Using Nudging for the Control of a Non-Local PDE Traffic Flow Model*

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Abstract— **The paper provides conditions that guarantee existence and uniqueness of classical solutions for a non-local conservation law on a ring-road with nudging (or "look behind") terms. The obtained conditions are novel, as they are not covered by existing results in the literature. The paper also provides results which indicate that nudging can increase the flow in a ring-road and, if properly designed, can have a strong stabilizing effect on traffic flow. The efficiency of the use of nudging terms is demonstrated by means of a numerical example.**

I. Introduction

Non-local traffic flow models with Partial Differential Equations (PDEs) are based on extensions of the well-known Lighthill-Whitham-Richards model (LWR model, see [20, 23]), where the speed is given by a non-local term. These models fall into the class of non-local conservation laws (see [5]) and possess some different features compared to the LWR model. Arrhenius "look-ahead" terms were considered in [25, 18, 19] as a result of stochastic microscopic dynamics, and it was shown that such models can develop shocks (and shock waves) in finite time. On the other hand, the fact that human drivers and automated vehicles adjust the vehicle speed based on a perception of downstream density, rather than the local density, motivated some researchers to express the perceived density by means of non-local (convolution) terms. Such models were studied in [2, 3, 4, 8, 15, 16], and it was shown that they may be producing smooth solutions.

In the era of automated vehicles, the real-time information fed to each vehicle on a road is exploited for the appropriate adjustment of the speed of the vehicle. In contrast to manual driving, this information may include upstream density data, in addition to downstream density data. Note that human drivers base their driving decisions only on the perceived *downstream* traffic state, something that leads to the celebrated anisotropy principle in traffic flow modeling [7]. The possible beneficiary role of the use of upstream density data was pointed out in [22], where the effect of the upstream density data on the speed adjustment was termed as "nudging". Such an effect was also studied in [18] (without reference to automated vehicles), where an Arrhenius "look-behind" non-local term was used for the mathematical expression of the use of upstream density data.

The selection of the nudging term for automated vehicles can be considered as a feedback design problem. Data are fed into the automated vehicles, based on which the vehicles adjust their speed. In other words, the density profile of the road changes over time, and this change is fed back to each automated vehicle. From a mathematically perspective, the use of upstream density data should not be performed in an arbitrary way, but so as to satisfy conditions for existence and uniqueness of solutions, together with further requirements for the closed-loop system (e.g., stability, optimality, etc.). It should be noticed here that the feedback design problem for the expression of "nudging" or "look-behind" effect can be considered as a special feedback design problem for non-local, hyperbolic PDEs (see [6, 14, 17]). However, this specific feedback design problem is different from other traffic control problems studied in the literature (see [11, 13, 26, 27, 28]).

The present paper addresses these questions for a ring-road. We first present conditions which guarantee existence and uniqueness of classical solutions for a non-local conservation law with nudging terms (Theorem 1). The obtained results are novel, as they are not covered by the results in [2, 3, 4, 8, 15, 16], where either the use of upstream density data is not allowed or a ring-road is not studied. In addition, the present paper studies the effects of nudging and it is shown that:

(i) nudging can increase the flow in a ring-road at any density value;

(ii) if properly designed, nudging can have a strong stabilizing effect on ring-road traffic.

Indeed, we present results (Theorem 2) which guarantee local exponential stability of the uniform equilibrium profile in the L^2 state norm even for cases where the uniform equilibrium profile in a ring-road without nudging is not asymptotically stable and the model admits density waves. The existence of travelling waves for non-local conservation laws was studied in [24], where it is shown that travelling waves may occur even in non-local conservation laws.

The paper is organized as follows. Section II is devoted to the presentation of the non-local traffic flow models which are studied in the paper; moreover, the statements of the existence and uniqueness results for non-local traffic flow models are also given in Section II. The effects of nudging on ring-road traffic are studied in Section III. Illustrative numerical experiments are presented in Section IV, where the strong stabilizing effect of nudging is demonstrated. Finally, concluding remarks are given in Section V. Due to space constraints the proofs are omitted and can be found in [12].

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Notation. Throughout this paper, we adopt the following notation.

* $\Re_+ := [0,+\infty)$. For a vector $y \in \Re^N$, $y^\top$ denotes its transpose and $|y|_\infty = \max_{i=1,...,N}(|y_i|)$ denotes its infinity norm. For two vectors $x, y \in \Re^N$ we write $x \le y$ if and only if $x_i \le y_i$ for $i = 1,...,N$. We also define $y^{(0)} = y = (y_1,...,y_N)^\top \in \Re^N$, $y^{(1)} = (y_2,...,y_N, y_1)^\top \in \Re^N$ and $y^{(k)}$ for $k \ge 2$ by means of the recursive formula $y^{(k)} = (y^{(k-1)})^{(1)}$.
* Let $S \subseteq \Re^n$ be an open set and let $A \subseteq \Re^n$ be a set that satisfies $S \subseteq A \subseteq cl(S)$. By $C^0(A;\Omega)$, we denote the class of continuous functions on A, which take values in $\Omega \subseteq \Re^m$. By $C^k(A;\Omega)$, where $k \ge 1$ is an integer, we denote the class of functions on $A \subseteq \Re^n$, which takes values in $\Omega \subseteq \Re^m$ and has continuous derivatives of order k. In other words, the functions of class $C^k(A;\Omega)$ are the functions which have continuous derivatives of order k in $S = \text{int}(A)$ that can be continued continuously to all points in $\partial S \cap A$. When $\Omega = \Re$, we write $C^0(A)$ or $C^k(A)$.
* Let $T \in (0,+\infty)$ and $\rho : [0,T] \times I \to \Re$ be given, where $I \subseteq \Re$ is an interval. We use the notation $\rho[t]$ to denote the profile at certain $t \in [0,T]$, i.e., $(\rho[t])(x) = \rho(t,x)$ for all $x \in I$. $L^p(I)$ with $p \ge 1$ denotes the equivalence class of measurable functions $f : I \to \Re$ for which $\|f\|_p = \left(\int_I |f(x)|^p \, dx \right)^{1/p} < +\infty$. $L^\infty(I)$ denotes the equivalence class of measurable functions $f : I \to \Re$ for which $\|f\|_\infty = \operatorname{ess\,sup}_{x \in I}(|f(x)|) < +\infty$. We use the notation $f'(x)$ for the derivative at $x \in I$ of a differentiable function $f : I \to \Re$.
* $W^{2,\infty}([0,1])$ is the Sobolev space of C^1 functions on $[0,1]$ with Lipschitz derivative.
* $Per(\Re)$ denotes the set of continuous, positive mappings $\rho : \Re \to (0,+\infty)$ which are periodic with period 1, i.e., $\rho(x+1) = \rho(x)$ for all $x \in \Re$.

II. NON-LOCAL TRAFFIC FLOW MODELS

Many non-local PDE traffic flow models which have appeared in the literature (see [2, 3, 4, 8, 15, 16]) have the form

$$\frac{\partial \rho}{\partial t}(t,x) + \frac{\partial}{\partial x}\big(\rho(t,x)v(t,x)\big) = 0 \text{, for } t \ge 0, x \in \Re \quad (2.1)$$

$$v(t,x) = f\left(\int_x^{x+\eta} \omega(s-x)\rho(t,s)ds \right) \text{, for } t \ge 0, x \in \Re \quad (2.2)$$

where $\rho(t,x)$ denotes the traffic density, $v(t,x)$ denotes the mean speed, $t \ge 0$ is time, x is the spatial variable, $\eta > 0$ is a constant (reflecting the visibility area), $f : \Re_+ \to \Re_+$ and $\omega : \Re_+ \to \Re_+$ are non-increasing functions with $\int_0^\eta \omega(x)dx = 1$. Model (2.1), (2.2) constitutes a generalization of the classical LWR traffic flow model, where $f : \Re_+ \to \Re_+$ is the function that relates density to speed (fundamental diagram) and $\int_x^{x+\eta} \omega(s-x)\rho(t,s)ds$ is the downstream density perceived by the human driver at spatial position x. Thus, the driver adapts the speed according to (2.2) on the basis of the perceived downstream density.

As a farther generalization, when automated vehicles are present on a highway, there may be a benefit by allowing the vehicle speed to depend on upstream density levels as well. Such an effect has been termed in the literature as “nudging” (see [22]) or “look-behind” effect (see [18]). In this case, the speed may be given by a relation of the form

$$v(t,x) = f\left(\int_x^{x+\eta} \omega(s-x)\rho(t,s)ds \right) g\left(\int_{x-\zeta}^{x} \tilde{\omega}(x-s)\rho(t,s)ds \right), \text{ for } t \ge 0, x \in \Re \quad (2.3)$$

where $\zeta > 0$ is a constant, $g : \Re_+ \to \Re_+$ is a non-decreasing, bounded function, and $\tilde{\omega} : \Re_+ \to \Re_+$ is a non-increasing function. Even more emphatically, in the era of automated vehicles, the functions $g : \Re_+ \to \Re_+$ and $\tilde{\omega} : \Re_+ \to \Re_+$ may be designed so that the traffic flow behavior of system (2.1), (2.3) has specific characteristics, e.g., so that the equilibrium point gives maximum flow of vehicles and is globally asymptotically stable. It is clear that in such a case the design problem for $g : \Re_+ \to \Re_+$ and $\tilde{\omega} : \Re_+ \to \Re_+$ is strongly reminiscent of the feedback design problem for control systems. Therefore, there is an interest to understand traffic flow models of the form (2.1), (2.3). The first thing that we need to understand is the set of properties that all the functions described above must possess in order to have a well-defined system with solutions that have physical meaning (e.g., $\rho(t,x)$, $v(t,x)$ have to be positive).

In this paper, we study traffic flow models on a ring-road; hence we impose the periodicity condition

$$\rho(t,x+1) = \rho(t,x) \text{, for } t \ge 0, x \in \Re. \quad (2.4)$$

Given $\rho_0 \in Per(\Re)$ we consider the initial-value problem (2.1), (2.4), (2.5) with initial condition

$$\rho[0] = \rho_0 \quad (2.5)$$

Our first main result is an existence and uniqueness result for the initial-value problem (2.1), (2.3), (2.4), (2.5).

Theorem 1: *Suppose that* $\eta,\zeta\in(0,1]$, $f,g\in C^3(\Re_+)$, $f'(\rho)\le 0$, $g'(\rho)\ge 0$, $f(\rho)\ge 0$, $g(\rho)\ge 1$ *for all* $\rho\ge 0$. *Moreover, suppose that there exists a constant* $M\ge 0$ *such that*

$$\sup_{\rho\ge 0}\left(\sum_{k=0}^{3}\left|f^{(k)}(\rho)\right|\right)+\sup_{\rho\ge 0}\left(\sum_{k=0}^{3}\left|g^{(k)}(\rho)\right|\right)\le M \tag{2.6}$$

Finally, suppose that the restrictions of $\omega,\tilde{\omega}:\Re_+\to\Re_+$ *on* $[0,\eta]$, $[0,\zeta]$, *respectively, are* C^1 *functions with* $\omega'(x)\le 0$ *for* $x\in[0,\eta]$, $\tilde{\omega}'(x)\le 0$ *for* $x\in[0,\zeta]$ *and that* $\omega(x)=0$ *for* $x>\eta$, $\tilde{\omega}(x)=0$ *for* $x>\zeta$. *Then for every* $\rho_0\in W^{2,\infty}(\Re)\cap Per(\Re)$ *the initial-value problem (2.1), (2.4), (2.3), (2.5) has a unique solution* $\rho\in C^1(\Re_+\times\Re)$ *with* $\rho[t]\in W^{2,\infty}(\Re)\cap Per(\Re)$ *for all* $t\ge 0$. *Moreover, the following inequality holds for all* $t\ge 0, x\in\Re$:

$$\min_{x\in[0,1]}\left(\rho_0(x)\right)\le\rho(t,x)\le\max_{x\in[0,1]}\left(\rho_0(x)\right) \tag{2.7}$$

Theorem 1 can be proved by applying [12, Theorem 2.3] to the case (2.3). Due to space constraints the proof is omitted and can be found in [12]. Finally, according to the proofs of Theorem 1 and [12, Theorem 2.3], the numerical scheme that approximates the unique classical solution of the initial value problem (2.1), (2.3), (2.4), (2.5) is

$$\rho_i((k+1)\delta)=\left(1-\lambda v_{i+1}(k\delta)\right)\rho_i(k\delta)+\lambda v_i(k\delta)\rho_{i-1}(k\delta),$$
$$\text{for } i=0,\pm1,\pm2,\ldots,\ k=0,1,\ldots \tag{2.8}$$

$$\rho_i(0):=\rho_0(ih),\ \text{for } i=0,\pm1,\pm2,\ldots \tag{2.9}$$

where $h:=1/N$, $N\ge 2$, $\delta:=\lambda h$

$$v_i(k\delta)=K_N((Q\rho(k\delta))^{(i)}),\ \text{for } i=0,\pm1,\pm2,\ldots,\ k=0,1,\ldots \tag{2.10}$$

$$Q\rho(k\delta)=(\rho_0(k\delta),\ldots,\rho_{N-1}(k\delta))^{\top}\in\Re^N,\quad k=0,1,\ldots \tag{2.11}$$

$$K_N(\rho)=f\left(\sum_{i=0}^{N-1}\rho_i\int_{ih}^{(i+1)h}\omega(s)ds\right)g\left(\sum_{i=1}^{N-1}\rho_i\int_{1-ih}^{1-(i-1)h}\tilde{\omega}(s)ds\right),$$
$$\text{for } \rho\in\Re_+^N \tag{2.12}$$

III. Controlling Non-Local Traffic Flow Models

The uniform equilibrium points $\rho(x)\equiv\rho>0$ of model (2.1), (2.2), (2.4) satisfy exactly the same density-flow relation $q=\rho f(\rho)$ of the classical LWR model (the so-called fundamental diagram). This is not true for the uniform equilibrium points $\rho(x)\equiv\rho>0$ of model (2.1), (2.3), (2.4). For this model, the density-flow relation is given by

$$q=\rho f(\rho)g(\sigma\rho) \tag{3.1}$$

where $\sigma:=\int_0^{\zeta}\tilde{\omega}(s)ds$. Since $g(\rho)\ge 1$, relation (3.1) shows

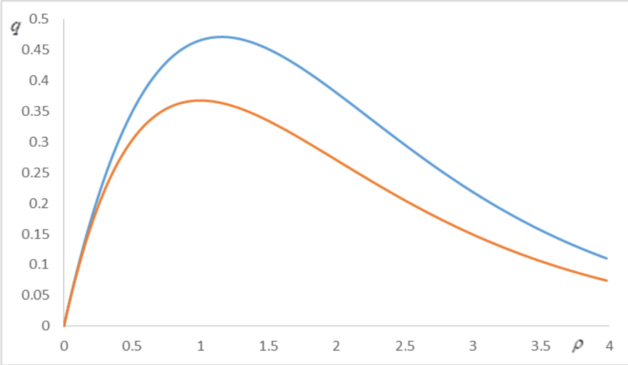


Fig. 1: The effect of nudging on the fundamental diagram. The blue line is the fundamental diagram with nudging and the red line is the fundamental diagram without nudging ($g(s)\equiv 1$).

that nudging can increase the flow. Moreover, the critical density, i.e., the density for which the flow becomes maximum, changes. This is demonstrated in Fig.1 for the case $f(\rho)=\exp(-\rho)$, $g(s)=(1+k)\dfrac{\exp(\gamma s)}{k+\exp(\gamma s)}$, $\gamma\sigma=1$, $k=1/2$. It may be seen that the flow values are increased, and the critical density is increased as well. It should be noted that Fig.1 is typical for many combinations of functions f,g with the characteristics required by the physics of traffic flow, i.e., $f'(\rho)\le 0$, $g'(\rho)\ge 0$, $f(\rho)>0$, $g(\rho)\ge 1$, for all $\rho\ge 0$, $\lim_{\rho\to+\infty}(f(\rho))=0$, $\lim_{\rho\to+\infty}(g(\rho))<+\infty$.

Theorem 1 guarantees that the uniform equilibrium points $\rho(x)\equiv\rho^*>0$ are neutrally stable in the sup norm of the state. However, Theorem 1 says nothing about (local or global) asymptotic stability and convergence to a uniform equilibrium point. Indeed, there are cases where the uniform equilibrium point $\rho(x)\equiv\rho^*>0$ for model (2.1), (2.2), (2.4) is not locally asymptotically stable, no matter what f is. More specifically, it is shown in [12], that model (2.1), (2.2), (2.4), for the important case $\omega(x)=\eta^{-1}$ for $x\in[0,\eta]$ and $\omega(x)=0$ for $x>\eta$ (case also studied in [17]), where $\eta\in(0,1]$ is a rational number, is characterized by lack of asymptotic stability of the uniform equilibrium point $\rho(x)\equiv\rho^*>0$ since its solutions are high density waves which move with constant speed $v(t,x)\equiv f(\rho^*)$. The reader should notice that, if convergence to the uniform equilibrium point $\rho(x)\equiv\rho^*>0$ is to be studied, then we should restrict our attention to initial conditions $\rho_0\in W^{2,\infty}(\Re)\cap Per(\Re)$ with $\int_0^1\rho_0(x)dx=\rho^*$, since only for this set of functions we can obtain solutions which converge to the uniform equilibrium point $\rho(x)\equiv\rho^*>0$ (notice that $\int_0^1\rho(t,x)dx=\int_0^1\rho_0(x)dx$ for all $t\ge 0$ for every solution of

(2.1), (2.4), (2.2), (2.5) or any solution of (2.1), (2.4), (2.3), (2.5)).

In the next result we show that nudging, if properly designed, can improve the stability properties of the system. The proof can be found in [12].

Theorem 2 (Local Stabilization by Means of Nudging): *Consider model (2.1), (2.3), (2.4) with* $\zeta=1$, $\omega(x)=\eta^{-1}$ *for* $x\in[0,\eta]$ *and* $\omega(x)=0$ *for* $x>\eta$, $\tilde{\omega}(x)=1-x$ *for* $x\in[0,1]$, *where* $\eta\in(0,1]$ *is a constant and* $f,g\in C^3(\Re_+)$ *are any functions with* $f'(\rho)<0$, $g'(\rho)>0$, $f(\rho)>0$, $g(\rho)\geq 1$ *for all* $\rho\geq 0$. *Moreover, suppose that there exists a constant* $M\geq 0$ *such that inequality (2.17) holds. Let* $\rho_0\in W^{2,\infty}(\Re)\cap Per(\Re)$ *with*

$$F_{\max}g_{\max}-F_{\min}g_{\min}<2\eta f_{\min}G_{\min} \quad (3.2)$$

where

$$F_{\max}:=\max\left\{|f'(s)|:\rho_{\min}\leq s\leq\min\left(\eta^{-1}\rho^*,\rho_{\max}\right)\right\} \quad (3.3)$$

$$F_{\min}:=\min\left\{|f'(s)|:\rho_{\min}\leq s\leq\min\left(\eta^{-1}\rho^*,\rho_{\max}\right)\right\} \quad (3.4)$$

$$f_{\min}:=f\left(\min\left(\eta^{-1}\rho^*,\rho_{\max}\right)\right) \quad (3.5)$$

$$g_{\max}:=g\left(\frac{1}{2}\min\left(2\rho^*-\rho_{\min},\rho_{\max}\right)\right) \quad (3.6)$$

$$g_{\min}:=g\left(\frac{1}{2}\max\left(2\rho^*-\rho_{\max},\rho_{\min}\right)\right) \quad (3.7)$$

$$G_{\min}:=\min\left\{\begin{array}{c} g'(s):\max\left(2\rho^*-\rho_{\max},\rho_{\min}\right) \\ \leq 2s\leq\min\left(2\rho^*-\rho_{\min},\rho_{\max}\right)\end{array}\right\} \quad (3.8)$$

$\rho^*=\int_0^1\rho_0(s)ds$, $\rho_{\min}:=\min_{x\in[0,1]}\left(\rho_0(x)\right)$ *and* $\rho_{\max}:=\max_{x\in[0,1]}\left(\rho_0(x)\right)$. *Then there exists a constant* $\bar{c}>0$ *such that the unique solution* $\rho\in C^1(\Re_+\times\Re)$ *of the initial-value problem (2.1), (2.4), (2.3), (2.5) satisfies the estimate:*

$$\int_0^1\left(\rho(t,x)-\rho^*\right)^2dx\leq\frac{\rho_{\max}}{\rho_{\min}}\exp(-\bar{c}t)\int_0^1\left(\rho_0(x)-\rho^*\right)^2dx\,, \quad \text{for } t\geq 0 \quad (3.9)$$

Remarks: (i) When $\rho_{\min}=\rho_{\max}=\rho^*$ then (3.2) holds automatically (by virtue of the fact that $f'(\rho)<0$, $g'(\rho)>0$, $f(\rho)>0$, $g(\rho)\geq 1$ for all $\rho\geq 0$). Due to continuity of $F_{\max},F_{\min},f_{\min},g_{\max},g_{\min},G_{\min}$ with respect to $\rho_{\min},\rho_{\max},\rho^*$, for every $\rho^*>0$, there exist $\rho_{\min}<\rho^*<\rho_{\max}$ such that (3.2) holds. This implies the existence of a neighborhood of the uniform equilibrium point $\rho(x)\equiv\rho^*>0$ in $Per(\Re)$ for which the $L^2(0,1)$ norm of the deviation of the solution from the equilibrium point converges exponentially to zero.
(ii) Condition (3.2) is a condition on the maximum deviation $\left\|\rho_0-\rho^*\right\|_\infty$ of the initial condition from the desired uniform equilibrium point $\rho(x)\equiv\rho^*>0$.
(iii) The nudging term with $\zeta=1$, $\tilde{\omega}(x)=1-x$ for $x\in[0,1]$, depends on the whole density profile of the ring-road. Such a term has no meaning when the vehicles are driven by human drivers. However, when automated vehicles are present in a highway, then such a term can be implemented by providing continuously information for the density profile to each vehicle. In such a case, the effect of nudging is not only the increase of the flow, but also the elimination of the well-known stop-and-go waves (see [1]).
(iv) The proof of Theorem 2 makes use of estimate (2.16) and the functional $V(\rho)=\int_0^1\left(\rho(x)\ln\left(\frac{\rho(x)}{\rho^*}\right)+\rho^*-\rho(x)\right)dx$. This functional, defined on the set of functions $\rho\in Per(\Re)$ with $\rho^*=\int_0^1\rho_0(x)dx$, is a non-coercive Control Lyapunov Functional for the control system (2.1), (2.4) with $v[t]\in Per(\Re)\cap C^1(\Re)$ as input. Indeed, for classical solutions of (2.1), (2.4) we get $\frac{d}{dt}V(\rho[t])=\int_0^1\left(\rho^*-\rho(t,x)\right)\frac{\partial v}{\partial x}(t,x)dx$. For non-coercive Lyapunov functionals, the reader can consult [9, 21]. It is possible that the use of other Lyapunov functionals can give less demanding conditions than (3.2) for exponential convergence to the uniform equilibrium point $\rho(x)\equiv\rho^*>0$.

IV. Illustrative Example

In this section we present some numerical examples that demonstrate the advantages and the stabilizing effects of nudging (Model 3) in comparison with the "look-ahead" model (2.1), (2.2) (Model 2) and the LWR model (2.1) with $v(t,x)=f(\rho(t,x))$ (Model 1). Let $f(\rho)=\exp(-\rho)$ for all models on a ring-road (2.4) with initial density given by

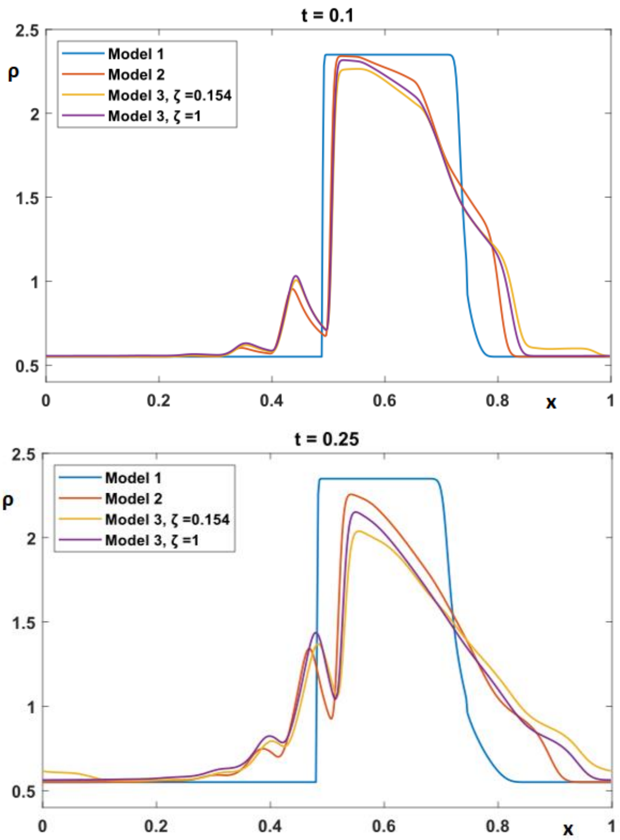
$$\rho_0(x)\begin{cases} =2.35, & 0.5<x<0.75 \\ \in[0.55,2.35], & x\in[0.5-\varepsilon,0.5]\cup[0.75,0.75+\varepsilon],\ \varepsilon>0 \\ =0.55, & else\end{cases}$$

Note that on the interval $x\in[0.5-\varepsilon,0.5]\cup[0.75,0.75+\varepsilon]$, for sufficiently small $\varepsilon>0$, the initial density profile $\rho_0(x)$ is smoothly extended to satisfy the requirement $\rho_0\in W^{2,\infty}(\Re)\cap Per(\Re)$ in Theorem 2. Notice also that the initial condition corresponds to a road with congestion belt at $[0.5,0.75]$ with the uniform equilibrium given by $\rho^* = \int_0^1 \rho_0(x)dx = 1$.

	Model 3
Velocity $v(t,x)$	Model (2.1) with $v(t,x)$ given by (2.3)
$g(\cdot)$	$g(x) = 1.6 \dfrac{\exp\left(\dfrac{1.8}{\zeta(2-\zeta)}x\right)}{0.6+\exp\left(\dfrac{1.8}{\zeta(2-\zeta)}x\right)}$
$\omega(\cdot)$	η^{-1}
$\tilde{\omega}(\cdot)$	$1-x$

Table 1. Model 3 of the simulation example.

We have used two different values for the upstream horizon: $\zeta = 1$ and $\zeta = 0.154$. While Theorem 2 guarantees local exponential stabilization for $\zeta = 1$, it is important for implementation purposes to consider small values for the upstream horizon (which do not require knowledge of the whole density profile of the ring road). The value of the downstream horizon in all experiments was set $\eta = 0.1$. While the numerical results for the LWR model are obtained by means of the Godunov numerical scheme, for the non-local PDEs we have used the numerical scheme (2.8)-(2.12) with $h = 1/500$, $\lambda = 0.25$. Figure 3 shows the density profiles at different times.



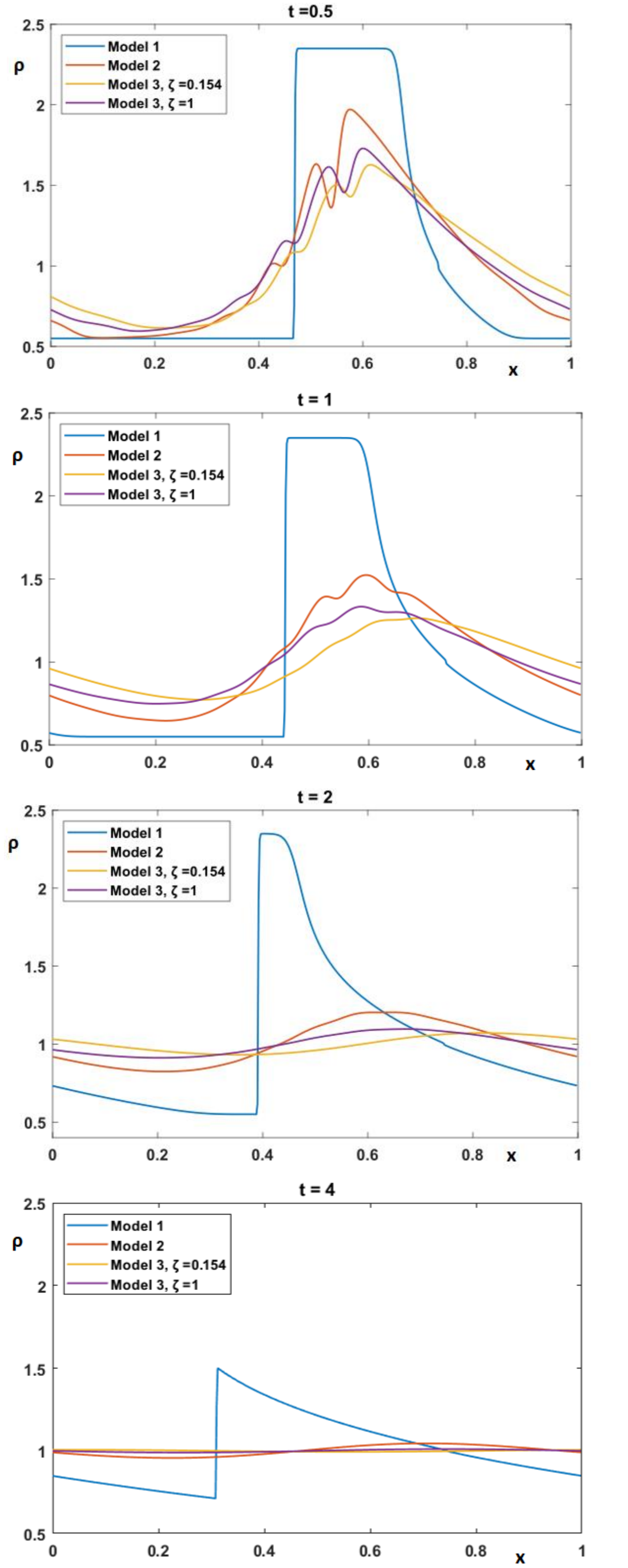


Fig.3 Density profiles of Model 1, Model 2, Model 3 with $\zeta = 1$ and Model 3 with $\zeta = 0.154$.

Notice that the rate of convergence of Model 3 for both values $\zeta = 1$ and $\zeta = 0.154$ is faster compared to the other models. This feature can also be verified in Figure 4, which depicts the evolution of the L^2 norm of the deviation from the equilibrium $\left\|\rho[t]-\rho^*\right\|_2 = \left(\int_0^1 \left(\rho(t,x)-\rho^*\right)^2 dx \right)^{1/2}$.

Figure 4 also shows the evolution of the logarithm of the L^2 norm $\left\|\rho[t]-\rho^*\right\|_2$ indicating exponential convergence.

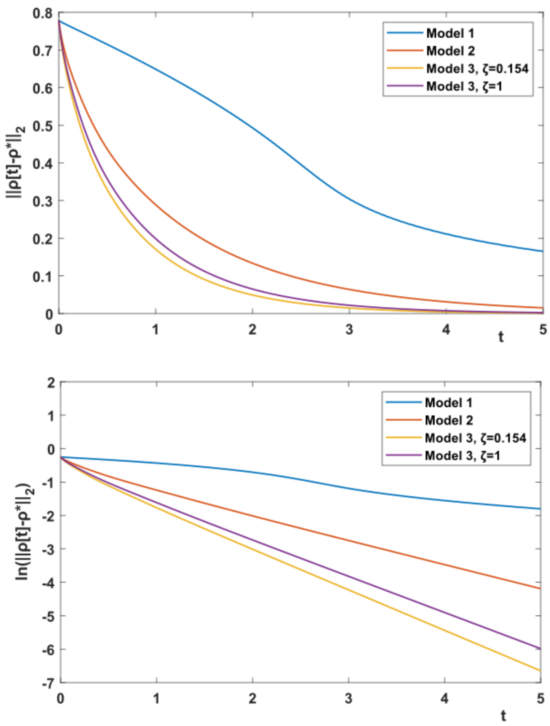


Fig.4 Evolution of the L^2 norm and its logarithm for Model 1, Model 2, Model 3 with $\zeta = 1$ and Model 3 with $\zeta = 0.154$.

V. CONCLUSION

The paper provided indications about the stabilizing effect of nudging in a ring-road when nudging is expressed by means of (2.3). It is also possible to study more complicated speed adjustment feedback laws which may allow the development of global stabilization results for ring-roads. This development has to be combined with the construction of appropriate Lyapunov functionals for the system. Another research direction is the study of the effect of boundary conditions in non-local conservation laws on bounded domains. It is (in principle) possible to combine boundary feedback stabilization approaches with nudging and obtain even better results.

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