's side, consisting of an additional qubit and a set of two-qubit unitary trasnfomations.



The whole diagram of teleporting one arbitrary entangled state (Time evolves from left to right)

2.5 2-Qubit Gate Teleportation

Suppose this time that Alice wants to teleport to Bob the action of a two qubit conditional (non-local) control-U gate U_{4x4} on two arbitrary qubits

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

and

$$\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

To achieve this we assume that Alice and Bob have prepared and shared two pairs of entangled qubits, which leads to the fact that the initial state of our system is:

$$|initial\rangle = |\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \otimes |\phi\rangle$$

so our whole system consists of six qubits$|\rangle_1 \otimes |\rangle_2 \otimes |\rangle_3 \otimes |\rangle_4 \otimes |\rangle_5 \otimes |\rangle_6$, from which {1,2,5,6} belongs to Alice and {3,4} belongs to Bob.

We know that the general form of a condition control-U gate (if the first qubit-target is equal to $|1\rangle$ act with U_{2x2} gate on the second one) is:

$$U_{CU} = P_0 \otimes I + P_1 \otimes U$$

We define $A_0 = P_0$, $A_1 = P_1$ and $B_0 = I$,$B_1 = U$, so the general form of U_{CU} can be rewritten in a compact form:

$$U_{CU} = A_0 \otimes B_0 + A_1 \otimes B_1 = \sum_{i=0}^{1} A_i \otimes B_i$$

embedding of U_{CU} gate into six-fold tensor product of state space. In particular we embed into 2nd and 5th space the controlled and target space respectively.

To achieve this we reformulate our gate to:

$$U_{25} = \sum_{i=0}^{1} I \otimes A_i \otimes I \otimes I \otimes B_i \otimes I$$

In order to apply the operator identity successfully Alice performs the U_{25}^T gate on her 2,5 qubits (on the particles of the shared entangled qubits) which are located in the total space of six qubits reads:

$$U_{25}^T = \sum_i \left(I \otimes A_i^T \otimes I \otimes I \otimes B_i^T \otimes I\right)$$

Acting on the initial state leads to the following equalities

$$(\sum_i I \otimes A_i^T \otimes I \otimes I \otimes B_i^T \otimes I)(|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \otimes |\phi\rangle)$$

due to:

$$A^T \otimes I|\frac{\mathbf{1}}{\sqrt{2}}\rangle\rangle = \ I \otimes A|\frac{\mathbf{1}}{\sqrt{2}}\rangle\rangle\rangle\rangle = |\frac{A^T}{\sqrt{2}}\rangle\rangle$$

and

$$I \otimes A^T|\frac{\mathbf{1}}{\sqrt{2}}\rangle\rangle = \ A \otimes I|\frac{\mathbf{1}}{\sqrt{2}}\rangle\rangle = |\frac{A}{\sqrt{2}}\rangle\rangle$$

Our equation can be rewritten

$$(\sum_i I\otimes A_i^T\otimes I\otimes I\otimes B_i^T\otimes I)(|\psi\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|\phi\rangle = (\sum_i I\otimes I\otimes A_i\otimes B_i\otimes I\otimes I)(|\psi\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|\phi\rangle)$$

At this point we observed that we have accomplished to transfer the A's and B's of the controlled gate U from Alice's spaces {2,5} to Bob's spaces {3,4} respectively

using **simplified teleportation equation (1)** twice:

$$\left(\sum_i I \otimes I \otimes A_i \otimes B_i \otimes I \otimes I\right) \frac{1}{\sqrt{2}} \sum_{\alpha,\beta} |V_{\alpha\beta}\rangle\rangle \otimes V^T_{\alpha,\beta}|\psi\rangle \otimes \frac{1}{\sqrt{2}} \sum_{\gamma,\delta} V^T_{\gamma,\delta}|\phi\rangle \otimes |V_{\gamma\delta}\rangle\rangle$$

We can further simplify the way we write the action of the gate knowing that

$$U_{CU} = \sum_{i=0}^{1} A_i \otimes B_i$$

So,

$$(I \otimes I \otimes U_{CU} \otimes I \otimes I)\frac{1}{2} \sum_{\alpha,\beta} |V_{\alpha\beta}\rangle\rangle \otimes V^T_{\alpha,\beta}|\psi\rangle \otimes \sum_{\gamma,\delta} V^T_{\gamma,\delta}|\phi\rangle \otimes |V_{\gamma\delta}\rangle\rangle$$

$$(I \otimes I \otimes U_{CU} \otimes I \otimes I)\frac{1}{2} \sum_{\alpha,\beta} |V_{\alpha\beta}\rangle\rangle \otimes \frac{\sigma_1^\alpha \sigma_3^\beta}{\sqrt{2}}|\psi\rangle \otimes \sum_{\gamma,\delta} \frac{\sigma_1^\gamma \sigma_3^\delta}{\sqrt{2}}|\phi\rangle \otimes |V_{\gamma\delta}\rangle\rangle$$

Hence:

$$(I \otimes I \otimes U_{CU} \otimes I \otimes I)\frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} |V_{\alpha\beta}\rangle\rangle \otimes \sigma_1^\alpha \sigma_3^\beta|\psi\rangle \otimes \sigma_1^\gamma \sigma_3^\delta|\phi\rangle \otimes |V_{\gamma\delta}\rangle\rangle$$

Exploiting the properties of tensor products on Bob's side we reformulate:

$$(I \otimes I \otimes U_{CU} \otimes I \otimes I)\frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} |V_{\alpha\beta}\rangle\rangle \otimes (\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)\,(|\psi\rangle \otimes |\phi\rangle) \otimes |V_{\gamma\delta}\rangle\rangle$$

Gate acts on the respect qubits

$$\frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} |V_{\alpha\beta}\rangle\rangle \otimes U_{CU}(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)\,(|\psi\rangle \otimes |\phi\rangle) \otimes |V_{\gamma\delta}\rangle\rangle$$

Taking advantage of the unitarity of gates ($U^\dagger_{CU} U_{CU} = I_4$)

$$\frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} |V_{\alpha\beta}\rangle\rangle \otimes \underbrace{U_{CU}(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)U^\dagger_{CU}}\, U_{CU}\,(|\psi\rangle \otimes |\phi\rangle) \otimes |V_{\gamma\delta}\rangle\rangle$$

Define:

$$W^{CU}_{a,\beta,\gamma,\delta} = U_{CU}(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)U^\dagger_{CU}$$

Our system takes the form:

$$(\sum_i I \otimes A_i^T \otimes I \otimes I \otimes B_i^T \otimes I)|\psi\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \otimes |\frac{I}{\sqrt{2}}\rangle\rangle \otimes |\phi\rangle = \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} |V_{\alpha\beta}\rangle\rangle \otimes W^{CU}_{a,\beta,\gamma,\delta} U_{CU}\,(|\psi\rangle \otimes |\phi\rangle) \otimes |V_{\gamma\delta}\rangle\rangle \tag{5}$$

This will be called the **simplified 2 qubit gate teleportation equation**

In order our system to collapse in one specific term Alice Performs two Bell measurements on her four qubits acting with

$$P_{V a'\beta'} = |V_{\alpha'\beta'}\rangle\rangle\langle\langle V_{a'\beta'}|$$

on the first and second qubit, and with:

$$P_{V \gamma'\delta'} = |V_{\gamma'\delta'}\rangle\rangle\langle\langle V_{\gamma'\delta'}|$$

on the fifth and sixth qubit.

where $\alpha^{'}, \beta', \gamma', \delta' \in \{0,1\}$ can be considered as four classical bits which exported from the two Bell measurements and indicate uniquely with which of four Bell Projection operator Alice acted on her qubits in each case.

So our initial state transforms to

$$(|V_{\alpha'\beta'}\rangle\rangle\langle\langle V_{a'\beta'}| \otimes I \otimes I \otimes |V_{\gamma'\delta'}\rangle\rangle\langle\langle V_{\gamma'\delta'}|)\frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta}|V_{\alpha\beta}\rangle\rangle \otimes W^{CU}_{a,\beta,\gamma,\delta}U_{CU}\left(|\psi\rangle \otimes |\phi\rangle\right) \otimes |V_{\gamma\delta}\rangle\rangle$$

due to orthonormality of Bell basis

$$\langle\langle V_{a'\beta'}||V_{\alpha\beta}\rangle\rangle = \delta_{aa',\beta\beta'} \text{ and } \langle\langle V_{\gamma'\delta'}||V_{\gamma\delta}\rangle\rangle = \delta_{\gamma\gamma',\delta\delta'}$$

Our system "collapses" to the below specific state:

$$\frac{1}{4}|V_{\alpha'\beta'}\rangle\rangle \otimes W^{CU}_{a',\beta',\gamma',\delta'}U_{CU}\left(|\psi\rangle \otimes |\phi\rangle\right) \otimes |V_{\gamma'\delta'}\rangle\rangle$$

Normalizing:

$$|System_{post-measurement}\rangle = |V_{\alpha'\beta'}\rangle\rangle \otimes W^{CU}_{a',\beta',\gamma',\delta'}U_{CU}\left(|\psi\rangle \otimes |\phi\rangle\right) \otimes |V_{\gamma'\delta'}\rangle\rangle$$

After the measurement Bob's qubits are in the state

$$|\psi_{Bob}\rangle = W^{CU}_{a',\beta',\gamma',\delta'}U_{CU}\left(|\psi\rangle \otimes |\phi\rangle\right)$$

In order for Bob to retrieve the target state he has to act with the recovery operator

$$W^{\dagger}_{a',\beta',\gamma',\delta'} = \left(U_{CU}(\sigma_1^{\alpha}\sigma_3^{\beta} \otimes \sigma_1^{\gamma}\sigma_3^{\delta})U^{\dagger}_{CU}\right)^{\dagger}$$

Which is a unitary as a product of unitary operators

$$W^{\dagger}_{a',\beta',\gamma',\delta'}|\psi_{Bob}\rangle = \underbrace{(W^{\dagger}_{a',\beta',\gamma',\delta'})W_{a',\beta',\gamma',\delta'}}U_{CU}\left(|\psi\rangle \otimes |\phi\rangle\right) = I_4 U_{CU}\left(|\psi\rangle \otimes |\phi\rangle\right) = U_{CU}|\psi\phi\rangle$$

Which is the outcome of the action of the U_{CU} gate on the initial arbitrary two qubits.

From resources perspective we achieved this using two pairs of entangled qubits the action of a gate by Alice and classical communication of four bits.

As a case we consider that we want to teleport the action of control-NOT gate on two arbitrary qubits and we calculate the recovery operators for this case.

We know that this gate can be written:

$$U_{CNOT} = P_0 \otimes I + P_1 \otimes \sigma_1$$

Recovery Operators are given by:

$$\begin{aligned}
W^{U_{CNOT}}_{\alpha,\beta,\gamma,\delta} &= U_{CNOT}(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta) U^\dagger_{CNOT} \\
W^{U_{CNOT}}_{\alpha,\beta,\gamma,\delta} &= (P_0 \otimes I + P_1 \otimes \sigma_1)(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)(P_0 \otimes I + P_1 \otimes \sigma_1)^\dagger \\
W^{U_{CNOT}}_{\alpha,\beta,\gamma,\delta} &= (P_0 \otimes I + P_1 \otimes \sigma_1)(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)\left(P_0^\dagger \otimes I^\dagger + P_1^\dagger \otimes \sigma_1^\dagger\right)
\end{aligned}$$

Projection & Pauli Operators are Hermitian

$$\begin{aligned}
W^{U_{CNOT}}_{\alpha,\beta,\gamma,\delta} &= (P_0 \otimes I + P_1 \otimes \sigma_1)(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)(P_0 \otimes I + P_1 \otimes \sigma_1) \\
W^{U_{CNOT}}_{\alpha,\beta,\gamma,\delta} &= \left(P_0 \sigma_1^a \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta + P_1 \sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1 \sigma_1^\gamma \sigma_3^\delta\right)(P_0 \otimes I + P_1 \otimes \sigma_1)
\end{aligned}$$

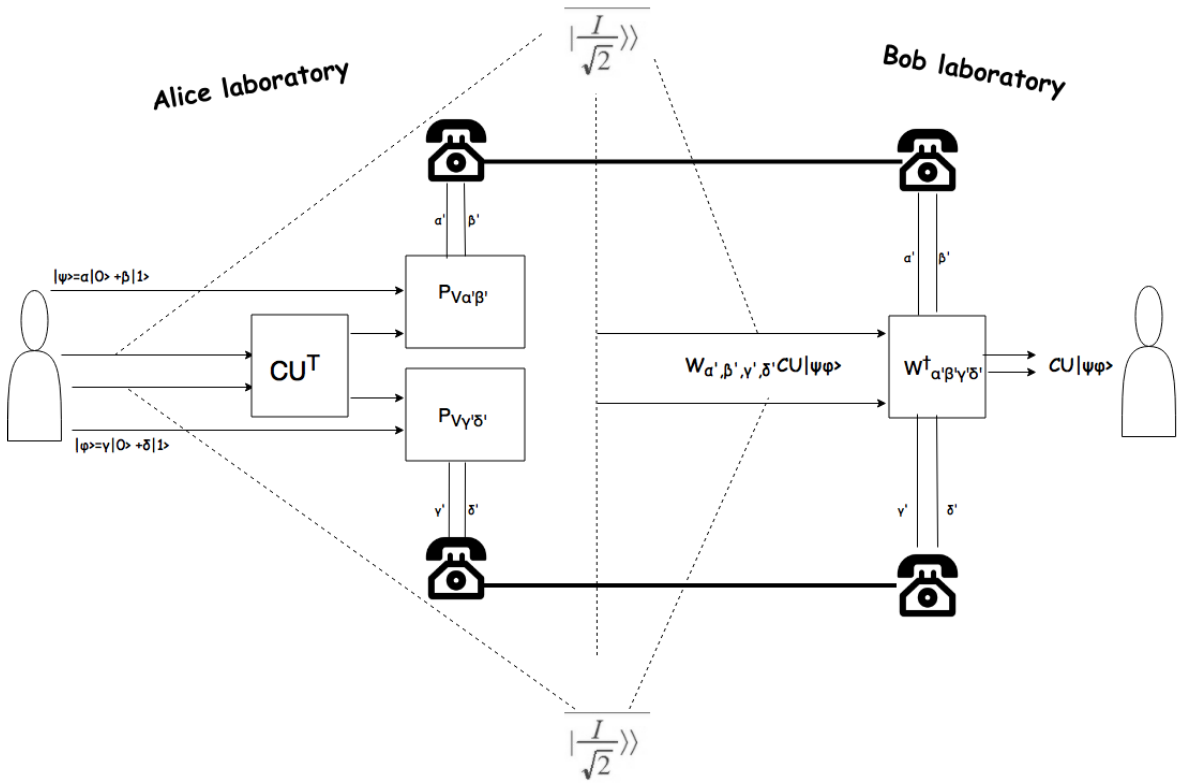
$$W^{U_{CNOT}}_{\alpha,\beta,\gamma,\delta} = P_0 \sigma_1^a \sigma_3^\beta P_0 \otimes \sigma_1^\gamma \sigma_3^\delta + P_0 \sigma_1^a \sigma_3^\beta P_1 \otimes \sigma_1^\gamma \sigma_3^\delta \sigma_1 + P_1 \sigma_1^\alpha \sigma_3^\beta P_0 \otimes \sigma_1 \sigma_1^\gamma \sigma_3^\delta + P_1 \sigma_1^\alpha \sigma_3^\beta P_1 \otimes \sigma_1 \sigma_1^\gamma \sigma_3^\delta \sigma_1$$

So taking the Hermitian of the matrix: $W^\dagger_{\alpha,\beta,\gamma,\delta}$

$$W^\dagger_{\alpha,\beta,\gamma,\delta} = P_0 \sigma_3^\beta \sigma_1^a P_0 \otimes \sigma_3^\delta \sigma_1^\gamma + P_1 \sigma_3^\beta \sigma_1^a P_0 \otimes \sigma_1 \sigma_3^\delta \sigma_1^\gamma + P_0 \sigma_3^\beta \sigma_1^\alpha P_1 \otimes \sigma_3^\delta \sigma_1^\gamma \sigma_1 + P_1 \sigma_3^\beta \sigma_1^\alpha P_1 \otimes \sigma_1 \sigma_3^\delta \sigma_1^\gamma \sigma_1$$

Calculating the recovery operators for all the cases of four classical bits we have these possible options:

$\alpha\beta \backslash \gamma\delta$	00	01	10	11
00	$\mathbf{1} \otimes \mathbf{1}$	$\sigma_3 \otimes \sigma_3$	$\mathbf{1} \otimes \sigma_1$	$\sigma_3 \otimes i\sigma_2$
01	$\sigma_3 \otimes I$	$I \otimes \sigma_3$	$\sigma_3 \otimes \sigma_1$	$I \otimes i\sigma_2$
10	$\sigma_1 \otimes \sigma_1$	$i\sigma_2 \otimes i\sigma_2$	$\sigma_1 \otimes I$	$i\sigma_2 \otimes \sigma_3$
11	$i\sigma_2 \otimes \sigma_1$	$\sigma_1 \otimes i\sigma_2$	$i\sigma_2 \otimes I$	$\sigma_1 \otimes \sigma_3$



The whole diagram of teleporting the action of one predetermined two qubit gate on two arbitrary qubits. (Time evolves from left to right)

2.6 2-Qubit Gate only Teleportation

The 2-Qubit Gate only Teleportation is a corollary of 2-qubit gate teleportation. The purpose is to teleport a whole gate, not only it.s action. On this end we will take advantage of the fact that the gate teleportation algorithm leaves unspecified the state $|\psi\phi\rangle$. Suppose that we want to teleport a 4x4 unitary gate to Bob. We need a way to make possible to Bob to reconstruct the U_{4x4} gate local on his laboratory. To this end Alice has to send the action of the U_{4x4} gate to Bob on four predetermined basis vectors $|\psi_{00}\rangle, |\psi_{01}\rangle, |\psi_{10}\rangle, |\psi_{11}\rangle$ using the protocol we introduced above ("2-Qubit Gate Teleportation") four times.

The initial qubits (basis vectors) that Alice has to send to Bob are e.g the canonical basis vectors $|\psi_{00}\rangle = |00\rangle, |\psi_{01}\rangle = |01\rangle, |\psi_{10}\rangle = |10\rangle, |\psi_{11}\rangle = |11\rangle$

We assume that Alice sends $U_{4x4}|00\rangle$, at the first 2-Qubit Gate Teleportation $U_{4x4}|01\rangle$ on the second one $U_{4x4}|10\rangle$ on the third one and $U_{4x4}|11\rangle$ on the last one.

So the simplified teleportation equation in this case using the **2 qubit gate teleportation equation (4)** will take the form:

$$(\sum_i I\otimes A_i^T\otimes I\otimes I\otimes B_i^T\otimes I)|i\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|j\rangle = \frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta}|V_{\alpha\beta}\rangle\rangle\otimes W^{CU}_{a,\beta,\gamma,\delta}U_{CU}\left(|i\rangle\otimes|j\rangle\right)\otimes|V_{\gamma\delta}\rangle\rangle \tag{6}$$

where $i, j \in \{0, 1\}$

This will be called the **simplified 2-qubit gate only teleportation equation.**

Therefore, after the completion of the first two qubit gate teleportation ($i = 0$ and $j = 0$) including recovery operation by Bob. He will has in his possession the action of U_{CU} on $|00\rangle$

$$|\psi'_{00}\rangle = U_{CU}|00\rangle$$

Similarly, after the completion of the second two qubit gate teleportation ($i = 0$ and $j = 1$). He will has in his possession

$$|\psi'_{01}\rangle = U_{CU}|01\rangle$$

Till the completion of the fourth two qubit gate teleportation($i = 1$ and $j = 1$). Where he will has in his possession

$$|\psi'_{11}\rangle = U_{CU}|11\rangle$$

Now he is ready to reconstruct the U_{CU} operator locally using these four vectors making some post processing calculations.

We have proved that the general matrix representation of two qubit gate knowing the action of the gate on basis ($|\psi'_{00}\rangle = U_{CU}|00\rangle, |\psi'_{01}\rangle = U_{CU}|01\rangle, |\psi'_{10}\rangle = U_{CU}|10\rangle, |\psi'_{11}\rangle = U_{CU}|11\rangle$) is:

$$\begin{pmatrix} \langle 00||\psi'_{00}\rangle & \langle 00||\psi'_{01}\rangle & \langle 00||\psi'_{10}\rangle & \langle 00||\psi'_{11}\rangle \\ \langle 01||\psi'_{00}\rangle & \langle 01||\psi'_{01}\rangle & \langle 01||\psi'_{10}\rangle & \langle 01||\psi'_{11}\rangle \\ \langle 10||\psi'_{00}\rangle & \langle 10||\psi'_{01}\rangle & \langle 10||\psi'_{10}\rangle & \langle 10||\psi'_{11}\rangle \\ \langle 11||\psi'_{00}\rangle & \langle 11||\psi'_{01}\rangle & \langle 11||\psi'_{10}\rangle & \langle 11||\psi'_{11}\rangle \end{pmatrix}$$

Where $|\psi'_{00}\rangle, |\psi'_{01}\rangle, |\psi'_{10}\rangle, |\psi'_{11}\rangle$ are the received vectors.

The post processing which Bob have to do is to calculate the inner products of the received vectors $|\psi'_{00}\rangle, |\psi'_{01}\rangle, |\psi'_{10}\rangle, |\psi'_{11}\rangle$ with the basis vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

Explicitly:

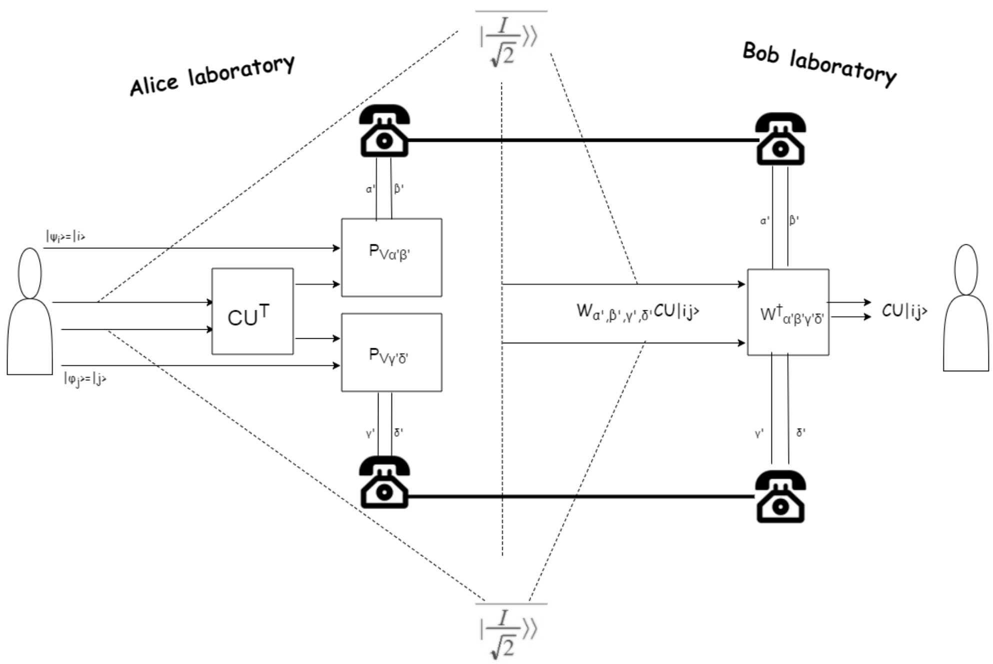
$$\alpha_{ijkl} = \langle ij||\psi'_{kl}\rangle = \langle ij|U_{CU}|\psi'_{kl}\rangle$$

Where i,j,k,l $\in \{0, 1\}$

And construct the final matrix representation of the gate

$$\begin{pmatrix} \alpha_{0000} & \alpha_{0001} & \alpha_{0010} & \alpha_{0011} \\ \alpha_{0100} & \alpha_{0101} & \alpha_{0110} & \alpha_{0111} \\ \alpha_{1000} & \alpha_{1001} & \alpha_{1010} & \alpha_{1011} \\ \alpha_{1100} & \alpha_{1101} & \alpha_{1110} & \alpha_{1111} \end{pmatrix}$$

In conclusion, from resources perspective to teleport a whole 2-qubit gate you need to repeat four times the 2-qubit gate teleportation which means $4 * (2$ ebit+4 cbit$) = 8$ ebits+16 cbits.



The whole diagram of teleporting one predetermined two qubits gate acting on a predetermined quantum state. (Time evolves from left to right)

Example:

Suppose that Alice wants teleport to Bob the U_{Bell} gate so he can create maximally entangled qubits on his laboratory. To achieve this they have to follow the "2Qubit Gate only Teleportation" protocol.

According to the steps of protocol, at first she initializes her qubits at state $|\psi_{00}\rangle = |00\rangle$ and acts with U_{Bell}^T on the other two qubits (the entangled ones).

So the **2-qubit gate only teleportation equation (6),** takes the form:

$$(\sum_i I\otimes A_i^T\otimes I\otimes I\otimes B_i^T\otimes I)|0\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|\frac{I}{\sqrt{2}}\rangle\rangle\otimes|0\rangle = \frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta}|V_{\alpha\beta}\rangle\rangle\otimes W_{a,\beta,\gamma,\delta}^{Bell}U_{Bell}\left(|0\rangle\otimes|0\rangle\right)\otimes|V_{\gamma\delta}\rangle\rangle$$

After the two Bell measurements Alice exports four classical bits $(a',\beta',\gamma',\delta')$ and our system collapses to the below state:

$$|System_{post-measurement}\rangle = |V_{\alpha'\beta'}\rangle\rangle \otimes W_{a',\beta',\gamma',\delta'}^{Bell}U_{Bell}|00\rangle \otimes |V_{\gamma'\delta'}\rangle\rangle$$

Where

$$W_{a,\beta,\gamma,\delta}^{Bell} = U_{Bell}(\sigma_1^{\alpha}\sigma_3^{\beta} \otimes \sigma_1^{\gamma}\sigma_3^{\delta})U_{Bell}^{\dagger}$$

So Bob's state is:

$$|y_{Bob}\rangle = W_{a',\beta',\gamma',\delta'}^{Bell}U_{Bell}|00\rangle$$

Knowing the four classical bits $(a',\beta',\gamma',\delta')$ and looking at the table below he acts with $W_{a',\beta',\gamma',\delta'}^{\dagger}$ to recover

$$\begin{aligned}
|\psi_{00}'\rangle &= W_{a',\beta',\gamma',\delta'}^{\dagger}W_{a',\beta',\gamma',\delta'}^{Bell}U_{Bell}|00\rangle \\
|\psi_{00}'\rangle &= IU_{Bell}|00\rangle \\
|\psi_{00}'\rangle &= U_{Bell}|00\rangle = |\phi_+\rangle
\end{aligned}$$

Similarly they repeat the process three more times (on initialized vectors $|01\rangle, |10\rangle, |11\rangle$) and he retrieves

$$\begin{aligned}
|\psi_{01}'\rangle &= U_{Bell}|01\rangle = |\psi_+\rangle \\
|\psi_{10}'\rangle &= U_{Bell}|10\rangle = |\phi_-\rangle \\
|\psi_{11}'\rangle &= U_{Bell}|11\rangle = |\psi_-\rangle
\end{aligned}$$

The matrix representation in this case is

$$\begin{pmatrix}
\langle 00||\phi_+\rangle & \langle 00||\psi_+\rangle & \langle 00||\phi_-\rangle & \langle 00||\psi_-\rangle \\
\langle 01||\phi_+\rangle & \langle 01||\psi_+\rangle & \langle 01||\phi_-\rangle & \langle 01||\psi_-\rangle \\
\langle 10|||\phi_+\rangle & \langle 10||\psi_+\rangle & \langle 10||\phi_-\rangle & \langle 10||\psi_-\rangle \\
\langle 11|||\phi_+\rangle & \langle 11||\psi_+\rangle & \langle 11||\phi_-\rangle & \langle 11|||\psi_-\rangle
\end{pmatrix}$$

Calculating these inner products the final matrix which Bob will have in his possession is

$$\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix}$$

which is the matrix representation of the Bell Operator.

Let's calculate the Recovery operators for this case (Bell Operator):

We know that the Bell Operator circuit in Dirac notation is described by this equation

$$\begin{aligned} U_{Bell} &= U_{CNOT}(H \otimes I) \\ U_{Bell} &= (P_0 \otimes I + P_1 \otimes \sigma_1)(H \otimes I) \\ U_{Bell} &= P_0 H \otimes I + P_1 H \otimes \sigma_1 \end{aligned}$$

Hence, Recovery Operators $W_{\alpha,\beta,\gamma,\delta}$ are:

$$\begin{aligned} W_{\alpha,\beta,\gamma,\delta} &= U_{Bell}(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta) U_{Bell}^\dagger \\ W_{\alpha,\beta,\gamma,\delta} &= (P_0 H \otimes I + P_1 H \otimes \sigma_1)\,(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)\,(P_0 H \otimes I + P_1 H \otimes \sigma_1)^\dagger \end{aligned}$$

Projection & Pauli & Hadamard operators are Hermitian

$$W_{\alpha,\beta,\gamma,\delta} = (P_0 H \otimes I + P_1 H \otimes \sigma_1)\,(\sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta)\,(H P_0 \otimes I + H P_1 \otimes \sigma_1)$$

Exploiting tensor's identities

$$\begin{aligned} W_{\alpha,\beta,\gamma,\delta} &= \left(P_0 H \sigma_1^a \sigma_3^\beta \otimes \sigma_1^\gamma \sigma_3^\delta + P_1 H \sigma_1^\alpha \sigma_3^\beta \otimes \sigma_1 \sigma_1^\gamma \sigma_3^\delta\right)(H P_0 \otimes I + H P_1 \otimes \sigma_1) \\ W_{\alpha,\beta,\gamma,\delta} &= (P_0 H \sigma_1^a \sigma_3^\beta H P_0 \otimes \sigma_1^\gamma \sigma_3^\delta + P_0 H \sigma_1^a \sigma_3^\beta H P_1 \otimes \sigma_1^\gamma \sigma_3^\delta \sigma_1 \\ &\quad + P_1 H \sigma_1^\alpha \sigma_3^\beta H P_0 \otimes \sigma_1 \sigma_1^\gamma \sigma_3^\delta + P_1 H \sigma_1^\alpha \sigma_3^\beta H P_1 \otimes \sigma_1 \sigma_1^\gamma \sigma_3^\delta \sigma_1) \end{aligned}$$

Taking the Hermitian to find the final recovery operator $W_{\alpha,\beta,\gamma,\delta}^\dagger$

$$\begin{aligned} W_{\alpha,\beta,\gamma,\delta}^\dagger = &\Big(P_0 H \sigma_3^\beta \sigma_1^a H P_0 \otimes \sigma_3^\delta \sigma_1^\gamma + P_1 H \sigma_3^\beta \sigma_1^a H P_0 \otimes \sigma_1 \sigma_3^\delta \sigma_1^\gamma \\ &+ P_0 H \sigma_3^\beta \sigma_1^\alpha H P_1 \otimes \sigma_3^\delta \sigma_1^\gamma \sigma_1 + P_1 H \sigma_3^\beta \sigma_1^\alpha H P_1 \otimes \sigma_1 \sigma_3^\delta \sigma_1^\gamma \sigma_1\Big) \end{aligned}$$

Calculating the recovery operators for all the cases of four classical bits we have these possible options:

$\alpha\beta \backslash \gamma\delta$	00	01	10	11
00	$\mathbf{1} \otimes \mathbf{1}$	$\sigma_3 \otimes \sigma_3$	$\mathbf{1} \otimes \sigma_1$	$\sigma_3 \otimes i\sigma_2$
01	$\sigma_1 \otimes \sigma_1$	$i\sigma_2 \otimes i\sigma_2$	$\sigma_1 \otimes I$	$i\sigma_2 \otimes \sigma_3$
10	$\sigma_3 \otimes I$	$I \otimes \sigma_3$	$\sigma_3 \otimes \sigma_1$	$I \otimes i\sigma_2$
11	$(-i\sigma_2) \otimes \sigma_1$	$\sigma_1 \otimes (-i\sigma_2)$	$(-i\sigma_2) \otimes I$	$\sigma_1 \otimes (-\sigma_3)$

3 Conclusions

Summing up in this thesis we investigated six algorithms for teleportation of quantum resources trying to keep a common representation on all of them making it more familiar to our readers. Central to this construction is the derivation for each case of the equation that we have called simplified teleportation equation. They summmarized below

1-Qubit State Teleportation
$\|\psi\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{\sqrt{2}} \sum_{\alpha,\beta} \|V_{\alpha\beta}\rangle\rangle \otimes V_{\alpha\beta}^T \|\psi\rangle.$
1-Qubit Gate Teleportation
$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)\|\psi\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2} \sum_{\alpha,\beta} \|V_{\alpha\beta}\rangle\rangle \otimes W_{\alpha\beta}^U U \|\psi\rangle$
1-Qubit Gate only Teleportation
$(\mathbf{1}_2 \otimes U^T \otimes \mathbf{1}_2)\|i\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle = \frac{1}{2} \sum_{\alpha,\beta} \|V_{\alpha\beta}\rangle\rangle \otimes W_{\alpha\beta}^U U \|i\rangle$
2-Qubit Entangled State Teleportation
$\|\psi_{sys}\rangle = \frac{1}{2\sqrt{2}} \sum_{\alpha,\beta} \sum_{\gamma} \|V_{\alpha,\beta}\rangle\rangle_{13} \otimes \|\gamma\rangle_4 \otimes W_{\alpha,\beta,\gamma} \|\psi_{send}\rangle\rangle_{25}$
2-Qubit Gate Teleportation
$(\sum_i I \otimes A_i^T \otimes I \otimes I \otimes B_i^T \otimes I)\|\psi\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle \otimes \|\phi\rangle$
$= \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} \|V_{\alpha\beta}\rangle\rangle \otimes W_{a,\beta,\gamma,\delta}^{CU} U_{CU} (\|\psi\phi\rangle) \otimes \|V_{\gamma\delta}\rangle\rangle$
2-Qubit Gate only Teleportation
$(\sum_i I \otimes A_i^T \otimes I \otimes I \otimes B_i^T \otimes I)\|i\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle \otimes \|\frac{I}{\sqrt{2}}\rangle\rangle \otimes \|j\rangle$
$= \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} \|V_{\alpha\beta}\rangle\rangle \otimes W_{a,\beta,\gamma,\delta}^{CU} U_{CU} (\|ij\rangle) \otimes \|V_{\gamma\delta}\rangle\rangle$

As we can easily check one equation is extension of the other with some modifications. Moreover all of them have a common way of representation. Also it is notable to observe that the information and the gates that we want to teleport transferred from spaces that belongs to Alice at the beginning to these one that belong to Bob at the end.

After the measurements on the simplified teleportation equations we are exporting some amount of classical information (cbits). A summarization of the resources (quatum-classical information) needed for each protocol in a table form exists below:

	1-Qubit	2-Qubits
State	1ebit 2cbits	1ebit 3cbits
Action	1ebit 2cbits	2ebits 4cbits
Gate	2ebits 4cbits	8ebits 16cbits

The prospects of this work are multiple and some of them are summarized below as follows:

• It would be feasible to extend these quantum algorithms to cases that our quantum systems are not represented by qubits but from larger dimensional state vectors such as qutrits,qudits and so on.

• Furthermore it would be interested to extend these algorithms to the cases of three partite state teleportation such as GHZ states and/or teleportation of 3-qubits gates such as Toffoli and Fredkin gates.

• It would be a step towards a more realistic treatment to relax the ideal mathematical assumptions that we have introduced and proceed to analyse the behaviour of our algorithms under the existence of noise and also try to quantify the performance of them.

• Last but not least, it would be desirable to modify the algorithms in order to accommodate teleportation of other quantum objects such as quantum channels.

4 Appendices

4.1 Appendix A:Orthonormal basis

In this appendix we are going to prove that canonical basis and Bell Basis are Orthonormal.

The basis vectors for canonical base which are a spanning set for the vector space C^2 is the set:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This set of vectors is orthogonal if and only if: $\langle 0||1\rangle = 0$
Actually:

$$\langle 0||1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

which means that vectors $|0\rangle$,$|1\rangle$ are orthogonal

This set of vectors is normal if and only if $\langle 0||0\rangle = 1$ and $\langle 1||1\rangle = 1$
Actually:

$$\langle 0||0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

and

$$\langle 1||1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

Summing up $|0\rangle, |1\rangle$ constitute an orthonormal set which means:

$$\langle i||j\rangle = \delta_{ij}$$

Also the tensor products of these vectors constitute a spanning set for the vector space C^4 the set is:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This set of vectors is orthogonal if and only if: $\langle ij||kl\rangle = 0$ for $i \neq k$ or $j \neq l$
Actually:

$$\langle 00||01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle 00||10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle 00||11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\langle 01||10\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

etc. for the other cases
which means that vectors $|00\rangle$,$|01\rangle$, $|10\rangle$,$|11\rangle$ are orthogonal

This set of vectors is normal if and only if

$$\langle 00||00\rangle = 1, \langle 01||01\rangle = 1, \langle 10||10\rangle = 1, \langle 11||11\rangle = 1$$

Actually:

$$\langle 00||00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$\langle 01||01\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$\langle 10||10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$\langle 11||11\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

which means that vectors $|00\rangle$,$|01\rangle$, $|10\rangle$,$|11\rangle$ are normal

Summing up $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ constitute an orthonormal set which means:

$$\langle ij||kl\rangle = \delta_{ik,,jl}$$

With a similar way we are going to prove that the Bell state vectors constitute a spanning set for the vector space C^4 the set:

$$|\phi_+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|\phi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|\psi_+\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$|\psi_-\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Orhogonality if an only if $\langle\phi_\pm||\psi_\pm\rangle = 0$ and $\langle\phi_+||\phi_-\rangle = 0$ and $\langle\psi_+||\psi_-\rangle = 0$ are all the possible

Actually:

$$\langle\phi_+|\phi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

$$\langle\phi_+|\psi_+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = 0$$

$$\langle\phi_+|\psi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = 0$$

$$\langle\psi_+|\psi_-\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = 0$$

which means that vectors $|\phi_+\rangle$,$|\phi_-\rangle$, $|\psi_+\rangle$,$|\psi-\rangle$ are orthogonal

This set of vectors is normal if and only if

$$\langle\phi_+||\phi_+\rangle = 1, \langle\phi_-||\phi_-\rangle = 1, \langle\psi_+||\psi_+\rangle = 1, \langle\psi_-||\psi_-\rangle = 1$$

Actually:

$$\langle\phi_+|\phi_+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1$$

$$\langle\phi_-|\phi_-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = 1$$

$$\langle\psi_+|\psi_+\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = 1$$

$$\langle\psi_+|\psi_+\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = 1$$

which means that vectors $|\phi_+\rangle$,$|\phi_-\rangle$,$|\psi_+\rangle$,$|\psi_-\rangle$ are normal.

Summing up $|\phi_+\rangle, |\phi_-\rangle, |\psi_+\rangle, |\psi_=\rangle$ constitute an orthonormal set.

Finally, we will prove also that $|V_{a,\beta}\rangle\rangle$ where $\alpha, \beta \in \{0,1\}$ are orthonormal which means that $\langle\langle|V_{a',\beta'}||V_{a,\beta}\rangle\rangle = \delta_{aa',\beta\beta'}$ something that is very useful in this thesis especially when Alice performs Bell Measurements

This set is orthogonal if and only if: $\langle\langle V_{a\beta}||V_{a'\beta'}\rangle\rangle = 0$ for $\alpha \neq \alpha'$ or $\beta \neq \beta'$

At section 1.5 "Double-Wedge" Notation we have prove the identity

$$\langle\langle A||B\rangle\rangle = Tr[A^\dagger B]$$

which is called trace inner product.

This set of matricies is orthogonal if and only if: $\langle V_{a',\beta'}||V_{a,\beta}\rangle = 0$ for $\alpha \neq \alpha'$ or $\beta \neq \beta'$

Actually:

$$\langle\langle V_{00}||V_{01}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = 0$$

$$\langle\langle V_{00}||V_{10}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0$$

$$\langle\langle V_{00}||V_{11}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = 0$$

$$\langle\langle V_{10}||V_{11}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = 0$$

etc. for the other cases

which means that matrices $|V_{00}\rangle\rangle, |V_{01}\rangle\rangle, |V_{10}\rangle\rangle, |V_{11}\rangle\rangle$ are orthogonal

This set of matricies is normal if and only if

$$\langle\langle V_{00}||V_{00}\rangle\rangle = 1, \langle\langle V_{01}||V_{01}\rangle\rangle = 1, \langle\langle V_{10}||V_{10}\rangle\rangle = 1, \langle\langle V_{11}||V_{11}\rangle\rangle = 1$$

Actually:

$$\langle\langle V_{00}||V_{00}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 1$$

$$\langle\langle V_{01}||V_{01}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 1$$

$$\langle\langle V_{10}||V_{10}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = 1$$

$$\langle\langle V_{11}||V_{11}\rangle\rangle = Tr\left[\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = 1$$

which means that matricies $|V_{00}\rangle\rangle, |V_{01}\rangle\rangle, |V_{10}\rangle\rangle, |V_{11}\rangle\rangle$ are normal.
Summing up $|V_{00}\rangle\rangle, |V_{01}\rangle\rangle, |V_{10}\rangle\rangle, |V_{11}\rangle\rangle$ constitute an orthonormal set.
which means:

$$\langle\langle V_{\alpha\beta}||V_{\alpha'\beta'}\rangle\rangle = \delta_{\alpha\alpha',,\beta\beta'}$$

4.2 Appendix B:Reduced Density Operator of Bell states

As we have proved at the section "1.7.2 Reduced Density Operator of Bipartite Systems".

Knowing the matrix representation of a bipartite quantum system

$$\rho_{AB} = \begin{pmatrix} \rho_{0000} & \rho_{0001} & \rho_{0100} & \rho_{0101} \\ \rho_{0010} & \rho_{0011} & \rho_{0110} & \rho_{0111} \\ \rho_{1000} & \rho_{1001} & \rho_{1100} & \rho_{1101} \\ \rho_{1010} & \rho_{1011} & \rho_{1110} & \rho_{1111} \end{pmatrix}$$

Grouping the above 4x4 matrix in four 2x2matrices with obvious identification:

$$\rho_{AB} = \begin{pmatrix} \boldsymbol{\rho}_{00} & \boldsymbol{\rho}_{01} \\ \boldsymbol{\rho}_{10} & \boldsymbol{\rho}_{11} \end{pmatrix}$$

The Reduced Density operators of this bipartitte system are given by

$$\rho_A = \begin{pmatrix} Tr(\boldsymbol{\rho}_{00}) & Tr(\boldsymbol{\rho}_{01}) \\ Tr(\boldsymbol{\rho}_{10}) & Tr(\boldsymbol{\rho}_{11}) \end{pmatrix}$$

and

$$\rho_B = (\boldsymbol{\rho}_{00} + \rho_{11})$$

In this appendix we will prove that each of the Bell states has

$$\rho_A = \rho_B = \frac{I_2}{2}$$

Explicitly:
For $|\phi_+\rangle$:

$$\rho_{AB} = |\phi_+\rangle\langle\phi_+| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\rho_A = \begin{pmatrix} Tr\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

and

$$\rho_B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

For $|\phi_-\rangle$:

$$\rho_{AB} = |\phi_-\rangle\langle\phi_-| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\rho_A = \begin{pmatrix} Tr\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

and

$$\rho_B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

For $|\psi_+\rangle$:

$$\rho_{AB} = |\psi_+\rangle\langle\psi_+| = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_A = \begin{pmatrix} Tr\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

and

$$\rho_B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

For $|\psi_-\rangle$:

$$\rho_{AB} = |\psi_-\rangle\langle\psi_=| = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_A = \begin{pmatrix} Tr\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}$$

and

$$\rho_B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{I}{2}.$$

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