Nonlinear Bilateral Output-Feedback Control for a Class of Viscous Hamilton-Jacobi PDEs

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Abstract

We tackle the boundary control and estimation problems for a class of viscous Hamilton-Jacobi PDEs, considering bilateral actuation and sensing, i.e., at the two boundaries of a 1-D spatial domain. We first solve the nonlinear trajectory generation problem for this type of PDEs, providing the necessary feedforward actions at both boundaries. We then design an observer-based output-feedback control law, which consists of two main elements–a nonlinear observer that is constructed utilizing measurements from both boundaries and state-feedback laws, which are employed at the two boundary ends. All of our designs are explicit since they are constructed interlacing a feedback linearizing transformation with backstepping. Due to the fact that the linearizing transformation is locally invertible, only a regional stability result is established, combining this transformation with backstepping, suitably formulated to handle the case of bilateral actuation and sensing. We illustrate the developed methodologies via application to traffic flow control and we present consistent simulation results.

1 Introduction

Contrary to linear parabolic Partial Differential Equations (PDEs), for which explicit boundary control and estimation designs are now largely available, see, for instance, [29], [35], in the nonlinear case, the design of explicit boundary control and estimation schemes is a more challenging problem. In addition, specific engineering applications, such as, for example, vehicular traffic [23], [46], plasma systems [8], fluids [9], chemical reactors [35], heat exchangers [35], and litium-ion batteries [44], [45], to name only a few, call for the development of systematic control and estimation design methodologies that, besides being able to efficiently exploit the capabilities of the available actuators and sensors, they can also be made fault tolerant.

Motivated by scalar, conservation law models for vehicular traffic flow that include a viscous term, in order to account for drivers' look-ahead ability [23], [25], [46], we consider the problems of boundary control and estimation of a certain class of viscous Hamilton-Jacobi (HJ) PDEs, which constitutes an alternative macroscopic description of traffic flow dynamics [11], [39]. In particular, we consider the case in which actuation and sensing is available at both boundaries (which we refer to as "bilateral" in our control and estimation approaches), aiming at constructing control and estimation schemes capable of utilizing efficiently both the available actuators and the available measurements. Since bilateral controllers and observers employ, in principle, smaller control and observer gains, compared to unilateral counterparts, it is expected that bilateral controllers may require less control effort, whereas bilateral observers may be more robust to measurement noise.

Arguably, the most relevant results to the ones presented here are those dealing with the controller and observer designs for viscous Burgers-type PDEs, which may be viewed as conservation law counterparts of the class of viscous HJ PDEs with quadratic Hamiltonian considered here. The trajectory generation problem for certain forms of viscous Burgers equations is considered in [28], [37], [41], whereas full-state boundary feedback laws are designed in [22], [26], [27], [31]. Observers and output-feedback controllers are presented in [3], [4], [9], [28]. Explicit boundary control and observer designs for other nonlinear parabolic PDEs also exist, see, e.g., [19], [36], [47], [50]. Although it is a different problem, for completeness, it should be mentioned that the control design problem of inviscid versions of Burgers or of specific HJ PDEs is considered in, e.g., [1], [6], [11], [26]. Bilateral controllers and observers for certain classes of linear parabolic and hyperbolic PDEs are recently developed in [2], [49]. Linear control laws constructed combining backstepping and flatness can be found in, e.g., [12], [38]. We should also mention here that, in comparison to [5], in the present paper we consider, 1) a more general class of

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viscous HJ PDE systems, 2) the problems of trajectory generation and tracking, and 3) the problems of bilateral control and estimation.

Our contributions are summarized as follows. First, we solve the nonlinear trajectory generation problem for the considered viscous HJ PDE, providing explicit feedforward actions at both boundaries. The key ingredient in our approach is the employment of a feedback linearizing transformation (inspired by the Hopf-Cole transformation [13], [21]) that we introduce, which allows us to convert the original nonlinear problem to a motion planning problem for a linear heat equation. We then establish the well-posedness of the feedforward controllers for the original nonlinear PDE system, for reference outputs that belong to Gevrey class (of certain order) with sufficiently small magnitude. One of the main contributions in our analysis is the determination of a bound for the transformation.

Second, we design observer-based output-feedback laws in order to achieve trajectory tracking, with an arbitrary decay rate, as the system is not, in general, asymptotically stable around a given reference trajectory. Two are the basic ingredients of the developed output-feedback control scheme, i) a nonlinear observer, which is constructed employing Dirichlet boundary measurements from both ends of the spatial domain and ii) state-feedback control laws, which are employed, via Neumann actuation, at each of the two boundaries. Our designs are based on the combination of a linearizing transformation together with backstepping [49], suitably formulated to the case of a one-dimensional spatial domain. We show that the bilateral, observer-based output-feedback control laws achieve local asymptotic stabilization of the reference trajectory in H^1 norm. Our stability result is local in H^1 norm due to the fact that the linearizing transformation is invertible only locally and, in particular, the size of the supremum norm of the transformed PDE state should be appropriately restricted.

Third, we apply the developed methodologies to a model of highway traffic flow. We illustrate, in simulation, the effectiveness of the proposed control design technique, including a comparison with the unilateral case.

We start presenting the class of viscous HJ PDEs under consideration and discussing its relation to a traffic flow model in Section 2. In Section 3 we present the nonlinear feedforward control designs. In Section 4 we present the nonlinear observer design and state-feedback controllers as well as we prove local stability of the closed-loop system under the observer-based output-feedback laws. We present an example of traffic flow control in Section 5. Concluding remarks and future research directions are provided in Section 6.

Notation and Definitions: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [24]. We denote by $L^2(0,1)$ and $H^1(0,1)$ the space of square-integrable scalar functions and the space of functions with square-integrable (weak) derivative, respectively, in the interval $[0,1]$. For a function $u \in L^2(0,1)$ we denote by $\|u\|_{L^2}$ the norm $\|u\|_{L^2} = \sqrt{\int_0^1 u(x)^2 dx}$. For $u \in H^1(0,1)$ we denote by $\|u\|_{H^1}$ the norm $\|u\|_{H^1} = \sqrt{\int_0^1 u(x)^2 dx} + \sqrt{\int_0^1 u_x(x)^2 dx}$. We denote by $C^j(A)$ the space of functions that have continuous derivatives of order j on A. We denote an initial condition as $u_0(x) = u(x,t_0)$ with some $t_0 \geq 0$, for all $x \in [0,1]$. With $C\left([t_0,+\infty); H^2(0,1)\right)$ we denote the class of continuous mappings on $[t_0,+\infty)$ with values into $H^2(0,1)$. We denote by $C_T^{2,1}([0,1]\times(t_0,T))$ the space of functions that have continuous spatial derivatives of order 2 and continuous time derivatives of order 1 on $[0,1]\times(t_0,T)$, and define $C_\infty^{2,1}([0,1]\times(t_0,+\infty)) = C^{2,1}([0,1]\times(t_0,+\infty))$.

Definition 1 *([32]) The function $f : \mathbb{S} \to \mathbb{R}$, belongs to $G_{F,M,\gamma}(\mathbb{S})$, the Gevrey class of order γ in $\mathbb{S}$, if $f(t) \in C^\infty(\mathbb{S})$ and there exist positive constants F, M such that $\sup_{t\in\mathbb{S}} \left|f^{(n)}(t)\right| \leq FM^n (n!)^\gamma$, for all $n = 0,1,2,\ldots$.*

2 Problem Formulation and Motivation

We consider the following viscous HJ PDE system

$$u_t(x,t) = \epsilon u_{xx}(x,t) - a u_x(x,t)\left(b + u_x(x,t)\right) \tag{1}$$
$$u_x(0,t) = U_0(t) \tag{2}$$
$$u_x(1,t) = U_1(t), \tag{3}$$

where u is the PDE state, $x \in [0,1]$ is the spatial variable, $t \geq t_0 \geq 0$ is time, $\epsilon > 0$ is a viscosity coefficient, $a \neq 0$ and $b \in \mathbb{R}$ are constant parameters, and U_0, U_1 are control variables. Our goal is to design an observer-based output-feedback law utilizing boundary measurements $y_{\mathrm{m}_1} = u(0)$ and $y_{\mathrm{m}_2} = u(1)$, such that the outputs $y_1 = u(x_0)$ and $y_2 = u_x(x_0)$, of the system, where x_0 is some fixed point within the interval $[0,1]$, track some desired reference outputs.

The motivation for considering the class of systems described by (1)–(3) comes from the fact that such PDE systems may serve as macroscopic models of vehicular traffic flow. To see this, consider a highway stretch with inlet at $x=0$ and outlet at $x=1$. We model the traffic density dynamics within the stretch with a conservation law PDE. In order to account for drivers' look-ahead ability, we incorporate in the expression for the traffic flow, in addition to the term that corresponds to a conventional fundamental diagram relation between speed and density of vehicles, an additional term that depends on the spatial derivative of the traffic density, giving rise to the following model, see, e.g., [23], [25], [46]

$$\rho_t(x,t) + \left(\rho(x,t)V\left(\rho(x,t)\right) - \epsilon\rho_x(x,t)\right)_x = 0 \tag{4}$$
$$\rho(0,t) = -U_0(t) \tag{5}$$
$$\rho(1,t) = -U_1(t), \tag{6}$$

where, for Greenshield's fundamental diagram [18] we have $V(\rho) = a(b-\rho)$, with a, b being free-flow speed and maximum density, respectively, whereas ρ denotes the traffic density. The density at the boundaries may be imposed manipulating either the flow or the speed of vehicles, via the employment of ramp-metering (RM) and variable speed limits (VSL), as well as exploiting the capabilities of connected and automated vehicles see, e.g., [10], [40].

In order to bring model (4)–(6) into the form (1)–(3) we define the following variable

$$u(x,t)=\int_x^1 \rho(y,t)dy + \int_0^t Q\left(\rho(1,s),\rho_x(1,s)\right)ds \quad (7)$$

$$Q\left(\rho,\rho_x\right)=\rho V\left(\rho\right)-\epsilon\rho_x. \quad (8)$$

It can be shown, by direct differentiation of (7) with respect to t and x, and by employing (4), that the variable u satisfies (1)–(3). In fact, the state u represents the so-called Moskowitz function, which constitutes an alternative macroscopic description of the dynamics of traffic flow in a highway. In particular, the value of the Moskowitz function $M=u(x,t)$ is interpreted as the "label" of a given vehicle at position x at time t, along a road segment [11], [39].

In fact, traffic flow models of the form (4)–(6) or (via (7)) of the form (1)–(3), belong to the class of the so-called first-order macroscopic traffic flow models, which incorporate only one PDE state for traffic density. However, systems of the form (1)–(3) (or (4)–(6)) differ from conventional first-order traffic flow models in that they incorporate a viscosity term. The reason for including a viscosity term is to model certain phenomena observed in real traffic empirically. In particular, this term may model the fact that drivers adjust their speed by also taking into account the downstream density. As a result, such models may capture important phenomena such as, for example, the capacity drop phenomenon [25], retaining, at the same time, their simplicity as compared to more complex models, which incorporate additional PDE states [25].

A typical aim of a traffic control scheme is to regulate the outlet flow to a certain set-point, say q^*, which may be the point that achieves the maximum flow (capacity flow) [10]. In terms of the u variable this corresponds to $u(1,t)$ tracking the reference trajectory q^*t. This motivates the trajectory generation and tracking problems for the class of systems described by (1)–(3). Moreover, since the value $u_x(1,t)$ could be also assigned, one may choose for reference value of $-u_x(1,t)$ the value of the density that corresponds to the critical density (i.e., the density at which capacity flow is achieved) of the nominal fundamental diagram relation (i.e., when there is no ρ_x term in (8)) between flow and density at the outlet of the considered stretch, which in turn would guarantee that the obtained desired profile for u_x (or, for ρ) is uniform with respect to space.

3 Trajectory Generation

In this section, we design the feedforward boundary control laws that generate the desired reference trajectory.

Theorem 1 *Let $y_1^{\rm r}(t)$ and $y_2^{\rm r}(t)$ be in $G_{F,M,\gamma}\left([0,+\infty)\right)$ class with $1\leq\gamma<2$. There exists a positive constant μ_1 such that if $F\leq\mu_1$ then the functions*

$$u^{\rm r}(x,t)=-\frac{\epsilon}{a}\ln\left(e^{\frac{ab}{2\epsilon}x}v^{\rm r}(x,t)+1\right) \quad (9)$$

$$U_0^{\rm r}(t)=-\frac{\epsilon}{a}\frac{v_x^{\rm r}(0,t)+\frac{ab}{2\epsilon}v^{\rm r}(0,t)}{1+v^{\rm r}(0,t)} \quad (10)$$

$$U_1^{\rm r}(t)=-\frac{\epsilon e^{\frac{ab}{2\epsilon}}}{a}\frac{v_x^{\rm r}(1,t)+\frac{ab}{2\epsilon}v^{\rm r}(1,t)}{1+e^{\frac{ab}{2\epsilon}}v^{\rm r}(1,t)}, \quad (11)$$

where

$$v^{\rm r}(x,t)=\sum_{k=0}^{\infty}\frac{1}{\epsilon^k}\frac{\left(x-x_0\right)^{2k}}{(2k)!}\sum_{m=0}^{k}\binom{k}{m}\left(\frac{a^2b^2}{4\epsilon}\right)^{k-m}\times {y_{1,v}^{\rm r}}^{(m)}(t)+\sum_{k=0}^{\infty}\frac{1}{\epsilon^k}\frac{\left(x-x_0\right)^{2k+1}}{(2k+1)!}\times\sum_{m=0}^{k}\binom{k}{m}\left(\frac{a^2b^2}{4\epsilon}\right)^{k-m}{y_{2,v}^{\rm r}}^{(m)}(t) \quad (12)$$

$$y_{1,v}^{\rm r}(t)=e^{-\frac{ab}{2\epsilon}x_0}\left(e^{-\frac{a}{\epsilon}y_1^{\rm r}(t)}-1\right) \quad (13)$$

$$y_{2,v}^{\rm r}(t)=e^{-\frac{ab}{2\epsilon}x_0}\left(-\frac{a}{\epsilon}e^{-\frac{a}{\epsilon}y_1^{\rm r}(t)}y_2^{\rm r}(t)-\frac{ab}{2\epsilon}\left(e^{-\frac{a}{\epsilon}y_1^{\rm r}(t)}-1\right)\right), \quad (14)$$

satisfy the boundary value problem (1)–(3) and, in particular, $u^{\rm r}\left(x_0,t\right)=y_1^{\rm r}(t)$ and $u_x^{\rm r}\left(x_0,t\right)=y_2^{\rm r}(t)$.

Proof The change of variables

$$v(x,t)=e^{-\frac{ab}{2\epsilon}x}\left(e^{-\frac{a}{\epsilon}u(x,t)}-1\right), \quad (15)$$

and the following choice for U_0, U_1

$$U_0(t)=-\frac{\epsilon}{a}e^{\frac{a}{\epsilon}u(0,t)}V_0(t)+\frac{b}{2}\left(e^{\frac{a}{\epsilon}u(0,t)}-1\right) \quad (16)$$

$$U_1(t)=-\frac{\epsilon}{a}e^{\frac{a}{\epsilon}u(1,t)+\frac{ab}{2\epsilon}}V_1(t)+\frac{b}{2}\left(e^{\frac{a}{\epsilon}u(1,t)}-1\right), \quad (17)$$

where V_0, V_1 are new control variables, transform (1)–(3) to

$$v_t(x,t)=\epsilon v_{xx}(x,t)-\frac{a^2b^2}{4\epsilon}v(x,t) \quad (18)$$

$$v_x(0,t)=V_0(t) \quad (19)$$

$$v_x(1,t)=V_1(t). \quad (20)$$

To generate the desired trajectory $u^{\rm r}$, providing the feedforward laws $U_0^{\rm r}$, $U_1^{\rm r}$, which achieve $u^{\rm r}\left(x_0\right)=y_1^{\rm r}$, $u_x^{\rm r}\left(x_0\right)=y_2^{\rm r}$, we first generate $v^{\rm r}$ satisfying (18) with $v^{\rm r}\left(x_0\right)=y_{1,v}^{\rm r}$ and $v_x^{\rm r}\left(x_0\right)=y_{2,v}^{\rm r}$, where $y_{1,v}^{\rm r}$, $y_{2,v}^{\rm r}$ are defined in (13), (14), respectively. Combining (19), (20) with (16), (17), we then get

$U_0^{\mathrm{r}}, U_1^{\mathrm{r}}$. Moreover, v^{r} should be restricted appropriately such that (9)–(11) are well-posed, which holds true whenever

$$\sup_{x\in[0,1]} |v^{\mathrm{r}}(x,t)| < \bar{c}e^{-\left|\frac{ab}{2\epsilon}\right|}, \quad \text{for all } t \geq t_0, \tag{21}$$

for some constant $\bar{c} \in (0,1)$, in addition to $v_x^{\mathrm{r}}(x,t)$ being bounded for all $x \in [0,1]$ and $t \geq t_0$.

Since (18)–(20) is a linear diffusion-reaction PDE the reference trajectory v^{r} is written as in (12), see, e.g., [17], [32], [33], [34], [37], [38]. Employing the results in, e.g., [37] (Remark 4), we conclude that series (12) is convergent (with infinite convergence radius) provided that $y_{1,v}^{\mathrm{r}}, y_{2,v}^{\mathrm{r}}$ belong to $G_{F_1^*,M_1^*,\gamma}([0,+\infty))$ for $1 \leq \gamma < 2$ and some F_1^*, M_1^*.

We derive next Gevrey estimates for $y_{1,v}^{\mathrm{r}}, y_{2,v}^{\mathrm{r}}$ as, in order to guarantee that (21) holds, one has to guarantee, in addition to $y_{1,v}^{\mathrm{r}}, y_{2,v}^{\mathrm{r}}$ belonging to $G_{F_1^*,M_1^*,\gamma}([0,+\infty))$ for $1 \leq \gamma < 2$, that F_1^* may be made small when F is small. Toward that end, from Lemmas A.1 and A.2 in Appendix A we obtain

$$\sup_{t\geq 0}\left|{y_{1,v}^{\mathrm{r}}}^{(n)}(t)\right| \leq \bar{F}_1\bar{M}_1^n (n!)^{\gamma}, \quad \text{for all } n = 0,1,\ldots \tag{22}$$

$$\sup_{t\geq 0}\left|{y_{2,v}^{\mathrm{r}}}^{(n)}(t)\right| \leq \bar{F}_2\bar{M}_2^n (n!)^{\gamma}, \quad \text{for all } n = 0,1,\ldots, \tag{23}$$

where

$$\bar{F}_1 = F\frac{|a|}{\epsilon}e^{F\frac{|a|}{\epsilon}}e^{-\frac{ab}{2\epsilon}x_0} \tag{24}$$

$$\bar{M}_1 = Me^{F\frac{|a|}{\epsilon}} \tag{25}$$

$$\bar{F}_2 = F\frac{|a|}{\epsilon}e^{F\frac{|a|}{\epsilon}}e^{-\frac{ab}{2\epsilon}x_0}\left(e^{-F\frac{|a|}{\epsilon}} + \frac{|ab|}{2\epsilon} + F\frac{|a|}{\epsilon}\right) \tag{26}$$

$$\bar{M}_2 = \left(1 + F\frac{|a|}{\epsilon}e^{F\frac{|a|}{\epsilon}}\right)\bar{M}_1, \tag{27}$$

and hence, one can choose $F_1^* = F\frac{|a|}{\epsilon}e^{F\frac{|a|}{\epsilon}}e^{-\frac{ab}{2\epsilon}x_0} \times \max\left\{1, e^{-F\frac{|a|}{\epsilon}} + \frac{|ab|}{2\epsilon} + F\frac{|a|}{\epsilon}\right\}$ and $M_1^* = \bar{M}_2$. Thus, series (12) is convergent. Combining (12), (22), (23) we get

$$|v^{\mathrm{r}}(x,t)| \leq \bar{F}_1\sum_{k=0}^{\infty}\frac{1}{\epsilon^k}(k!)^{\gamma-2}\left(\frac{a^2b^2}{4\epsilon} + \bar{M}_1\right)^k + \bar{F}_2\sum_{k=0}^{\infty}\frac{1}{\epsilon^k}(k!)^{\gamma-2}\left(\frac{a^2b^2}{4\epsilon} + \bar{M}_2\right)^k, \tag{28}$$

as $(k!)^2 \leq (2k)!$. For all $x \in [0,1]$ the general term, say ζ_k, in the first series satisfies $\left|\frac{\zeta_{k+1}}{\zeta_k}\right| = \frac{1}{\epsilon}\left(\frac{a^2b^2}{4\epsilon} + \bar{M}_1\right)(k+1)^{\gamma-2}$, and thus, since $\gamma < 2$, we conclude that $\lim_{k\to\infty}\left|\frac{\zeta_{k+1}}{\zeta_k}\right| = 0 < 1$, which in turn implies that the infinite sum converges to a positive number, say l_1. Similarly, the second sum converges to a positive number, say l_2. Therefore, from (28) we arrive at

$$|v^{\mathrm{r}}(x,t)| \leq \max\left\{\bar{F}_1, \bar{F}_2\right\}(l_1 + l_2), \quad \text{for all } x \in [0,1] \text{ and } t \geq t_0, \tag{29}$$

and hence, choosing μ_1 such that $\max\left\{\bar{F}_1, \bar{F}_2\right\}(l_1 + l_2) < \bar{c}e^{-\left|\frac{ab}{2\epsilon}\right|}$, for some $\bar{c} \in (0,1)$, which, according to (24), (26) is always possible (note that l_1, l_2 are continuous functions of F since the two series in (28) converge uniformly and from (25), (27) it follows that $\bar{M}_1$, $\bar{M}_2$ are continuous with respect to F), condition (21) is satisfied. It follows from (9) that u^{r} is uniformly bounded with respect to t and x. Uniform boundedness, with respect to t and x, of v_x^{r}, v_{xx}^{r}, and v_t^{r}, which, from (9), (21), imply the uniform boundedness of u_x^{r}, u_{xx}^{r}, and u_t^{r}, follows differentiating (12) and employing almost identical arguments (see also, e.g., [32]). □

Remark 1 The developed approach allows the system outputs to be located either at an intermediate point (see also, e.g., [37]) or at the boundary as well as it allows the consideration of constant or time-varying reference outputs. The proposed methodology also allows the construction of explicit (and even closed-form, e.g., for sinusoidal reference outputs) feedforward control designs for a class of nonlinear PDEs, which are new and haven't appeared in literature. In fact, $(y_1, y_2) = (u(x_0), u_x(x_0))$ is a "flat" output for system (1)–(3), and thus, generation of a reference trajectory for Gevrey-class reference outputs, which may be expressed algebraically as a function of the reference outputs and their derivatives, is possible. Moreover, although part of the design procedure is devoted to trajectory generation for a linear PDE, the main difficulty in showing that (9)–(14) solve (1)–(3), satisfying $u^{\mathrm{r}}(x_0) = y_1^{\mathrm{r}}$, $u_x^{\mathrm{r}}(x_0) = y_2^{\mathrm{r}}$, is to show that the reference trajectory derived is such that transformation (15) is invertible. Toward that end, we first prove that the transformed reference outputs belong to Gevrey class of same order as the original ones and we then show that their magnitude (namely, F in Definition 1) can be made small when the magnitudes of the original functions are small. The key results used in these proofs are Lemmas A.1 and A.2 in Appendix A, which are novel and of interest on their own.

4 Trajectory Tracking

Having available the reference trajectory for (1)–(3), which is not in general asymptotically stable, we design boundary output-feedback laws and prove that they asymptotically stabilize the reference trajectory for any initial condition. The control laws are based on the combination of an observer, which utilizes measurements from both boundaries, with state-feedback designs that employ the observer state.

4.1 Observer design

We introduce the following observer

$$\hat{\tilde{v}}_t(x,t) = \epsilon\hat{\tilde{v}}_{xx}(x,t) - \frac{a^2b^2}{4\epsilon}\hat{\tilde{v}}(x,t)$$

$$+p_2(x)\left(\bar{y}_0(t)-\hat{\tilde{v}}(0,t)\right)$$
$$+p_1(x)\left(\bar{y}_1(t)-\hat{\tilde{v}}(1,t)\right) \quad (30)$$
$$\hat{\tilde{v}}_x(0,t)=\tilde{V}_0(t)+p_{00}\left(\bar{y}_0(t)-\hat{\tilde{v}}(0,t)\right) \quad (31)$$
$$\hat{\tilde{v}}_x(1,t)=\tilde{V}_1(t)+p_{11}\left(\bar{y}_1(t)-\hat{\tilde{v}}(1,t)\right), \quad (32)$$

where we define

$$\bar{y}_0(t)=\left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)}-1\right)e^{-\frac{a}{\epsilon}u^{\mathrm{r}}(0,t)} \quad (33)$$
$$\bar{y}_1(t)=\left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)}-1\right)e^{-\frac{ab}{2\epsilon}-\frac{a}{\epsilon}u^{\mathrm{r}}(1,t)} \quad (34)$$
$$\tilde{u}(x,t)=u(x,t)-u^{\mathrm{r}}(x,t) \quad (35)$$
$$\tilde{U}_i(t)=U_i(t)-U_i^{\mathrm{r}}(t), \quad i\in\{0,1\}, \quad (36)$$

and $\tilde{V}_0$, $\tilde{V}_1$ are related to $\tilde{U}_0$, $\tilde{U}_1$ (yet to be chosen) via

$$\tilde{U}_0(t)=-\frac{\epsilon}{a}e^{\frac{a}{\epsilon}u(0,t)}\tilde{V}_0(t)$$
$$+\left(\frac{b}{2}+U_0^{\mathrm{r}}(t)\right)\left(e^{\frac{a}{\epsilon}\tilde{u}(0,t)}-1\right) \quad (37)$$
$$\tilde{U}_1(t)=-\frac{\epsilon}{a}e^{\frac{ab}{2\epsilon}+\frac{a}{\epsilon}u(1,t)}\tilde{V}_1(t)$$
$$+\left(\frac{b}{2}+U_1^{\mathrm{r}}(t)\right)\left(e^{\frac{a}{\epsilon}\tilde{u}(1,t)}-1\right). \quad (38)$$

The gains $p_2(x)$, $p_1(x)$, p_{00}, and p_{11} are designed as [49]

$$p_2(x)=-\epsilon P_\xi(x,0), \quad p_1(x)=-\epsilon P_\xi(x,1) \quad (39)$$
$$p_{00}=-P(0,0), \quad p_{11}=-P(1,1), \quad (40)$$

where P is given explicitly, for (x,ξ) in $E=E_1\cup E_2$, where $E_1=\left\{(x,\xi):\frac{1}{2}\le\xi\le 1,-\xi+1\le x\le\xi\right\}$ and $E_2=\left\{(x,\xi):0\le\xi\le\frac{1}{2},\xi\le x\le 1-\xi\right\}$, by

$$P(x,\xi)=-\frac{1}{2}\sqrt{\frac{c_2}{\epsilon}}\frac{\mathrm{I}_1\left(\sqrt{\frac{c_2}{\epsilon}\left(\left(\xi-\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right)}\right)}{\sqrt{\left(\xi-\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}$$
$$\times(\xi+x-1), \quad (41)$$

where I_1 denotes the modified Bessel function of the first kind of first order and $c_2>0$ is arbitrary.

For making clear the structure of the observer (30)–(32), it should be noted that the change of variables

$$\tilde{v}(x,t)=e^{-\frac{ab}{2\epsilon}x-\frac{a}{\epsilon}u^{\mathrm{r}}(x,t)}\tilde{\tilde{v}}(x,t) \quad (42)$$
$$\tilde{\tilde{v}}(x,t)=e^{-\frac{a}{\epsilon}\tilde{u}(x,t)}-1, \quad (43)$$

transform $\tilde{u}$ to $\tilde{v}$ that satisfies (18)–(20) with inputs $\tilde{V}_0$, $\tilde{V}_1$. Thus, observer (30)–(32) may be viewed as copy of $\tilde{v}$ system plus output injection, where the output-injection terms are linear in $\tilde{v}$. This reduces the implementation complexity of the proposed observer. Moreover, the time-varying part in (42), which is, to the best of our knowledge, new, is crucial because it allows one to transform the nonlinear, time-varying $\tilde{u}$ system into a linear, time-invariant system. This is possible because the overall transformation (42) may be expressed as the difference of two nonlinear functions of u and u^{r}, respectively, which both satisfy a linear PDE (18).

4.2 Feedback control design

The boundary feedback laws are designed as

$$U_0(t)=-\frac{\epsilon}{a}e^{\frac{a}{\epsilon}\tilde{u}(0,t)}\left(\left(k(0,0)+\frac{ab}{2\epsilon}\right)\left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)}-1\right)\right.$$
$$\left.-e^{\frac{a}{\epsilon}u^{\mathrm{r}}(0,t)}\int_0^1 k_x(0,\xi)\,\hat{\tilde{v}}(\xi,t)\,d\xi\right)$$
$$+U_0^{\mathrm{r}}(t)e^{\frac{a}{\epsilon}\tilde{u}(0,t)} \quad (44)$$
$$U_1(t)=-\frac{\epsilon}{a}e^{\frac{a}{\epsilon}\tilde{u}(1,t)}\left(\left(k(1,1)+\frac{ab}{2\epsilon}\right)\left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)}-1\right)\right.$$
$$\left.+e^{\frac{ab}{2\epsilon}+\frac{a}{\epsilon}u^{\mathrm{r}}(1,t)}\int_0^1 k_x(1,\xi)\,\hat{\tilde{v}}(\xi,t)\,d\xi\right)$$
$$+U_1^{\mathrm{r}}(t)e^{\frac{a}{\epsilon}\tilde{u}(1,t)}, \quad (45)$$

where k is given explicitly, for (x,ξ) in $D=D_1\cup D_2$, where $D_1=\left\{(x,\xi):\frac{1}{2}\le x\le 1,-x+1\le\xi\le x\right\}$ and $D_2=\left\{(x,\xi):0\le x\le\frac{1}{2},x\le\xi\le 1-x\right\}$, by

$$k(x,\xi)=-\frac{1}{2}\sqrt{\frac{c_1}{\epsilon}}\frac{\mathrm{I}_1\left(\sqrt{\frac{c_1}{\epsilon}\left(\left(x-\frac{1}{2}\right)^2-\left(\xi-\frac{1}{2}\right)^2\right)}\right)}{\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\xi-\frac{1}{2}\right)^2}}$$
$$\times(x+\xi-1), \quad (46)$$

with I_1 denoting the modified Bessel function of the first kind of first order and $c_1>0$ being arbitrary. The inspiration of (44), (45) comes from the fact that the $\tilde{v}$ variable satisfies the linear diffusion-reaction PDE (18)–(20) (with inputs $\tilde{V}_0$, $\tilde{V}_1$), and thus, $\tilde{V}_0$, $\tilde{V}_1$ may be chosen as [49]

$$\tilde{V}_0(t)=k(0,0)\,\tilde{v}(0,t)-\int_0^1 k_x(0,\xi)\,\hat{\tilde{v}}(\xi,t)\,d\xi \quad (47)$$
$$\tilde{V}_1(t)=k(1,1)\,\tilde{v}(1,t)+\int_0^1 k_x(1,\xi)\,\hat{\tilde{v}}(\xi,t)\,d\xi. \quad (48)$$

4.3 Stability analysis

In order to show asymptotic stability of the closed-loop system, under the observer-based output-feedback laws, in the original variable $\tilde{u}$, we have to ensure that the linearizing transformation $\tilde{\tilde{v}}$, defined in (43), is invertible. Its inverse is

$$\tilde{u}(x,t)=-\frac{\epsilon}{a}\ln\left(\tilde{\tilde{v}}(x,t)+1\right), \quad (49)$$

which is well-defined when the initial conditions and solutions of the system satisfy for some $c \in (0,1]$

$$\sup_{x\in[0,1]} |\tilde{\bar{v}}(x,t)| < c, \quad \text{for all } t \geq t_0. \tag{50}$$

Due to the feasibility condition (50), only a local stability result can be obtained, which is stated next.

Theorem 2 *Consider a closed-loop system consisting of system (1)–(3), the control laws (44), (45), and the observer (30)–(32) with (47), (48). Under the conditions of Theorem 1 for the reference outputs, there exist a positive constant μ^* and a class $\mathcal{KL}$ function β^* such that for all initial conditions $\left(u_0, \hat{\bar{v}}_0\right) \in H^2(0,1) \times H^2(0,1)$ which are compatible with boundary conditions and which satisfy*

$$\|\tilde{u}(t_0)\|_{H^1} + \|\hat{\bar{v}}(t_0)\|_{H^1} < \mu^*, \tag{51}$$

the following holds

$$\Omega(t) \leq \beta^*(\Omega(t_0), t - t_0), \quad \text{for all } t \geq t_0, \tag{52}$$

where

$$\Omega(t) = \|\tilde{u}(t)\|_{H^1} + \|\hat{\bar{v}}(t)\|_{H^1}. \tag{53}$$

Moreover, the closed-loop system has a unique solution $u, \hat{\bar{v}} \in C\left([t_0,+\infty); H^2(0,1)\right)$ with $u, \hat{\bar{v}} \in C^{2,1}([0,1] \times (t_0,+\infty))$.

The proof of Theorem 2 is based on the following three lemmas. The proof of the third lemma can be found in Appendix B, whereas the proofs of the first two lemmas may be established using similar arguments to the corresponding proofs in [5] (which considers, however, only the case $a = b = 1$), and thus, they are omitted due to space limitation.

In certain instances, some of the conditions of Theorem 1 on the reference outputs may be relaxed (see, e.g., Section 5). For example, for a reference trajectory that incorporates a linear function of time with positive slope, although the corresponding reference output may not be uniformly bounded, stabilization may still be achieved provided that the convergence rate of the linear $\tilde{v}$ system is sufficiently large (e.g., choosing large c_1, c_2), as it may be seen from (42).

Lemma 1 *There exists a class $\mathcal{K}_\infty$ function α_1 such that if $\tilde{u} \in H^1(0,1)$ then $\tilde{\bar{v}} \in H^1(0,1)$ and the following holds*

$$\|\tilde{\bar{v}}(t)\|_{H^1} \leq \alpha_1(\|\tilde{u}(t)\|_{H^1}). \tag{54}$$

Lemma 2 *For all solutions of the system that satisfy (50) for some $0 < c < 1$, if $\tilde{\bar{v}} \in H^1(0,1)$ then $\tilde{u} \in H^1(0,1)$ and the following holds*

$$\|\tilde{u}(t)\|_{H^1} \leq \frac{\epsilon}{|a|(1-c)} \|\tilde{\bar{v}}(t)\|_{H^1}. \tag{55}$$

Lemma 3 *Under the conditions of Theorem 1 for the reference outputs, if $\tilde{\bar{v}} \in H^1(0,1)$ then $\tilde{v} \in H^1(0,1)$ and there exists a positive constant ξ_1 such that the following holds*

$$\|\tilde{v}(t)\|_{H^1} \leq \xi_1 \|\tilde{\bar{v}}(t)\|_{H^1}. \tag{56}$$

In reverse, if $\tilde{v} \in H^1(0,1)$ then $\tilde{\bar{v}} \in H^1(0,1)$ and there exists a positive constant ξ_2 such that the following holds

$$\|\tilde{\bar{v}}(t)\|_{H^1} \leq \xi_2 \|\tilde{v}(t)\|_{H^1}. \tag{57}$$

Proof of Theorem 2 The proof is divided into three parts.

Part 1: *Backstepping transformation of the estimation error*

We start defining the state estimation error $\tilde{e} = \tilde{v} - \hat{\tilde{v}}$. Using the fact that the $\tilde{v}$ variable satisfies (18)–(20) and relations (30)–(32), we get with the definition of $\tilde{v}$ in (42) that the state estimation error $\tilde{e}$ satisfies the PDE $\tilde{e}_t(x,t) = \epsilon\tilde{e}_{xx}(x,t) - \frac{a^2b^2}{4\epsilon}\tilde{e}(x,t) - p_2(x)\tilde{e}(0,t) - p_1(x)\tilde{e}(1,t)$ with boundary conditions $\tilde{e}_x(0,t) = -p_{00}\tilde{e}(0,t)$ and $\tilde{e}_x(1,t) = -p_{11}\tilde{e}(1,t)$. Since it turns out to be convenient to shift from the variable x to the variable $z = x - \frac{1}{2}$, we write the error system as

$$\begin{aligned}\bar{e}_t(z,t) = \epsilon\bar{e}_{zz}(z,t) - \frac{a^2b^2}{4\epsilon}\bar{e}(z,t)\\ -\bar{p}_2(z)\bar{e}\left(-\frac{1}{2},t\right) - \bar{p}_1(z)\bar{e}\left(\frac{1}{2},t\right)\end{aligned} \tag{58}$$

$$\bar{e}_z\left(-\frac{1}{2},t\right) = -p_{00}\bar{e}\left(-\frac{1}{2},t\right) \tag{59}$$

$$\bar{e}_z\left(\frac{1}{2},t\right) = -p_{11}\bar{e}\left(\frac{1}{2},t\right), \tag{60}$$

where $\bar{e}(z,t) = \tilde{e}\left(z + \frac{1}{2}, t\right)$, $\bar{p}_i(z) = p_i\left(z + \frac{1}{2}\right)$, $i = 1, 2$.

In order to derive a suitable bilateral backstepping transformation for the estimation error we may proceed as in [49]. Note, however, that the results in [49] are obtained only for Dirichlet actuation and Neumann sensing. Moreover, although the case of a one-dimensional spatial domain may be viewed as a special case of a one-dimensional ball, it may be useful and intriguing for a potential reader, who wishes to focus only on one-dimensional domains, to explicitly derive the backstepping transformation and its inverse for the case of an interval, utilizing tools from 1-D backstepping only. The backstepping transformation is written as

$$\begin{aligned}\bar{e}(z,t) = \bar{w}(z,t) - \int_z^{\frac{1}{2}} p(z,y)\,\bar{w}(y,t)\,dy\\ + \int_{-\frac{1}{2}}^{-z} p(z,y)\,\bar{w}(y,t)\,dy, \quad 0 \leq z \leq \frac{1}{2}\end{aligned} \tag{61}$$

$$\begin{aligned}\bar{e}(z,t) = \bar{w}(z,t) + \int_{-\frac{1}{2}}^{z} p(z,y)\,\bar{w}(y,t)\,dy\\ - \int_{-z}^{\frac{1}{2}} p(z,y)\,\bar{w}(y,t)\,dy, \quad -\frac{1}{2} \leq z \leq 0,\end{aligned} \tag{62}$$

where $p(z,y) = P\left(z+\frac{1}{2}, y+\frac{1}{2}\right)$ and the kernel P is defined in (41). The inverse of (61), (62) may be expressed as

$$\bar{w}(z,t) = \bar{e}(z,t) + \int_{z}^{\frac{1}{2}} \bar{p}(z,y)\,\bar{e}(y,t)\,dy - \int_{-\frac{1}{2}}^{-z} \bar{p}(z,y)\,\bar{e}(y,t)\,dy, \quad 0 \le z \le \frac{1}{2} \tag{63}$$

$$\bar{w}(z,t) = \bar{e}(z,t) - \int_{-\frac{1}{2}}^{z} \bar{p}(z,y)\,\bar{e}(y,t)\,dy + \int_{-z}^{\frac{1}{2}} \bar{p}(z,y)\,\bar{e}(y,t)\,dy, \quad -\frac{1}{2} \le z \le 0, \tag{64}$$

where the kernel $\bar{p}(z,y)$ has a similar structure to p.

Invertibility of (61), (62) could be shown as follows. Replacing z by $-z$ in (62) and performing the change of variables $y \to -y$ in the second integral of (61) and in the first integral of (62), transformation (61), (62) may be viewed as a 2×2 backstepping transformation (i.e., as a 2×2 Volterra transformation, thus invertible) of the state $(\bar{w}_1, \bar{w}_2)$, where $\bar{w}_1(z) = \bar{w}(z)$ and $\bar{w}_2(z) = \bar{w}(-z)$, $z \in [0, \frac{1}{2}]$, into $(\bar{e}_1, \bar{e}_2)$, where $\bar{e}_1(z) = \bar{e}(z)$ and $\bar{e}_2(z) = \bar{e}(-z)$, $z \in [0, \frac{1}{2}]$, with boundary conditions at $z = 0$ for the $(\bar{w}_1, \bar{w}_2)$ system as $\bar{w}_1(0) = \bar{w}_2(0)$ and $\bar{w}_{2_z}(0) = -\bar{w}_{1_z}(0)$ (which lead to the corresponding boundary conditions for $(\bar{e}_1, \bar{e}_2)$, namely, $\bar{e}_1(0) = \bar{e}_2(0)$ and $\bar{e}_{2_z}(0) = -\bar{e}_{1_z}(0)$).

Standard backstepping-style computations and (41) give

$$\bar{w}_t(z,t) = \epsilon \bar{w}_{zz}(z,t) - \left(\frac{a^2 b^2}{4\epsilon} + c_2\right) \bar{w}(z,t) \tag{65}$$

$$\bar{w}_z\left(-\frac{1}{2}, t\right) = \bar{w}_z\left(\frac{1}{2}, t\right) = 0, \tag{66}$$

as well as

$$\|\bar{w}(t)\|_{H^1} \le m_3 \|\bar{e}(t)\|_{H^1} \tag{67}$$

$$\|\bar{e}(t)\|_{H^1} \le m_4 \|\bar{w}(t)\|_{H^1}, \tag{68}$$

for some positive constants m_3 and m_4.

Part 2: *Backstepping transformation of the observer state*

We consider the backstepping transformation [49]

$$\hat{w}_1(z,t) = \hat{\tilde{v}}_1(z,t) - \int_{-z}^{z} K(z,y)\,\hat{\tilde{v}}_1(y,t)\,dy, \tag{69}$$

where $\hat{\tilde{v}}_1(z,t) = \hat{\tilde{v}}\left(z+\frac{1}{2}, t\right)$ and $K(z,y) = k\left(z+\frac{1}{2}, y+\frac{1}{2}\right)$, with k being defined in (46). Its inverse is defined as

$$\hat{\tilde{v}}_1(z,t) = \hat{w}_1(z,t) + \int_{-z}^{z} L(z,y)\,\hat{w}_1(y,t)\,dy, \tag{70}$$

where $L(z,y) = l\left(z+\frac{1}{2}, y+\frac{1}{2}\right)$ and l has a similar structure to k. It can be shown that $\hat{w}_1$ satisfies

$$\hat{w}_{1_t}(z,t) = \epsilon \hat{w}_{1_{zz}}(z,t) - \left(\frac{a^2 b^2}{4\epsilon} + c_1\right) \hat{w}_1(z,t) + \left(\bar{p}_2(z) - \int_{-z}^{z} K(z,y)\,\bar{p}_2(y)\,dy\right) \times \bar{w}\left(-\frac{1}{2}, t\right) + \left(\bar{p}_1(z) - \int_{-z}^{z} K(z,y)\,\bar{p}_1(y)\,dy\right) \bar{w}\left(\frac{1}{2}, t\right) \tag{71}$$

$$\hat{w}_{1_z}\left(-\frac{1}{2}, t\right) = (k(0,0) + p_{00})\,\bar{w}\left(-\frac{1}{2}, t\right) \tag{72}$$

$$\hat{w}_{1_z}\left(\frac{1}{2}, t\right) = (k(1,1) + p_{11})\,\bar{w}\left(\frac{1}{2}, t\right), \tag{73}$$

where we also used the facts that $\bar{e}\left(\frac{1}{2}, t\right) = \bar{w}\left(\frac{1}{2}, t\right)$ and $\bar{e}\left(-\frac{1}{2}, t\right) = \bar{w}\left(-\frac{1}{2}, t\right)$, which follow from (61) and (62), respectively. From transformations (69), (70) it also follows that there exist positive constants m_5 and m_6 such that

$$\|\hat{w}_1(t)\|_{H^1} \le m_5 \|\hat{\tilde{v}}_1(t)\|_{H^1} \tag{74}$$

$$\|\hat{\tilde{v}}_1(t)\|_{H^1} \le m_6 \|\hat{w}_1(t)\|_{H^1}. \tag{75}$$

Part 3: *Stability estimates and well-posedness*

The $(\bar{w}, \hat{w}_1)$ system is a cascade in which, the homogenous part of both subsystems is an exponentially stable (also in the H^1 norm) heat equation and the non-autonomous part, i.e., the $\hat{w}_1$ subsystem, is driven by the autonomous $\bar{w}$ subsystem. Therefore, employing similar arguments to the proof of Theorem 5.1 in [43] (see also, e.g., [14], [15], [42], [48], [49]) one can conclude that the $(\bar{w}, \hat{w}_1)$ system is exponentially stable in the H^1 norm, and hence, so is system $\left(\bar{e}, \hat{\tilde{v}}_1\right)$ (based on estimates (67), (68), (74), and (75)). Thus, $\|\hat{\tilde{v}}(t)\|_{H^1} + \|\tilde{e}(t)\|_{H^1} \le \bar{\nu}\left(\|\hat{\tilde{v}}(t_0)\|_{H^1} + \|\tilde{e}(t_0)\|_{H^1}\right) e^{-\bar{\mu}(t-t_0)}$, for all $t \ge t_0$, for some positive constants $\bar{\nu}$ and $\bar{\mu}$. Therefore, with relation $\tilde{e} = \tilde{v} - \hat{\tilde{v}}$ and employing Lemma 3 we arrive at

$$\|\hat{\tilde{v}}(t)\|_{H^1} + \|\tilde{\tilde{v}}(t)\|_{H^1} \le \bar{\nu}_1 \left(\|\hat{\tilde{v}}(t_0)\|_{H^1} + \|\tilde{\tilde{v}}(t_0)\|_{H^1}\right) \times e^{-\bar{\mu}(t-t_0)}, \quad \text{for all } t \ge t_0, \tag{76}$$

for some positive constant $\bar{\nu}_1$. From Lemma 1 (relation (54)) we conclude that

$$\|\hat{\tilde{v}}(t)\|_{H^1} + \|\tilde{\tilde{v}}(t)\|_{H^1} \le \rho\left(\|\hat{\tilde{v}}(t_0)\|_{H^1} + \|\tilde{u}(t_0)\|_{H^1}\right) \times e^{-\bar{\mu}(t-t_0)}, \quad \text{for all } t \ge t_0, \tag{77}$$

where the class $\mathcal{K}_\infty$ function ρ is given by $\rho(s) = \bar{\nu}_1 s + \bar{\nu}_1 \alpha_1(s)$. Since $\sup_{x \in [0,1]} |\theta(x,t)| \le 2\|\theta(t)\|_{H^1}$, for any

$\theta \in H^1(0,1)$, choosing any positive constant μ^* such that $\mu^* \leq \rho^{-1}\left(\frac{c}{2}\right)$, for some $0 < c < 1$, we get that (50) holds. Thus, using Lemma 2 (relation (55)) we get (52).

Due to the regularity properties of all control and observer kernels as well as of the reference trajectory, using the fact that $\tilde{v}$ is related to $\tilde{u}$ via (42), (43), (49) it follows that well-posedness of the closed-loop system may be studied using the $(\bar{w}, \hat{w}_1)$ system (65), (66), (71)–(73), with initial condition $(\bar{w}_0, \hat{w}_{1_0}) \in H^2\left(-\frac{1}{2}, \frac{1}{2}\right) \times H^2\left(-\frac{1}{2}, \frac{1}{2}\right)$, which satisfies the compatibility conditions. Well-posedness of $(\bar{w}, \hat{w}_1)$, with regularity as in Theorem 2, may be established with, e.g., [7], [16], following the arguments employed in, e.g., [49], (see also [27], [42]) and exploiting the cascade form of $(\bar{w}, \hat{w}_1)$ together with the regularity of $\bar{w}$. □

5 Application to Traffic Flow Control

We consider the model presented in Section 2. Setting $a = b = 1$, we obtain $y_1^{\mathrm{r}}(t) = \frac{1}{4}t$ and $y_2^{\mathrm{r}}(t) = -\frac{1}{2}$, since the maximum value of $\rho(1-\rho)$ is achieved at $\rho = \frac{1}{2}$ and is equal to $\frac{1}{4}$ (see Section 2). Employing (9)–(11), the reference trajectory and reference inputs are given in closed form as

$$u^{\mathrm{r}}(x,t) = \frac{1}{4}t + \frac{1-x}{2} \quad (78)$$

$$U_0^{\mathrm{r}}(t) = U_1^{\mathrm{r}}(t) = -\frac{1}{2}. \quad (79)$$

The control laws are given in (44), (45) with $c_1 = c_2 = 1$.

We choose $\epsilon = 0.25$, whereas the initial condition for the plant is defined as $u(x,0) = u^{\mathrm{r}}(x,0) + 0.1\sin(\pi x) = \frac{1-x}{2} + 0.1\sin(\pi x)$ and for the observer as $\hat{\tilde{v}}(x,0) = -0.1\sin(\pi x)$. Fig. 1 shows the output $u(1,t)$. It is evident that trajectory tracking is achieved. Fig. 2 shows the highway density ρ. One observes that ρ converges to the desired reference profile, namely, to the uniform profile $\rho^{\mathrm{e}}(x) = \frac{1}{2}$, $\forall\, x \in [0,1]$. Note that the output $u_x(1,t)$ equals $-\rho(1,t)$ and, according to Fig. 2, converges to $u_x^{\mathrm{r}}(1,t) = -\frac{1}{2}$. Fig. 3 compares the bilateral control efforts with the control efforts in the unilateral case (see, e.g., [5], [29]). It is evident that the unilateral design results in larger control effort, which may lead to practically unrealistic ordered values for flows or speeds.

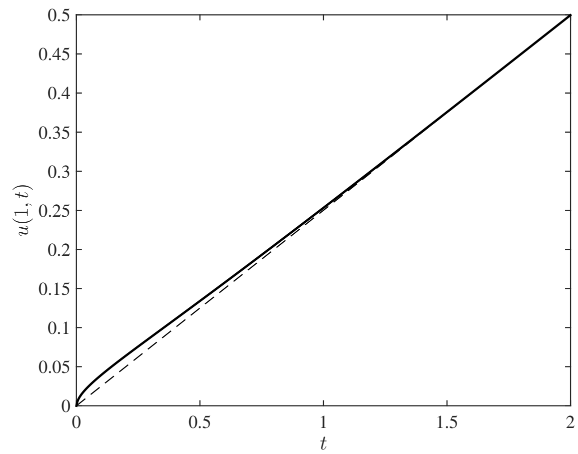

Fig. 1. Solid line: Output $u(1,t)$ of system (1)–(3) with $a = b = 1$, $\epsilon = 0.25$, under the feedback laws (44), (45) with (30)–(32), (47), (48) for $c_1 = c_2 = 1$ and initial conditions $u(x,0) = \frac{1-x}{2} + 0.1\sin(\pi x)$, $\hat{\tilde{v}}(x,0) = -0.1\sin(\pi x)$. Dashed line: The reference output $u^{\mathrm{r}}(1,t) = \frac{1}{4}t$.

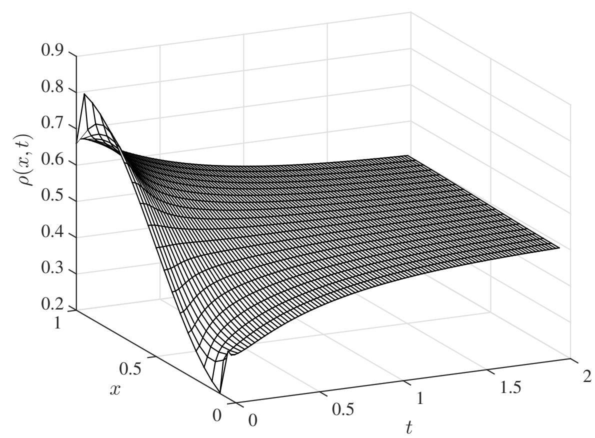

Fig. 2. The density evolution of the highway stretch.

6 Discussion and Future Work

In principle, a bilateral control design, besides requiring, in general, less total energy, it can be also made fault tolerant. To see this note that a bilateral control design can be made robust to the failure of one of the actuators, that is, one could switch to a unilateral control law as long as a fault is detected. Furthermore, it is expected that bilateral observers would be more robust with respect to, e.g., measurement noise, since, as in the case of bilateral control designs, in general, smaller output-injection gains are required compared to unilateral observers. More sophisticated fault-tolerant and robust control/observer designs could be pursued as future research.

As another potential topic of future research one may consider problems that involve interconnections of viscous HJ PDEs with Ordinary Differential Equations (ODEs), as it is the case, for example, in [20], which considers an interconnected system consisting of a viscous Burgers PDE and a linear ODE. The bilateral backstepping design used in this work can potentially deal with more complex PDE-ODE couplings than the standard unilateral design, thus we expect to be able to consider new families of previously unexplored systems. Another possible next step may be problems that incorporate viscous HJ PDE systems with actuator (or sensor) dynamics governed by certain types of ODEs or PDEs, as it is the case with, e.g., [31], which are dealing with viscous Burgers PDEs with ODE input dynamics.

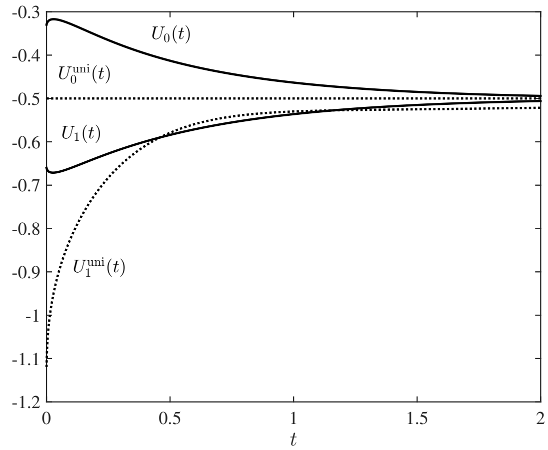

Fig. 3. Solid lines: Bilateral control efforts (44), (45). Dotted lines: Control efforts in the unilateral case.

Appendix A

Lemma A.1 *Let $f(t)$ be in $G_{F,M,\gamma}([0,+\infty))$ with $\gamma \in [1,2)$. Then the function $g(t) = e^{f(t)} - 1$ belongs to $G_{F_1,M_1,\gamma}([0,+\infty))$ with $F_1 = Fe^F$ and $M_1 = Me^F$.*

Proof From the power series expansion of the exponential function and the triangular inequality we obtain that

$$\left|g^{(n)}(t)\right| \le \sum_{k=1}^{\infty} \frac{\left|\frac{d^n f(t)^k}{dt^n}\right|}{k!}. \tag{A.1}$$

We claim that for any k-th power of f the following holds for all $n = 0, 1, \ldots$

$$\sup_{t\ge 0}\left|\frac{d^n f(t)^k}{dt^n}\right| \le (n+1)^{k-1} F^k M^n (n!)^\gamma, \tag{A.2}$$

which we prove by induction. For $k = 1$ our claim is true by assumption. Assume next that (A.2) holds for $k > 1$. We show that it holds for $k+1$. Employing Leibniz formula for the n-th derivative of the product of two functions we get

$$\left|\frac{d^n f(t)^{k+1}}{dt^n}\right| = \left|\sum_{i=0}^{n}\binom{n}{i} f^{(i)}(t) \frac{d^{n-i}\left(f(t)^k\right)}{dt^{n-i}}\right|, \tag{A.3}$$

and hence, using (A.2) we obtain

$$\left|\frac{d^n f(t)^{k+1}}{dt^n}\right| \le \sum_{i=0}^{n}\binom{n}{i}\left|f^{(i)}(t)\right|(n-i+1)^{k-1}F^k \times M^{n-i}((n-i)!)^\gamma, \tag{A.4}$$

which, under the assumption that $f \in G_{F,M,\gamma}([0,+\infty))$, in turn implies that

$$\left|\frac{d^n f(t)^{k+1}}{dt^n}\right| \le F^{k+1}M^n(n+1)^{k-1} \times \sum_{i=0}^{n}\binom{n}{i}(i!)^\gamma((n-i)!)^\gamma. \tag{A.5}$$

Thus, from the definition of the binomial coefficient we get

$$\left|\frac{d^n f(t)^{k+1}}{dt^n}\right| \le (n+1)^{k-1}F^{k+1}M^n(n!)^\gamma \times \sum_{i=0}^{n}\binom{n}{i}^{1-\gamma}, \tag{A.6}$$

which gives $\left|\frac{d^n f(t)^{k+1}}{dt^n}\right| \le (n+1)^k F^{k+1}(M)^n (n!)^\gamma$, where we used the fact that $\sum_{i=0}^{n}\binom{n}{i}^{1-\gamma} \le \sum_{i=0}^{n} 1 = n+1$, for $\gamma \in [1,2)$. Hence, (A.2) is proved, which gives with (A.1)

$$\left|g^{(n)}(t)\right| \le M^n (n!)^\gamma \frac{1}{n+1}\sum_{k=1}^{\infty}\frac{F^k (n+1)^k}{k!}, \tag{A.7}$$

and hence, $\left|g^{(n)}(t)\right| \le M^n (n!)^\gamma \frac{1}{n+1}\left(e^{F(n+1)} - 1\right)$. Since $e^r - 1 \le re^r$, $\forall\, r \ge 0$, we get the following estimate

$$\sup_{t\ge 0}\left|g^{(n)}(t)\right| \le Fe^F\left(e^F M\right)^n (n!)^\gamma, \tag{A.8}$$

for all $n = 0, 1, 2, \ldots$, which concludes the proof. □

Lemma A.2 *Let $\bar{f}(t)$ and $\bar{g}(t)$ be in $G_{F,M,\gamma}([0,+\infty))$ with $\gamma \in [1,2)$. Then the function $h(t) = e^{\bar{f}(t)}\bar{g}(t)$ belongs to $G_{F_2,M_2,\gamma}([0,+\infty))$ with $F_2 = F\left(1 + Fe^F\right)$ and $M_2 = \left(1 + Fe^F\right)Me^F$.*

Proof We start by writing the function h as $h(t) = \left(e^{\bar{f}(t)} - 1\right)\bar{g}(t) + \bar{g}(t)$. Using Leibniz formula for the n-th derivative of the product of two functions we get $\left|h^{(n)}(t)\right| = \left|\sum_{i=0}^{n}\binom{n}{i}\bar{g}^{(i)}(t)\frac{d^{n-i}\left(e^{\bar{f}(t)}-1\right)}{dt^{n-i}} + \bar{g}^{(n)}(t)\right|$, and hence, from Lemma A.1 we obtain

$$\left|h^{(n)}(t)\right| \le FF_1 M^{*n}\sum_{i=0}^{n}\binom{n}{i}(i!)^\gamma((n-i)!)^\gamma + FM^{*n}(n!)^\gamma, \tag{A.9}$$

where $M^* = \max\{M, M_1\}$. With similar arguments to the proof of Lemma A.1 and employing Bernoulli's inequality we get the following estimate for all $n = 0, 1, 2, \ldots$

$$\sup_{t\ge 0}\left|h^{(n)}(t)\right| \le F(1+F_1)M^{*n}(1+F_1)^n(n!)^\gamma. \tag{A.10}$$

The proof is completed with M_1 and F_1 from Lemma A.1. □

Appendix B

From (42) and (9) it follows that

$$\tilde{v}(x,t) = \tilde{\bar{v}}(x,t)e^{-\frac{ab}{2\epsilon}x}\left(e^{\frac{ab}{2\epsilon}x}v^{\mathrm{r}}(x,t)+1\right). \tag{B.1}$$

Under the conditions of Theorem 1, which also guarantee that v^{r} is uniformly bounded with respect to t and x (see (21)), we obtain for all $x \in [0,1]$ and $t \geq t_0$ that $\tilde{v}(x,t)^2 \leq \nu_1 \tilde{\bar{v}}(x,t)^2$, for some positive constant ν_1, and hence,

$$\|\tilde{v}(t)\|_{L^2}^2 \leq \nu_1 \|\tilde{\bar{v}}(t)\|_{L^2}^2. \tag{B.2}$$

Moreover, differentiating (B.1) with respect to x we get that

$$\begin{aligned}\tilde{v}_x(x,t) = &\tilde{\bar{v}}_x(x,t)\left(v^{\mathrm{r}}(x,t)+e^{-\frac{ab}{2\epsilon}x}\right)\\ &+\tilde{\bar{v}}(x,t)\left(v_x^{\mathrm{r}}(x,t)-\frac{ab}{2\epsilon}e^{-\frac{ab}{2\epsilon}x}\right).\end{aligned} \tag{B.3}$$

Mimicking the arguments of boundedness for v^{r} in the proof of Theorem 1, it is shown that v_x^{r} is uniformly bounded with respect to t and x. Hence, it follows from (B.3) that $\|\tilde{v}_x(t)\|_{L^2}^2 \leq \nu_2 \|\tilde{\bar{v}}(t)\|_{H^1}^2$, for some positive constant ν_2, and thus, with (B.2) we arrive at (56) with $\xi_1 = \sqrt{\nu_1}+\sqrt{\nu_2}$. Under the conditions of Theorem 1, using relations (21), (B.1) we obtain from (B.1) for all $x \in [0,1]$ and $t \geq t_0$

$$\|\tilde{\bar{v}}(t)\|_{L^2}^2 \leq e^{\left|\frac{ab}{\epsilon}\right|}(1-\bar{c})^{-2}\|\tilde{v}(t)\|_{L^2}^2. \tag{B.4}$$

With (21), solving (B.3) for $\tilde{\bar{v}}_x$ and using (B.4), there exists a positive constant ν_3 such that $\|\tilde{\bar{v}}_x(t)\|_{L^2}^2 \leq \nu_3 \|\tilde{v}(t)\|_{H^1}^2$, and hence, with (B.4) we get (57) with $\xi_2 = \sqrt{\nu_3}+\frac{e^{\left|\frac{ab}{2\epsilon}\right|}}{1-\bar{c}}$.

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