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Three-Dimensional Elastic Analysis of Rock Excavations by Using the g2 Constant Displacement Discontinuity Method

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Abstract

A fast computational code is presented that is dedicated for the elastic analysis of three-dimensional excavations and cracks in rocks. The problem is solved on the boundaries that are discretized with a new triangular leaf constant displacement discontinuity element with one collocation point. The creation of the new triangular element was inspired from Mindlin's special version of grade-2 or strain-gradient elasticity theory (second gradient of displacement, g2). This element is characterized by a much better measure of the average stress at the center of gravity of the triangular element compared to that of the classical elasticity element close to regions with stress or strain gradients (e.g. notches, cracks etc). In a verification stage, the accuracy of the computational algorithm for the pressurized penny-shaped and mixed-mode elliptical crack problems that have analytical solutions is demonstrated. More specifically, it is shown that the average error of the crack tip Stress Intensity Factor predicted by the gradient modified method for nine discretizations of varying density is around 3.5 % with a máximum error of 5 %, while the constant displacement discontinuity element displays errors varying around 14 %. Moreover, the new method preserves the simplicity and hence the high speed of the constant displacement discontinuity with only one collocation point per element, but it is far more efficient compared to it, especially close to the crack tips and corners of excavations where the displacement and stress gradients are highest.

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Keywords: displacement discontinuity method; dislocation; crack; stress intensity factor; rock excavation

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Nomenclature

δ_{ij}	Kronecker's delta function
ε_{ij}	second-order Cartesian strain tensor
E	Young's modulus
λ , G	Lamé constants
ν	Poisson's ratio
σ_{ij}	second-order Cartesian stress tensor
∇^2	Laplacian operator
D_i	displacement discontinuity vector
ℓ^2	strain-gradient term with dimensions of length squared
u_i	displacement vector

1. Introduction

It is almost certain that any planned underground excavation in the scale of 10^1 m or more will transect a fault or persistent joint. The problem in the design phase is to examine the effect of the faults on the behavior of the rock mass during excavation and then to optimize the design of the underground openings and pillars. In Geophysics there remains the reasonable trend to explain earthquake mechanisms by means of dislocation or fracture mechanics models. Also, in petroleum engineering as well as in Rock Mechanics, the hydraulic fracturing technique where a pressurized mode-I crack propagates from a shallow or deep borehole, is widely used for permeability enhancement and measurement of in situ stresses, respectively. There many more problems involving three-dimensional excavations and fractures that should be attacked with computational methods capable to tackle in a formal and accurate manner crack tip or corner singularities.

This paper is concerned with the upgrade of the classical Constant Displacement Discontinuity Method (CDDM) developed by Crouch [1] and Crouch and Starfield [2] and later nicely extended in three-dimensions by Kuriyama and Mizuta [3]. It is worth noting that this is essentially a boundary element method in order to avoid employing the cumbersome two-dimensional singular integral techniques [4] or three-dimensional finite element methods to solve crack and rock excavations problems. This is achieved, in a first phase of research, by developing a new triangular leaf (extremely thin) element for more accurate shape approximation of excavation and fault boundaries in three-dimensions of space. The element has one collocation point located at its centre of gravity for the fast solution of three-dimensional elastostatic problems of fractured rock masses. Kuriyama and Mizuta have already presented such an analytic solution based on Green's function or "influence function" approach [14]. Their work is extraordinary since they give the solution of the influence functions and their derivatives in closed form (instead of using numerical integration); however, one may note the following two intricate points in this paper:

- as is shown in Section 3, this solution overestimates the Stress Intensity Factors (SIF's) at the crack tips with at least an error of 6 % depending on the density of the grid,
- as is demonstrated in Fig. 1a below, the analytical expressions for the influence function and its derivatives given in the Appendix in [3], yield infinite values when the field point **x** is located on any extension line of an edge of the triangular element on which integration is performed.

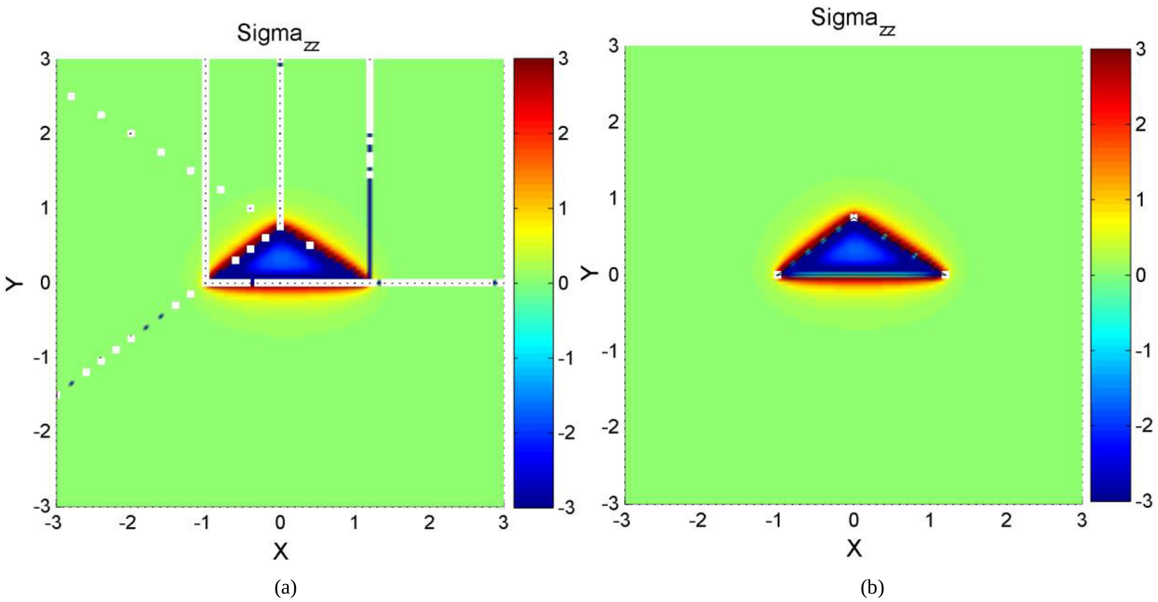


Fig. 1. Normal stress distribution around a uniformly pressurized triangular dislocation (a) singularities along the extension lines of the three edges; (b) elimination of singularities.

To overcome the first bullet above, in contrast to CDDM, the stress at the centre of gravity of a triangular dislocation is derived from the strain-gradient elasticity theory in its simplest possible grade-2 (second gradient of displacement or g2 theory) variant extracted from the milestone work by Mindlin [5]. This was already demonstrated for plane crack problems, in a series of three papers [6–8]. There, the g2 constant displacement discontinuity method (G2CDDM) was presented for the solution of mode-I, -II and -III crack problems in a plane or half-plane isotropic elastic body. This g2 formulation applies only on the elements close to crack tip or other type of geometric singularities and not outside them, where it is assumed that the classical stresses are valid and CDDM gives accurate results; furthermore, it does not alter the nature of the classical elasticity problem of a cracked body we are aiming to solve. That is to say, no extra boundary conditions along the crack are imposed that are necessary for the solution of a strain-gradient elasticity problem, apart from those prescribed by classical elasticity theory. Then, the value of the strain gradient coefficient or length-scale ℓ is found once, in such manner to achieve minimization of the discrepancy between the numerical and analytical solutions of crack tip opening displacements for the benchmark problem of the penny-shaped crack. As is expected the calibrated length scale is not a constant, but rather it depends on the element size and aspect ratio, as well as on the distance from the crack tip. This length scale being calibrated once, is subsequently used to solve any other crack or corner problem.

The second bullet is also of concern, that is annihilated here by finding analytically the limits of these terms of the g2 solution that give infinite values along the extension lines of the triangle's sides. The result of this approach is illustrated in Fig. 1b. The algorithm of the numerical code is then based on the established algorithm of the CDDM already presented in [2, 3].

The accuracy of the computational algorithm created using the new element is demonstrated for the cases of the uniformly pressurized penny-shaped crack in Section 3 and the mixed-mode elliptic crack in Section 4.

2. Derivation of the elementary solutions for the opening and sliding triangular dislocations

2.1. Governing equations of G2 theory

The fundamental concept of the CDDM is that the mid-element displacements and stresses adequately represent the displacements and stresses over the face of a dislocation or element [1, 2]. Use is made of the concept of displacement discontinuity – that is the difference in displacement between the two sides of the dislocation – occurring over the area of a given discontinuity (i.e. crack, joint, fault, bedding plane) positioned with respect to a given Cartesian coordinate system Oxyz, i.e.:

$$D_i \equiv u_i^- - u_i^+, \quad i = x, y, z \tag{1}$$

where the (+) and (-) superscripts refer to the sides directed along the positive and negative sides of the coordinate axis, respectively. The strain tensor in the linear theory is given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

The equations of equilibrium of g2 theory, with vanishing body force, are the usual Cauchy's equations:

$$\sigma_{ij,i} = 0 \tag{3}$$

However, the constitutive equations are different. In a first place it is assumed that the stresses are derived from a simplified form of the strain energy density *ansatz* of an elastic solid with microstructure that has been postulated by Mindlin [5]. Based on an appropriate and simple strain energy expression the constitutive relations linking stresses with strains have the following form [9]:

$$\sigma_{ij} = \left(1 - \ell^2 \nabla^2\right)\left[\lambda \delta_{ij} \varepsilon_{kk} + 2G\varepsilon_{ij}\right] \tag{4}$$

2.2. Influence coefficients for opening mode dislocation

We seek the fundamental solutions for the stresses and displacements outside a planar triangular dislocation normal to Oz-axis with prescribed uniform opening (aperture). For this purpose let us consider a triangular leaf (planar) element BCD representing the dislocation that is lying on a plane perpendicular to Oz-axis and exhibiting a uniform opening displacement discontinuity $2\varepsilon_z = D_z$ as is shown in Fig. 2, that is:

$$u_z\left(x, y, 0^-\right) = \begin{cases} \varepsilon_z, \mathit{if}\ \cot\delta \cdot y - d \le x \le b - \cot\beta \cdot y \ \mathit{and}\ 0 \le y \le c \\ 0, \mathit{otherwise} \end{cases} \tag{5}$$

Applying the direct two-dimensional (2D) Fourier transform on the displacements at the horizontal plane (i.e. $z = 0$) one obtains the induced displacements in the transformed coordinates (ξ, η) due to triangular dislocations BOC and DOC, respectively:

$$\begin{aligned} \bar{u}_i^{BOC}\left(\xi, n, 0^-\right) &= -\frac{1}{2} \cdot \frac{\varepsilon_z \cdot \left[(c \cdot n - b \cdot \xi) - c \cdot \exp\left(i \cdot b \cdot \xi\right) + b \cdot \exp\left(i \cdot c \cdot n\right)\right]}{\pi \cdot \xi \cdot n \cdot \left(c \cdot n - b \cdot \xi\right)} \\ \bar{u}_i^{DOC}\left(\xi, n, 0^-\right) &= \frac{1}{2} \cdot \frac{\varepsilon_z \cdot \left[(c \cdot n + d \cdot \xi) - c \cdot \exp\left(-i \cdot d \cdot \xi\right) - d \cdot \exp\left(i \cdot c \cdot n\right)\right]}{\pi \cdot \xi \cdot n \cdot \left(c \cdot n + d \cdot \xi\right)} \end{aligned} \tag{6}$$

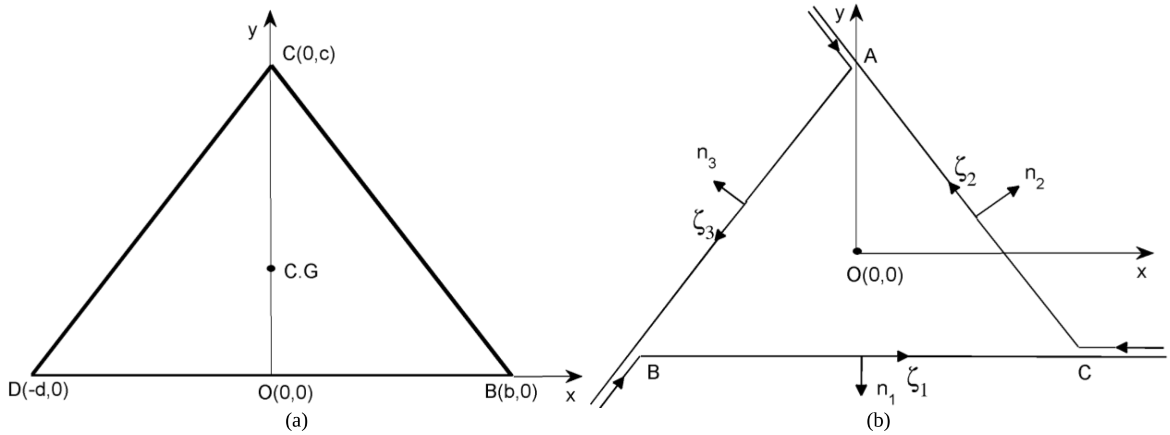


Fig. 2. Triangular leaf element: (a) Dimensions of the triangular element that is perpendicular to Oz-axis; and (b) Yoffe's sketch for the triangular disclocation.

Based on this splitting scheme of the original triangular element, as well as on the superposition principle, it is sufficient to solve first for the orthogonal triangle BOC and then from this solution to find the solution for the triangle DOC based on the following transformation:

$$\bar{u}_i^{DOC} = -\bar{u}_i^{BOC}\left(b = -d\right) \tag{7}$$

From Eq. (6) it may be observed that the final solution in the transformed domain will be the superposition of the following three solutions of the respective wedges with apexes at O, B and C, respectively, (e.g. Fig. 2a):

$$\begin{aligned} &\bar{u}_i^{O}(\xi,n,0) = -\frac{\varepsilon_z}{2\cdot\pi\cdot\xi\cdot n}, \quad \bar{u}_i^{B}(\xi,n,0) = -\frac{c\cdot\varepsilon_z\cdot\exp(i\cdot b\cdot\xi)}{2\cdot\pi\cdot\xi\cdot(b\cdot\xi - c\cdot n)}, \\ &\bar{u}_i^{C}(\xi,n,0) = \frac{b\cdot\varepsilon_z\cdot\exp(i\cdot c\cdot n)}{2\cdot\pi\cdot n\cdot(b\cdot\xi - c\cdot n)} \end{aligned} \tag{8}$$

It is worth noticing that the above decomposition exactly agrees with Yoffe's [10] approach for the triangular dislocation BOC obtained from the superposition of solutions of three angular dislocations (i.e. dislocations composed of two semi-infinite straight arms meeting at the origin), namely: (a) the orthogonal wedge with an apex at origin of coordinates O, (b) the wedge with angle β with an apex at B, and (c) the wedge with apex angle 90-β with apex at C, as is illustrated in Fig. 2a. Yoffe has shown that these disclocations form exterior angles to the triangle indicated by shading in Fig. 2b, with their infinite segments cancelling one another. The influence coefficients for the mode-I, -II and -III triangular dislocations have been found analytically using the Fourier transform. For brevity of the presentation the analytical expressions and method of solution of the dislocations are not shown here.

As is known the SIF's at the crack tip are found from the opening, shear and anti-plane shear displacement of the later based on the following relations and using a cylindrical coordinate system *Or θz:*

$$\begin{aligned} &K_I = \frac{G}{4(1-v)}\sqrt{\frac{2\pi}{r_{tip}}}D_z\left(r_{tip}\right), K_{II} = \frac{G}{4(1-v)}\sqrt{\frac{2\pi}{r_{tip}}}D_r\left(r_{tip}\right), \\ &K_{III} = \frac{G}{4}\sqrt{\frac{2\pi}{r_{tip}}}D_\theta\left(r_{tip}\right) \end{aligned} \tag{9}$$

where r_{tip} denotes the distance from the crack tip.

3. The penny-shaped crack under uniform internal pressure

The natural and essential boundary conditions for the circular (penny-shaped) crack of radius c subjected to uniform pressure w.r.t. a cylindrical coordinate system $Or\theta z$ are the following:

$$\begin{aligned} &\sigma_{rz} = 0 \qquad -\infty < r < \infty, z = 0 \\ &\sigma_{zz} = -p_o \quad 0 < r < c, z = 0 \\ &u_z = 0 \qquad r \geq 0, z = 0 \end{aligned} \tag{10}$$

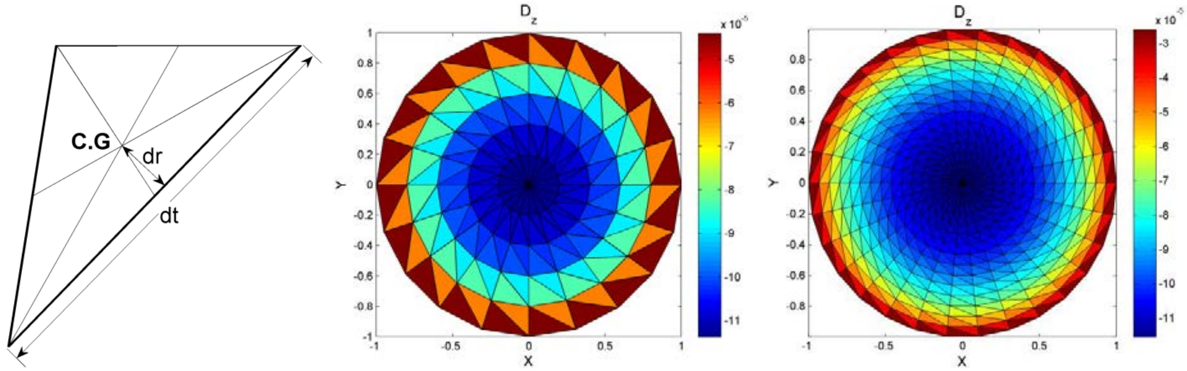
In addition, it is required that the stresses and displacements vanish at infinity (radiation condition). This axisymmetric mixed boundary value problem has been solved exactly with the opening discontinuity and the stresses to be given by the following relations [11] (as is expected all the shear stresses are null):

$$\begin{aligned} &D_z(\rho,0) = 4cp_o\frac{1-\nu}{\pi G}\sqrt{1-\rho^2},\ \rho \leq 1, \\ &\sigma_{zz} = -p_o,\ \sigma_{rr} = \sigma_{\theta\theta} = -\left(\nu + \frac{1}{2}\right)p_o,\ \rho < 1 \end{aligned} \tag{11}$$

where $\rho = r / c$ denotes the scaled radial distance from the center of the crack. The calibration of the only free parameter of the problem, namely the length scale ℓ of the g2 elasticity theory, is performed next in such a manner to minimize the error between the analytical solution for opening displacement along the half penny-shaped crack and the numerical solution. The best-fitted relationship of the normalized length scale ℓ / dr with the radial scaled distance r_{tip} / c measured from the tip of the crack and the geometry of the element (e.g. Fig. 3a) has the following form:

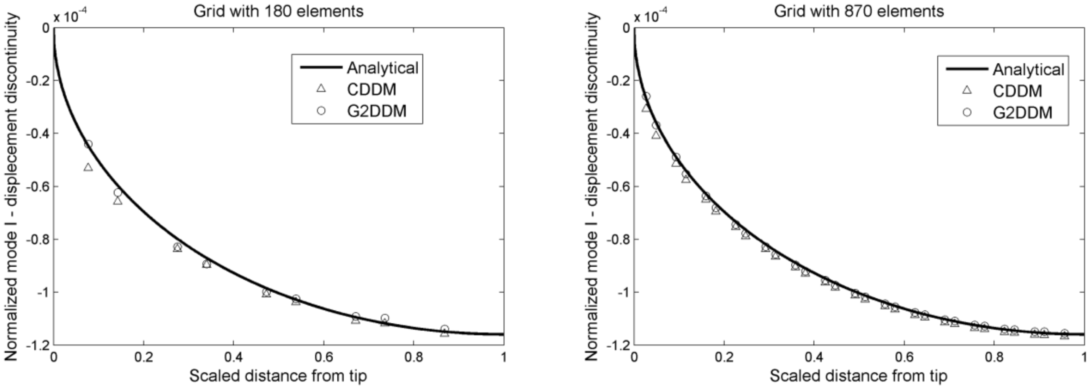
$$\ell / dr = \left(1 - \frac{r_{tip}}{c}\right)^8 \frac{\ell_{tip}}{dr} + 0.175; \quad \frac{\ell_{tip}}{dr} = 9.261\frac{drdt}{c^2} - 0.311\frac{dr}{c} - 1.328\frac{dt}{c} + 0.457 \tag{12}$$

By setting the internal length equal to zero the numerical solution corresponds to the classical solution presented by Kuriyama and Mizuta. Fig. 3b and Fig. 3c, as well as Fig 4a and Fig. 4b display the distribution of the normal displacement discontinuity on the crack surfaces for the two discretizations at hand, and for E = 20 GPa, $\nu = 0.3$, internal pressure p_0 = -1 MPa and crack radius c = 1 m. Also Table 1 displays the mode-I SIF predicted by the classical and G2 DDM and the error of each method with regards to the analytical solution of K_I = 1.128 MPa $\sqrt{m}$.



(a) (b) (c)

Fig. 3. Geometry of the leaf element and numerical solution for the normal displacement discontinuity for two grids for the penny-shaped crack at hand; (a) element geometry (C.G. is center of gravity); (b) grid with 180 elements; (c) grid with 870 elements.



(a) (b)

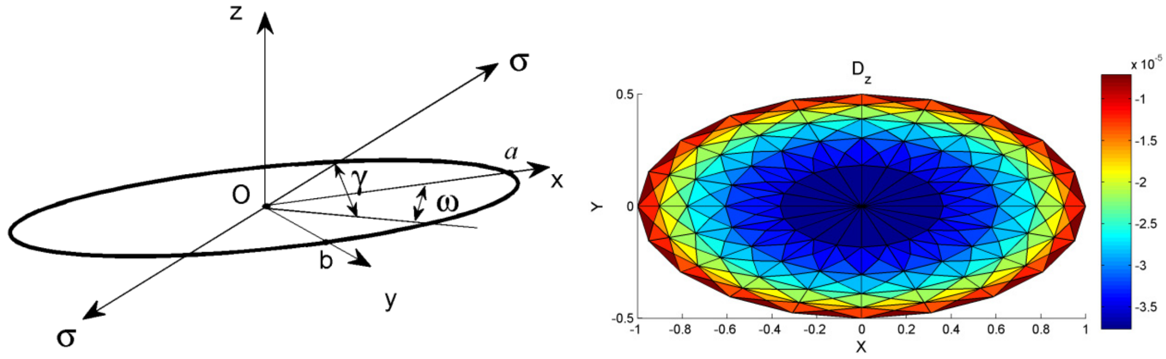
Fig. 4. Distribution of normal displacement discontinuity along the half-penny-shaped crack; (a) sparse grid with 180 elements; (b) dense grid with 870 elements.

Table 1. Comparison of CDDM and G2DDM with the analytical solution $K_I = 1.128\ MPa\sqrt{m}$.

SIF			Error (%)	
Number of grid elements	CDDM	G2DDM	CDDM	G2DDM
108	1.166	1.089	-3.3	3.5
180	1.314	1.090	-16.4	3.4
216	1.346	1.081	-19.3	4.2
270	1.377	1.071	-22.0	5.1
380	1.231	1.107	-9.1	1.9
456	1.281	1.106	-13.6	2.0
570	1.329	1.096	-17.8	2.8
580	1.163	1.072	-3.1	5.0
870	1.285	1.088	-13.9	3.6

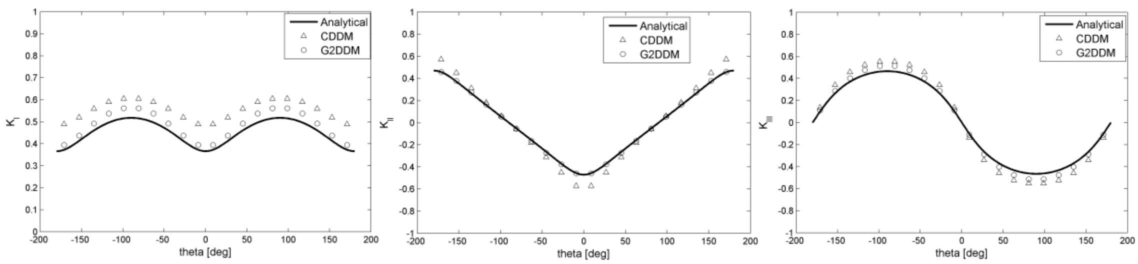
4. The elliptic crack under far-field tension

In a subsequent stage the numerical method is compared with the closed form solutions [12] of mode-I, -II and -III SIF's around an elliptic crack subjected to tension loading along an oblique direction with the normal to the crack, as is illustrated in Fig. 5a. The aspect ratio of the crack is $b / a = 0.5$, the loading angles are $\gamma = 0.25\pi$ and $\omega = 0$, the far-field tension is $\sigma = 1$ MPa and the rest of parameters as they given for the example of the penny-shaped crack. The values of the gradient parameter estimated from Eq. (12) on the crack surface discretized with 340 elements are shown in Fig. 5b. Also the comparison of the analytical solution with CDDM and G2DDM numerical solutions, are illustrated in Fig 6 (a–c).



(a) (b)

Fig. 5. Comparison of analytical solution for the mixed-mode elliptic crack with numerical model predictions; (a) geometry and loading configuration; (b) distribution of the gradient length scale on the crack surface.



(a) (b) (c)

Fig. 6. Comparison of analytical solution for the mixed-mode elliptic crack with predictions of CDDM and G2DDM (a) angular distribution of mode-I SIF; (b) angular distribution of mode-II SIF; and (c) angular distribution of mode-III SIF.

5. Conclusions

A computing algorithm for 3D elastic analysis of cracks and excavations in rocks by virtue of the boundary element method was developed and is outlined here. In order to minimize significantly the overprediction of the crack opening displacements at the crack tips and hence the SIF's by using the classical CDD method, a new trianglular element was constructed that is based on a simple version of the grade-2 elasticity theory. In this manner a significantly better average stress measure is achieved which gives accurate solutions in regions with high gradients of stress and crack displacements. The higher accuracy of this G2CDD element compared to the CDD elements was demonstrated with the uniformly pressurized penny-shaped crack and elliptic crack problems for which exact solutions exist. It was shown that the use of this element allows accurate analysis of a crack tip without recourse to a special crack tip element or elements with more than one collocation points (e.g. linear, quadratic etc).

Here the study of more elaborate problems like 3D interaction between cracks or cracks and excavations are not presented due to limited space of the paper. This will be done in a shortcoming publication.

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