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Phase space regions operators and ISp(2) maps

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Abstract. Region operators (generalized observable) (RO), defined as operator valued probability measures over classical phase space ensuing from quantization of phase space characteristic functions are investigated. The problem of constructing of RO is treated by means of positive trace increasing maps (pti maps). Exemplary domains of RO include: sets of points, line segments and rotated line segments, circles and disks, straight and rotated lines, canonical polygons, squeezed line segments, and lines related to Radon transform in phase space. Phase plane RO constructing pti maps, in their operator sum representation, are generated by elements of the quantum symplectic in-homogeneous group ISp(2). The group ISp(2) and its double cover metaplectic group Mp(2) are the kinematical groups for RO construction on the plane.

1. Introduction

The general probabilistic framework of quantum mechanics in phase space requires the evaluation of (quasi) probability measures obtained by integration of a (quasi) probability distribution function such as the P, the Q, or the Wigner function, over phase space areas[1]. This integration problem is equivalent to an integration of a corresponding operator valued measure (OVM) over a given phase space region. Alternatively RO as operators ensuing from the quantization of phase space characteristic functions various regions can be constructed following various quantization schemes related to ordering of the creation-annihilation operators of the canonical algebra[2]. In either case the resulting operators are called region operators (RO), and their construction can be addressed by techniques that generate them starting from RO for points. E.g. the RO for the zero point of phase plane ([3]-[6]) is the parity operator of the canonical algebra. Creating RO for extended regions is shown to require the action of positive trace increasing maps (pti maps), which in their operator sum representation have generators identified with (sub)group elements of the quantum symplectic in-homogeneous group $ISp(2)$. Explicitly what is used is the adjoint action of the symplectic group that is provided by its double covering group known as the metaplectic group $Mp(2)$[7]. Examples of phase space region with constructed RO include sets of points on axes, line segments and rotated line segments, circles and disks, straight and rotated lines, canonical polygons, and squeezed line segments, as well as lines related to Radon transform reconstruction problem in phase space ([10]-[16]). This work shows that the symplectic group unifies the construction technique of all those cases and employs the group $ISp(2)$ as the

appropriate kinematical group for all region operator (generalized observable) constructions on the phase plane.

2. Phase space OVM: Region Operators

General setting: Let Γ a symplectic phase space and let an operator valued measure (OVM) taken to be a map $\widehat{R} : \mathcal{F}(\Gamma) \to \mathcal{L}(\mathcal{H})$, from the sigma algebra $\mathcal{F}(\Gamma)$ of Γ to the set of bounded linear operators $\mathcal{L}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$. The OVM has the following properties: for vectors $\Psi \in \mathcal{H}$ and elements $A \in \mathcal{F}(\Gamma)$, the function $\mu_\Psi(A) \equiv \langle \Psi | \widehat{R}(A)\Psi \rangle$, is normalized i.e. $\mu_\Psi(\Gamma) = 1$; also $\mu_\Psi(A)$ is a generalized probability measure in Γ, i.e. its (not necessarily) positive and σ additive set function. If $\widehat{R}(A) > 0$ i.e. $\mu_\Psi(A) > 0, \forall \Psi \in \mathcal{H}$, then we have a positive OVM (POVM); further if $\widehat{R}(A)^\dagger = \widehat{R}(A)$ and $\widehat{R}(A)^2 = \widehat{R}(A)$, i.e. we have a projective and positive OVM (PVM), namely a quantum mechanical observable, [2].

Let the special case of canonical phase space $\Gamma \approx \mathbf{C}$, with $\mathcal{H}_F = span_{\mathbf{C}}\{|n\rangle, n \in \mathbf{N}_0\}$, the Fock Hilbert (FH) space which spanned by the number state basis. Let the creation $\widehat{a}^\dagger$, annihilation $\widehat{a}$, and *number* operator $\widehat{N} = \widehat{a}^\dagger \widehat{a}$, respectively. The unitary displacement operator

$$\widehat{D}(x, p) = \exp(ix\widehat{P} - ip\widehat{X}) = \widehat{D}(\alpha) = \exp(\alpha \widehat{a}^\dagger - \alpha^* \widehat{a}),$$

with $\alpha = \frac{1}{\sqrt{2}}(x + ip)$, and $\widehat{P} = \frac{1}{i2}(\widehat{a}^\dagger - \widehat{a})$, $\widehat{X} = \frac{1}{2}(\widehat{a}^\dagger + \widehat{a})$, respectively the *momentum* and *position* operator. Also let the parity operator $\widehat{\Pi} = (-1)^{\widehat{N}} = e^{i\pi\widehat{N}}$, determined via the number operator, expressed in terms of position and momentum eigen-states respectively

$$\widehat{\Pi} = \int_{\mathbf{R}} dx | -x \rangle \langle x | = \int_{\mathbf{R}} dp | -p \rangle \langle p |.$$

The region operator with support on phase space region $A \in \Gamma$ reads

$$\widehat{R}(A) \equiv \int_A \widehat{D}(\alpha) \widehat{\Pi} \widehat{D}(\alpha)^\dagger \frac{d^2\alpha}{\pi}. \tag{1}$$

It is an OVM in Γ, which for a given quantum system described by $|\Psi\rangle \in \mathcal{H}_F$ provides the quasi-probability accumulated over region A and under the Wigner function $W_{|\Psi\rangle}(\alpha) = \langle \Psi | \widehat{\Delta}(\alpha) |\Psi\rangle$, according to the formula, ([8],[9],[19]),

$$\int_A W_{|\Psi\rangle}(\alpha) \frac{d^2\alpha}{\pi} = Tr(|\Psi\rangle \langle \Psi | \widehat{R}(A)) = \langle \Psi | \widehat{R}(A) |\Psi\rangle. \tag{2}$$

An alternative way of introducing region operators is to invoke the quantization program which, according to certain quantization rules, maps functions of classical phase space Γ to operators, and then apply such maps specifically to characteristic functions of regions in Γ. Indeed let χ_A the characteristic function of region $A \in \mathcal{F}(\Gamma)$, defined as $\chi_A(\alpha) = 1$ for $\alpha \in A$, or $\chi_A(\alpha) = 0$ for $\alpha \in A$, and let the following orthonormal operator basis $\{\widehat{\Delta}(x, p), (x, p) \in \Gamma\}$. This is constructed in terms of the parity operator $\widehat{\Pi}$ and the displacement operator as follows $\widehat{\Delta}(x, p) := \widehat{D}(x, p) \widehat{\Pi} \widehat{D}(x, p)^\dagger$.

This generalized operator basis satisfies, in terms of the trace inner product, the orthonormality relation

$$Tr(\widehat{\Delta}(x, p), \widehat{\Delta}(x', p')) = \frac{\pi}{2} \delta(x - x') \delta(p - p'),$$

as well as the completeness relation

$$\int \widehat{\Delta}(x, p) Tr(\widehat{\Delta}(x, p) \widehat{F}) dx dp = \frac{\pi}{2} \widehat{F},$$

for any well behaved operator $\widehat{F}$ acting on $\mathcal{H}$. By means of the operator basis the so called Wigner-Weyl transform (WWT) $\mathcal{W}_s$ and its inverse $\mathcal{W}_s^{-1}$ (known as Weyl map), are introduced. The inverse Weyl map maps classical phase space functions $F(q,p)$ to operators $\widehat{F}$, implementing the following quantization map

$$\mathcal{W}^{-1} : F(x,p) \rightarrow \mathcal{W}^{-1}(F) = \widehat{F} = \frac{1}{\pi} \int \widehat{\Delta}(x,p)F(x,p))dxdp$$

The opposite de-quantization map establishes a correspondence from operators $\widehat{F}$ to classical phase space functions as follows,

$$\mathcal{W} : \widehat{F} \rightarrow \mathcal{W}(\widehat{F}) = F(x,p) = 2Tr(\widehat{\Delta}(x,p)\widehat{F}).$$

The (de)-quantization rule upon quantization gives rise to normally ordered operators, i.e. to expressions in which the creation operators are on the right of annihilation operators ([3]-[6]).

For the particular case of density operators $\widehat{F} \equiv \rho$, which are Hermitian, positive and have the property $Tr\rho = 1$, the de-quantization maps $\rho \rightarrow F_\rho(x,p)$ provide the Wigner function W i.e. $F(x,p) \equiv W(x,p)$. The following identity is used

$$\widehat{\Delta}(x,p) = \widehat{D}(x,p)\widehat{\Pi}\widehat{D}(x,p)^\dagger = \widehat{D}(2x,2p)\widehat{\Pi}.$$

Let χ_A to be the characteristic function of region $A \in \mathcal{F}(\Gamma)$, and let its quantization by means of the operator

$$\widehat{R}(\chi_A) \equiv \widehat{R}(A) = \int_\Gamma \chi_A(\alpha)\widehat{\Delta}(\alpha)\frac{d^2\alpha}{\pi}. \tag{3}$$

This is an OVM on Γ ([20]), which shares the properties

$$\begin{aligned} Tr\widehat{R}(A) &= area(supp(\widehat{R}(A))) = area(supp(\chi_A)) \\ &= area(A) = \int_\Gamma \chi_A(\alpha)\frac{d^2\alpha}{\pi}, \end{aligned}$$

obtained by virtue of the trace of parity operator $Tr\widehat{\Pi} = Tr \int dx| -x\rangle\langle x| = \int dx\delta(2x) = 1$.

Transformation maps and duality. Let a phase space area $A \in \Gamma$ and the respective region operator $\widehat{R}(A)$. We consider the group $G = ISp(2,\mathbf{C})$, of translations and symplectic (area preserving) transformations on the plane (see below for details).

Let $g \in G$ and T_g a unitary faithful representation of the group carried by an appropriate Hilbert space $\mathcal{H}$ with properties $T_g^\dagger = T_{g^{-1}}$, and $T_g^\dagger T_g = T_{g^{-1}}T_g = T_{g^{-1}g} = T_\mathbf{1} = \mathbf{I}_\mathcal{H}$. We consider the phase plane as a representation module of group G implemented by the action $m : G \times \Gamma \rightarrow \Gamma$, $(g,\alpha) \rightarrow m(g,\alpha) = g(\alpha)$. The action m induces on the phase space point $\alpha = (x,p)$ a combined transformation of symplectic rotation followed by a phase space translation.

Concerning the operator valued measure $\widehat{\Delta}(\alpha)$, it is also transformed in its argument i.e. $\alpha \rightarrow g(\alpha)$, by the action of metaplectic representation of $Sp(2,\mathbf{C})$ denoted $Mp(2)$ [7],[17](c.f. [7] for the related theory of the metaplectic group $Mp(2)$ the double cover group $Sp(2,\mathbf{C})$ and the three ways of introducing the metaplectic operators in the context of phase space quantum mechanics, as well for related prior bibliography on the subject), $\widehat{\Delta}(\alpha) \rightarrow T_g^\dagger \widehat{\Delta}(\alpha)T_g = \widehat{\Delta}(g(\alpha))$.

The integration measure is invariant under map m i.e. $\frac{d^2(g^{-1}(\alpha))}{\pi} = \frac{d^2\alpha}{\pi}$. As a consequence the transformation of a RO under $ISp(2,\mathbf{C})$ reads

$$\hat{R}(A) \longrightarrow T_g^\dagger \hat{R}(A) T_g = \int_\Gamma \chi_A(\alpha) T_g^\dagger \hat{\Delta}(\alpha) T_g \frac{d^2\alpha}{\pi}$$

$$= \int_\Gamma \chi_A(\alpha) \hat{\Delta}(g(\alpha)) \frac{d^2\alpha}{\pi} = \int_\Gamma \chi_A(g^{-1}(\alpha)) \hat{\Delta}(\alpha) \frac{d^2\alpha}{\pi}.$$

Hence under a $ISp(2,\mathbf{C})$ unitary action, an initial RO becomes another RO while its support transformed by the induced action of fundamental matrix action of $ISp(2,\mathbf{C})$.

Unitary transformations on RO are further generalized to positive maps $\mathcal{E}_t(.)$ defined for functions $t(g)$, $g \in G$, with generators $T_g \in ISp(2,\mathbf{C})$ ([18]),

$$\mathcal{E}_t(.) = \int_\Gamma t(g) T_g^\dagger(.) T_g d\mu(g),$$

acting on RO as ([15])

$$\hat{R}(A) \longrightarrow \mathcal{E}_t(\hat{R}(A)) = \int_G t(g) \; T_g^\dagger \hat{R}(A) T_g \; d\mu(g) \tag{4}$$

$$= \int_\Gamma \left[\int_G t(g) \chi_A(g^{-1}(\alpha)) \; d\mu(g) \right] \hat{\Delta}(\alpha) \frac{d^2\alpha}{\pi}.$$

This action can be expressed in terms of the $ISp(2,\mathbf{C})$ convolution

$$(t * \chi_A)(\alpha) \equiv \int_G t(g) \chi_A(g^{-1}(\alpha)) d\mu(g)$$

as

$$\mathcal{E}_t(\hat{R}(A)) = \int_\Gamma (t * \chi_A)(\alpha) \hat{\Delta}(\alpha) \frac{d^2\alpha}{\pi}. \tag{5}$$

Maps $\mathcal{E}_t$ are in general taken to be positive, and not necessarily trace preserving. Positivity is required as these maps, by their dual action (see below), are required to preserve the positivity of density operator describing the state of quantum system under investigation. On the other hand the property of non trace preservation, should be allowed since $\mathcal{E}_t$ can be introduced not only as maps transforming the area A of the RO they act upon, but also as generators of RO for various domains in Γ. In particular trace increasing maps, for which the property $\tilde{t} := \int_G t(g) \; d\mu(g) \geq 1$, is satisfied, have been introduced and investigated for a variety of phase space domains c.f.

The general property of the family of trace increasing maps $\mathcal{E}_t$, with trace (area) increasing factor $\tilde{t} \geq 1$,

$$Tr\mathcal{E}_t(\hat{R}(A)) = \int_\Gamma (t * \chi_A)(\alpha) \frac{d^2\alpha}{\pi} = \tilde{t} \; \times Tr(\hat{R}(A)) = \tilde{t} \; \times area(g(A)) \geq Tr(\hat{R}(A)), \tag{6}$$

is satisfied for any RO $\hat{R}(A)$.

In view of eqs.(3,5), the transformed by map $\mathcal{E}_t$ RO, is expressed as $\mathcal{E}_t(\hat{R}(\chi_A)) = \hat{R}(t * \chi_A)$. The convoluted characteristic function χ_A characterizing the mapped RO has the support

$$supp(t * \chi_A) = \int_G t(g) \chi_{g(A)} d\mu(g) = \bigcup_{g \in supp(t)} supp(\chi_{g(A)}), \tag{7}$$

where $\chi_{g(A)}(\alpha) = 1$ if $\alpha \in g(A)$, or 0 if $\alpha \notin g(A)$, with $g(A) = \{g(\alpha) \mid \alpha \in A \subset \Gamma \}$. This in turn implies for the support of the transformed RO $\mathcal{E}_t(\widehat{R}(\chi_A))$ that

$$supp(\mathcal{E}_t(\widehat{R}(\chi_A))) = supp(\widehat{R}(t * \chi_A) = supp(t * \chi_A).$$

Finally regarding the state-observable duality we notice that in terms of the trace inner product $\left\langle \rho, \widehat{R}(A) \right\rangle = Tr(\rho(\widehat{R}(A)))$,the quasi-probability mass (qpm) is expressed as qpm $=\langle density\,operator, region\,operator \rangle$. From this relation the dual expression of the qpm follows [21]

$$qpm = \left\langle \rho, \mathcal{E}_t(\widehat{R}(A)) \right\rangle = Tr(\rho \mathcal{E}_t(\widehat{R}(A))), \tag{8}$$

$$qpm = \left\langle \mathcal{E}_t^*(\rho), \widehat{R}(A) \right\rangle = Tr(\mathcal{E}_t^*(\rho)\widehat{R}(A)). \tag{9}$$

3. Generating region operators

In this section we intend to show how a general group element of the $ISp(2, \mathbf{C})$ group transforms a generic region operator of certain phase space domain $A \in \Gamma$, and in particular to study the change induced by such a transformation into the characteristic function of domain A. By means of the commutation relations among the two subalgebra generators issued in equations (15,16,17) and the bosonic realization of those generators in eq. (18,19), we obtain the following similarity transformation of region operator $\widehat{R}(A)$ by the rotation element $e^{i\phi\widehat{N}}$,

$$\widehat{R}(A) \longrightarrow e^{i\phi\widehat{N}}\widehat{R}(A)e^{-i\phi\widehat{N}} = \int_{\mathbf{C}} \chi_A(\alpha)\widehat{\Delta}(e^{i\phi}\alpha)\frac{d^2\alpha}{\pi}$$

$$= \int_{\mathbf{C}} \chi_A(e^{-i\phi}\alpha)\widehat{\Delta}(\alpha)\frac{d^2\alpha}{\pi}. \tag{10}$$

Last equation indicates that the transformed region operator is the same as the original one except that its domain is rotated by an angle ϕ.

The analogues question is addressed for the transformation of region operator by a general $W_2(\mathbf{C})/U(1)$ element $\widehat{D}(\beta)$, and the result is

$$\widehat{R}(A) \longrightarrow \widehat{D}(\beta)\widehat{R}(A)\widehat{D}(\beta)^\dagger = \int_{\mathbf{C}} \chi_A(\alpha)\widehat{\Delta}(\alpha + \beta)\frac{d^2\alpha}{\pi}$$

$$= \int_{\mathbf{C}} \chi_A(\alpha - \beta)\widehat{\Delta}(\alpha)\frac{d^2\alpha}{\pi} \tag{11}$$

Last equation indicates that the transformed region operator remains the same to the initial one except that its domain is shifted by a length β.The proof of this is based on the property $\widehat{D}(\mu)\widehat{D}(\nu) = \exp(\frac{1}{2}\begin{vmatrix} \mu & \mu^* \\ \nu & \nu^* \end{vmatrix})\widehat{D}(\mu + \nu)$. This property indicates that the $\widehat{D}$ operator carries a projective representation of the abelian group of translations in $\mathbf{C}$, and is used to show the necessary formula for the validity of eq. (11), namely that $\widehat{D}(\beta)\widehat{\Delta}(\alpha)\widehat{D}(\beta)^\dagger = \widehat{\Delta}(\alpha + \beta)$, as follows

$$\widehat{D}(\beta)\widehat{\Delta}(\alpha)\widehat{D}(\beta)^\dagger = \widehat{D}(\beta)\widehat{D}(2\alpha)\widehat{\Pi}\widehat{D}(\beta)^\dagger = \widehat{D}(\beta)\widehat{D}(2\alpha)\widehat{D}(\beta)\widehat{\Pi}$$

$$= \exp(\frac{1}{2}\begin{vmatrix} \beta & \beta^* \\ 2\alpha & 2\alpha^* \end{vmatrix})$$

$$\times \exp(\frac{1}{2}\begin{vmatrix} 2\alpha + \beta & 2\alpha^* + \beta^* \\ \beta & \beta^* \end{vmatrix})\widehat{D}(2\alpha + 2\beta)\widehat{\Pi}$$

$$= \widehat{D}(2\alpha + 2\beta)\widehat{\Pi} = \widehat{D}(\alpha + \beta)\widehat{\Pi}\widehat{D}(\alpha + \beta)^\dagger = \widehat{\Delta}(\alpha + \beta).$$

Finally transformations of RO by coset elements $Sp(2, \mathbf{C})/U(1)$ parametrized by $\gamma = re^{i\phi}$, the so called squeezing transformations $\widehat{S}(r, \phi)$, c.f. eq. (20), can been worked out utilizing the commutation relations of the generators of the Lie($ISp(2, \mathbf{C})$) algebra,

$$\widehat{S}(r, \phi)^\dagger \widehat{D}(\alpha) \widehat{S}(r, \phi) = \widehat{D}(\alpha \cosh r - a^* e^{i\phi} \sinh r),$$

in particular for $\phi = 0$, and $\alpha = \alpha_R + i\alpha_I$, and if we let $\widehat{S}(r, \phi = 0) = \widehat{S}(r)$, we obtain a new transformation $\widehat{S}(r)^\dagger \widehat{D}(\alpha) \widehat{S}(r) = \widehat{D}(\alpha_R e^r + ia_I e^{-r})$.

Since the squeezing operator is even i.e. $\widehat{S}(r, \phi)\widehat{\Pi} = \widehat{\Pi}\widehat{S}(r, \phi)$ then in view of the definition of eq. (1), we obtain

$$\widehat{R}(A) \longrightarrow \widehat{S}(\gamma)\widehat{R}(A)\widehat{S}(\gamma)^\dagger = \int_{\mathbf{C}} \chi_A(\alpha)\widehat{\Delta}(\alpha \cosh r - a^* e^{i\phi} \sinh r)\frac{d^2\alpha}{\pi}. \qquad (12)$$

To elaborate on this formula consider an element $g \in Sp(2) \approx SU(1, 1)$ in the fundamental 2D representation of that matrix group

$$g = \begin{pmatrix} \cosh r & -e^{i\phi} \sinh r \\ -e^{-i\phi} \sinh r & \cosh r \end{pmatrix}.$$

This element satisfies the (pseudo)unitary condition $g\sigma_3 g^\dagger = \sigma_3$, where $\sigma_3 = diag(1, -1)$; also $g^{-1} = \sigma_3 g^\dagger \sigma_3$. Next let $B = (\alpha \ \alpha^*)^T$ be a column vector, and let the transformation $B \longrightarrow B' = gB$, this yields $\alpha \cosh r - a^* e^{i\phi} \sinh r = (gB)_1 = (B')_1$, for the first component of the transform vector which is identified with the argument of the delta operator in eq. (12). Inverting the action of g elements cast the transformed region operator of equation (12) in the form

$$\widehat{R}(A) \longrightarrow \widehat{S}(\gamma)\widehat{R}(A)\widehat{S}(\gamma)^\dagger = \int_{\mathbf{C}} \chi_A(\alpha \cosh r + a^* e^{i\phi} \sinh r)\widehat{\Delta}(\alpha)\frac{d^2\alpha}{\pi}. \qquad (13)$$

The choice $\phi = 0$, yields $\chi_A(\alpha_R e^{-r} + ia_I e^r)$ for the characteristic function of region A, which implies that the effect of $\widehat{S}(r)$ operator on $\widehat{R}(A)$, is to stretch its horizontal and vertical components by the squeezing factors e^{-r} and e^r respectively.

Combining elements $\widehat{D}(\beta)$, and $e^{i\phi\widehat{N}}$, we construct a general $H(1)$ Heisenberg group element, and if we further combine this with $\widehat{S}(\gamma)$ we construct a general $ISp(2, \mathbf{C})$ group element. Therefore in order to find the effect of transforming general $\widehat{R}(A)$ region operators with general $ISp(2, \mathbf{C})$ group elements, it suffices to combine the transformations issued in eqs.(10,11,13), to obtain the total change effected on the characteristic function of $\widehat{R}(A)$, namely a combination of rotation, with translation and squeezing of the region A respectively, viz.

$$\begin{aligned} \chi_A(\alpha_R + i\alpha_I) & \rightarrow & \chi_A(\alpha_R \cos\phi - i\alpha_I \sin\phi), \\ & \rightarrow & \chi_A(\alpha_R - \beta_R + i(\alpha_{I-I}))\beta, \\ & \rightarrow & \chi_A(\alpha_R e^{-r} + ia_I e^r). \end{aligned} \qquad (14)$$

These actions are induced by unitary similarity transformations in FH space, the irrep module of bosonic realization of $ISp(2, \mathbf{C})$ group.

4. Region operators constructed by $ISp(2, \mathbf{C})$ maps

For the various domains in the table below the pti maps creating the respective RO (c.f. eq. (4), have Kraus generators given in the 2nd column identified with $ISp(2, \mathbf{C})$ elements. The $ISp(2, \mathbf{C})$ group via its metaplectic group $Mp(2)$ serves as the kinematical (motion) group for building any RO in phase plane.

Region operator	$ISp(2, \mathbf{C})$ generators T_g	Act on	Integration domain
Points	$e^{i\frac{x}{2}(A+A^\dagger)}$	$\widehat{\Pi}$	$x \in \{1, 2, ..., n\}$
Line segment	$e^{i\frac{x}{2}(A+A^\dagger)}$	$\widehat{\Pi}$	$x \in [-\frac{L}{2}, \frac{L}{2}]$
Rotated line segment	$e^{i\frac{x}{2}(A+A^\dagger)}, e^{i\theta(2K_0-\frac{1}{2}\mathbf{1})}$	$\widehat{\Pi}$	$x \in [-\frac{L}{2}, \frac{L}{2}], \theta \in [0, \Theta]$
Circle	$e^{i\frac{x}{2}(A+A^\dagger)}, e^{i\theta(2K_0-\frac{1}{2}\mathbf{1})}$	$\widehat{\Pi}$	$x = \frac{r}{2}, \theta \in [0, 2\pi)$
Disk	$e^{i\frac{x}{2}(A+A^\dagger)}, e^{i\theta(2K_0-\frac{1}{2}\mathbf{1})}$	$\widehat{\Pi}$	$x \in [-\frac{r}{2}, \frac{r}{2}], \theta \in [0, 2\pi)$
Straight line segm.	$e^{i\frac{x}{2}(A+A^\dagger)}, e^{i\frac{y}{2i}(A-A^\dagger)},$	$\widehat{\Pi}$	$x \in R, y = q$
Squeezed line segm.	$e^{\frac{r}{2}(e^{i\theta}A^{\dagger 2}-e^{-i\theta}A^2)}, e^{i\frac{x}{2}(A+A^\dagger)}$	$\widehat{\Pi}$	$x \in [-\frac{L}{2}, \frac{L}{2}], \frac{r}{2}e^{i\theta}, r \in R$
Canonical polygon	$e^{is\phi(2K_0-\frac{1}{2}\mathbf{1})}$	$\widehat{R}_\triangledown$	$s \in \{0, 1, ..., 2^m-1\}$
Radon line	$e^{\frac{1}{\sqrt{2}}it\Omega(Ae^{-i\theta}+A^\dagger e^{i\theta})}$	$\widehat{\Delta}(u, \theta)$	$t \in R$

5. Appendix: The group $ISp(2, \mathbf{C})$

The symplectic group in n dimensions is the semi-direct product of the group $ISp(n, \mathbf{C})$ with the n dimensional Heisenberg-Weyl (HW) group $W_n(\mathbf{C})$, i.e. $Sp(n, \mathbf{C}) = Sp(n, \mathbf{C}) \times W_n(\mathbf{C})$. For $n = 2$, the simplest group is $ISp(2, \mathbf{C}) \approx Sp(2, \mathbf{C}) \times W_2(\mathbf{C})$. If alternatively we use the metaplectic group $Mp(2)$ which is the double cover of $Sp(2, \mathbf{C})$. The root vector space of the algebra of $ISp(2, \mathbf{C})$ is one dimensional, i.e. the group is of rank one and is not semisimple. The generators are $\text{Lie}(ISp(2, \mathbf{C})) \equiv \{K_+, K_-, K_0, A_+, A_-, \mathbf{I}\}$, $\text{Lie}(Sp(2, \mathbf{C})) \equiv \{K_+, K_-, K_0\}$, and $\text{Lie}(W_2(\mathbf{C})) \equiv \{A_+, A_-, \mathbf{I}\}$, commutation relations,

$$[K_-, K_+] = 2K_0, [K_0, K_\pm] = \pm K_\pm,$$

and $[*, \mathbf{I}] = 0, [A_-, A_+] = \hbar\mathbf{I}$, where $*$ stands for any generators. The inhomogeneous part of $ISp(2, \mathbf{C})$ HW generators A_-, A_+, which form the central extension of the Euclidean group of translations on the plane with extension parameter $\hbar$, to be taken hereafter $\hbar = 1$. The commutation relations among the sub-algebras read

$$[K_-, A_-] = 0, [K_-, A_+] = A_-, \tag{15}$$

$$[K_+, A_+] = 0, [K_+, A_-] = -A_+, \tag{16}$$

$$[K_0, A_\pm] = \pm\frac{1}{2}A_\pm. \tag{17}$$

The quadratic central element of the algebra (Casimir operator) reads, $C := K_0^2 - \frac{1}{2}(K_-K_+ + K_+K_-) = K_0(K_0 - \mathbf{I}) - K_+K_-$. In irreps the central element equals $C = k(k-1)\mathbf{I}$, where $\mathbf{I}$ is the unit operator. The discrete series representations are labelled by $k \in \mathbf{Q}$, and have module $\text{span}(\{|k, m\rangle, m = \mathbf{N}_0\})$, with the action of the generators

$$K_- |k, m\rangle = \sqrt{m(m-1+2k)} |k, m-1\rangle,$$

$$K_+ |k, m\rangle = \sqrt{(m+1)(m+2k)} |k, m+1\rangle,$$

$$K_0 |k, m\rangle = (m+k) |k, m\rangle, C |k, m\rangle = k(k-1) |k, m\rangle.$$

Two representations corresponding to $C = \frac{3}{16}\mathbf{I}$, where $\mathbf{I} = \sum_{m=0}^{\infty} |m\rangle \langle m|$ is the unit operator in the $\mathcal{H}$, for $k = \frac{1}{4}$ and $\frac{3}{4}$, are obtained by introducing the so called *bosonic realization* of the Lie($ISp(2, \mathbf{C})$) algebra. To this end let the Hilbert space $\mathcal{H} = span_{\mathbf{C}}\{|n\rangle, n = \mathbf{N}_0\} = \mathcal{H}_e \oplus \mathcal{H}_o$, where $\mathcal{H}_e = span_{\mathbf{C}}\{|2n\rangle, n = \mathbf{N}_0\}$ the even subspace and $\mathcal{H}_o = span_{\mathbf{C}}\{|2n+1\rangle, n = \mathbf{N}_0\}$ the odd subspace.

The bosonic realization of the Lie($ISp(2, \mathbf{C})$) reads

$$A_- = \widehat{a}, \ A_+ = \widehat{a}^\dagger, \tag{18}$$

$$K_- = \frac{1}{2}\widehat{a}^2, \ K_+ = \frac{1}{2}\widehat{a}^{\dagger 2}, \ K_0 = \frac{1}{4}(\widehat{a}\widehat{a}^\dagger + \widehat{a}^\dagger\widehat{a}) = \frac{1}{4}(2\widehat{N} + \mathbf{I}), \tag{19}$$

The two irreps obtained by the bosonic realization are labelled by $k = \frac{1}{4}$, and $k = \frac{3}{4}$. The Lie($ISp(2, \mathbf{C})$) generators act in these irreps in the even and odd subspaces respectively as follows: for the even subspace $\left|k = \frac{1}{4}, m\right\rangle \equiv |2m\rangle$ and the action is,

$$K_- |2m\rangle = \sqrt{m(m - \frac{1}{2})} |2m - 2\rangle,$$

$$K_+ |2m\rangle = \sqrt{(m + 1)(m + \frac{1}{2})} |2m + 2\rangle,$$

$$K_0 |2m\rangle = (m + \frac{1}{2}) |2m\rangle,$$

similarly for the odd subspace $\left|k = \frac{3}{4}, m\right\rangle \equiv |2m + 1\rangle$, and the action is,

$$K_- |2m + 1\rangle = \sqrt{m(m + \frac{1}{2})} |2m - 1\rangle,$$

$$K_+ |2m + 1\rangle = \sqrt{(m + 1)(m + \frac{3}{2})} |2m + 3\rangle,$$

$$K_0 |2m + 1\rangle = (m + \frac{3}{2}) |2m + 1\rangle.$$

For these irreps the following Hermitian conjugation relations are valid $K_-^\dagger = K_+$, $K_+^\dagger = K_-$, $K_0^\dagger = K_0$, and $(A_\mp)^\dagger = A_\pm$. Further the bosonic realization provides the relation between operators involved in the definition and the canonical transformations of region operators (see the table). Relations concerning rotations and translations of RO and elements of the $W_2(\mathbf{C})$ group are respectively

$$e^{i\phi\widehat{N}} = e^{i\phi(2K_0 - \frac{1}{2})} \ and \ \widehat{D}(\alpha) = \exp(\alpha A_+ - \alpha^* A_-),$$

while the generator of squeezing of RO is given in terms of coset elements $Sp(2, \mathbf{C})/U(1)$

$$\widehat{S}(r, \phi) = \exp(\frac{1}{2}r[e^{i\phi}\widehat{a}^{\dagger 2} - e^{-i\phi}\widehat{a}^2]) = \exp(r[e^{i\phi}K_+ - e^{-i\phi}K_-]). \tag{20}$$

6. Conclusion-Prospects

Generating generalized operator observables determined by various domains of interest in phase space is a project that addresses the question of going beyond the usual measurement theory in QM. As these generalized observables may take the form of operator valued measure that can be treated as a quantized form of classical characteristic functions supported on regions of

interest, the problem of construction becomes that of generating such region operators (RO) from known ones. The parity operator being the RO of 0 point serves as a start for building RO with increasing area. The need of maps which by their action on a RO accumulate area and on its defining region, suggests introducing positive trace increasing maps. Such maps admit unitary generators from a motion group. Identification such group leads to ISp(2) groups and metaplectic group Mp(2), in their bosonic realization and the FockHilbert space representations. Previous works of that have succeeded in constructing RO for various highly symmetric phase space domains find a unifying framework in the setting of kinematical group ISp(2). This would allow constructing more general RO in terms of various shapes dictated by applications. Finally the problem of addressing quantum entanglement in terms of RO can be addressed in this mathematical framework. The question of existence of RO having or not classical analogue for their supported regions in the presence of entanglement between quantum systems, can be addressed and investigated. The findings would provide a novel manifestation of quantum entanglement in the language of phase space region operators. [22].

7. References

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[22] Ellinas D and Bracken A J 2019 forthcoming work.