A Novel Nonlinear Adaptive Cruise Controller for Vehicular Platoons

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***Abstract*— The paper deals with the design of nonlinear adaptive cruise controllers for vehicular platoons. The constructed feedback controllers are nonlinear functions of the distance between successive vehicles and their speeds. It is shown that the proposed novel controllers guarantee safety (collision avoidance) and bounded vehicle speeds by explicitly characterizing the set of allowable inputs. Moreover, we guarantee global asymptotic stability of the platoon to a desired configuration as well as string stability. Certain macroscopic properties are also investigated. The efficiency of the nonlinear adaptive cruise controllers is demonstrated by means of numerical examples.**

I. INTRODUCTION

Adaptive cruise control (ACC) systems are designed to enhance the standard cruise control by allowing an equipped vehicle to maintain certain distance from the preceding vehicle.

A large variety of spacing policies and controllers for ACC vehicles and platoons of ACC vehicles have appeared, see [4], [5], [7], [12], [16], [18], [20], [21], [22], [23], [28], [31]. The most common policies considered in the related literature are the constant spacing policy [26], where the distance between successive vehicles remains constant at all speeds; and the Constant-Time Gap (CTG) policy [12], where the spacing varies linearly with speed. To evaluate a spacing policy and its associated controller, the following criteria were proposed, see [21]: (i) individual vehicle stability, which characterizes the convergence towards a desired equilibrium; (ii) string stability, which focuses on the dissipation of small perturbations along a string of vehicles ([5], [17], [25], [27]); and (iii) traffic flow stability which deals with the evolution of density when all vehicles use the same spacing policy ([21], [22], [24]).

The notion of string stability has been widely studied and several definitions have appeared in the literature, see [5], [17], [20], [27], [25], [30]. A detailed overview of the various string stability definitions and their properties can be found in [8], [17]. To address the ambiguity over the different definitions used in the literature, a novel definition was proposed in [17] for both linear and nonlinear systems based on L_p stability, which encompasses the upstream disturbance attenuation, the external input of the leading vehicle, as well as perturbations on initial conditions.

While string stability ensures that disturbances in position, speed or acceleration do not accentuate while propagating along the platoon, it does not guarantee collision avoidance between vehicles in the platoon, see [6]. Indeed, the majority of spacing policies and ACC controllers focus on stability and string stability properties, ([10], [16], [31], [27], [32], [33]), which, however, may result in negative spacing error and negative speeds. On the other hand, approaches considering safety can be found in [1], [4], but they do not formally study string stability, and also in [9], [11], [28], which mainly deal with boundedness of spacing error rather than convergence to a desired value. In [15], different control configurations and conditions for a CTG policy are derived that guarantee string stability and collision avoidance when the platoon is initiated from an equilibrium position with zero speed and sufficiently large initial spacing between vehicles. Safety criteria were also presented in [2], [3], where, collisions are avoided whenever the platoon does not exceed a given relative speed threshold regardless of the behavior of the leader.

It is clear from the above that a methodology that simultaneously guarantees safety, stability, string stability under predecessor-follower (i.e. autonomous) control architecture is missing in the literature. In this paper, we present conditions which guarantee safety, stability and L_p string stability of a vehicular platoon using nonlinear adaptive cruise controllers. The proposed nonlinear controller incorporates and integrates the car-following and cruise control tasks without any need for heuristic switching logic and has the following features:
1. It provides safe platoon operation without collisions, negative speeds or speeds exceeding speed limits.
2. It guarantees global asymptotic stability of the spacing/speed equilibrium for a platoon on an open road.
3. It guarantees L_p string stability for the platoon.

Moreover, we explicitly characterize the set of feasible initial states for safe operation in terms of collision avoidance and bounded vehicle speeds as well as the class of inputs (maneuvers of the leading vehicle) that can be allowed for the safe operation of the platoon. Finally, certain macroscopic properties related to traffic flow stability, design of the fundamental diagram, and the reduction of the microscopic model to the standard Lighthill-Witham-Richards (LWR) model [14], [19] are studied.

The structure of the paper is as follows. Section II is devoted to the presentation of the properties of adaptive cruise controllers, such as safety criteria and appropriate stability notions. To motivate the use of nonlinear controllers,

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simulation scenarios are also presented in Section II using the standard CTG controller (see [18]) and the Variable Time-Gap (VTG) controller (see [29]), which demonstrate that certain safety criteria may fail. A general form of a nonlinear adaptive cruise controller is provided in Section III together with sufficient conditions for the safe operation of a platoon of vehicles. Section IV provides results for the L_p string stability of the proposed adaptive cruise controller. It is also shown that the sufficient conditions for string stability and the existence of a fundamental diagram also guarantee global asymptotic stability of the unique equilibrium point of a platoon operating in an open road. Numerical examples are presented in Section V to demonstrate the efficiency of the proposed nonlinear adaptive cruise controller. Due to space constraints the proofs are omitted and can be found in [13] together with the case of platoons operating on a ring-road.

Notation. Throughout this paper, we adopt the following notation: $\mathbb{R}_+ := [0, +\infty)$ denotes the set of non-negative real numbers. By $|x|$ we denote both the Euclidean norm of a vector $x \in \mathbb{R}^n$ and the absolute value of a scalar $x \in \mathbb{R}$. By K we denote the class of strictly increasing C^0 functions $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$. By K_∞ we denote the class of strictly increasing C^0 functions $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$ and $\lim_{s\to+\infty} a(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is of class K; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{s\to+\infty} \sigma(s, t) = 0$. By $C^0(A, \Omega)$, we denote the class of continuous functions on $A \subseteq \mathbb{R}^n$, which take values in $\Omega \subseteq \mathbb{R}^m$. By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \mathbb{R}^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \mathbb{R}^m$. When $\Omega = \mathbb{R}$ the we write $C^0(A)$ or $C^k(A)$. By L^p with $p \geq 1$ we denote the equivalence class of measurable functions $f : \mathbb{R}_+ \to \mathbb{R}^n$, for which $\|f\|_{[0,t],p} = \left(\int_0^t |f(x)|^p \, dx\right)^{1/p} < +\infty$. L^∞ denotes the equivalence class of measurable functions $f : \mathbb{R}_+ \to \mathbb{R}^n$, for which $\|f\|_{[0,t],\infty} = \underset{x\in[0,t)}{ess\sup}\,(|f(x)|) < +\infty$. For a set $S \subseteq \mathbb{R}^n$, $\bar{S}$ denotes the closure of S.

II. Motivation

A commonly used model for vehicle dynamics in vehicular platoons consists of the following ODEs:

$$\begin{aligned} \dot{s}_i &= v_{i-1} - v_i \\ \dot{v}_i &= u_i \end{aligned} \quad , i = 1, \dots, n \tag{1}$$

where we consider a platoon of n identical vehicles on a road, s_i ($i = 1, \dots, n$) is the back-to-back distance of the i-th vehicle from the $(i-1)$-th vehicle, v_i ($i = 1, \dots, n$) is the speed of the i-th vehicle, u_i ($i = 1, \dots, n$) is the control input (acceleration) of the i-th vehicle, and v_0 is the speed of the leader and is an external input.

For autonomous vehicles (no communication), the so-called predecessor-follower control architecture is used, i.e., there exists a function $F : \mathbb{R}_+^3 \to \mathbb{R}$ so that

$$u_i = F(s_i, v_{i-1}, v_i), i = 1, \dots, n. \tag{2}$$

The function $F : \mathbb{R}_+^3 \to \mathbb{R}$ is a feedback law that constitutes the Adaptive Cruise Controller. This function must be selected in such a way that the following requirements hold.

1) Safe Operation Requirement: *There exists constant $a > 0$, a non-empty set of inputs $J \subseteq \left\{ v_0 \in C^1(\mathbb{R}_+) : 0 < v_0 < v_{\max} \right\}$, where $v_{\max} > 0$ is the speed limit of the road, and a set valued map $(0, v_{\max}) \ni v_0 \to D(v_0) \subseteq \mathbb{R}^{2n}$ with*

$$\begin{aligned} D(v_0) \subseteq \{(s_1, \dots, s_n, v_1, \dots, v_n) \in \mathbb{R}^{2n} :& \\ 0 < v_i < v_{\max},, s_i > a\,,\, i = 1, \dots, n\,\}& \end{aligned} \tag{3}$$

with the following property:
"For each $v_0 \in J$, if $(s_1(0), \dots, s_n(0), v_1(0), \dots, v_n(0)) \in D(v_0(0))$, then the solution of the initial-value problem (1), (2) *with initial condition $(s_1(0), \dots, s_n(0), v_1(0), \dots, v_n(0))$ exists for all $t \geq 0$ and satisfies $(s_1(t), \dots, s_n(t), v_1(t), \dots, v_n(t)) \in D(v_0(t))$."*

Notice that the requirement of safe operation is actually a well-posedness requirement, i.e., we require that the solution exists and takes values on a physically meaningful set. However, the requirement of safe operation is not only a well-posedness characterization of the solution; we further require that $s_i(t) > a$, where the constant $a > 0$ is the minimum allowable distance between two vehicles. This is a safety requirement which implies the absence of collisions.

2) Technical Requirement: *For a given constant $A > 0$, we have*

$$|F(s, w, v)| \leq A, \text{ for all } s > a, v, w \in (0, v_{\max}). \tag{4}$$

The constant $A > 0$ appearing in the technical requirement is the maximum acceleration that the vehicle can have and depends on the technical characteristics of the vehicles and the road.

3) Stability Requirement: *For every $v^* \in (0, v_{\max})$, there exists $s^* \in (a, +\infty)$ with $F(s^*, v^*, v^*) = 0$ such that (i) $(s^*, \dots, s^*, v^*, \dots, v^*) \in D(v^*)$, (ii) the constant input $v_0(t) \equiv v^*$ is in the allowable input set J, and (iii) the equilibrium point $(s^*, \dots, s^*, v^*, \dots, v^*) \in D(v^*)$ of* (1), (2) *with $v_0(t) \equiv v^*$ defined on $\overline{D(v^*)}$ is Globally Asymptotically Stable and Locally Exponentially Stable, i.e., there exist constants $M, \sigma, \delta > 0$ and a function $\omega \in KL$ so that for every $(s_1(0), \dots, s_n(0), v_1(0), \dots, v_n(0)) \in \overline{D(v^*)}$ the solution of* (1), (2) *with $v_0(t) \equiv v^*$ satisfies*

$$\begin{aligned} &|(s_1(t) - s^*, \dots, s_n(t) - s^*, v_1(t) - v^*, \dots, v_n(t) - v^*)| \\ &\leq \omega(\,|(s_1(0) - s^*, \dots, s_n(0) - s^*, \\ &\qquad v_1(0) - v^*, \dots, v_n(0) - v^*)|\,,\, t\,), \text{for all } t \geq 0; \end{aligned} \tag{5}$$

and if in addition $|(s_1(0) - s^*, \dots, s_n(0) - s^*, v_1(0) - v^*, \dots, v_n(0) - v^*)| < \delta$ then

$$\begin{aligned} &|(s_1(t) - s^*, \dots, s_n(t) - s^*, v_1(t) - v^*, \dots, v_n(t) - v^*)| \\ &\leq M \exp(-\sigma t)(\,|(s_1(0) - s^*, \dots, s_n(0) - s^*, \\ &\qquad v_1(0) - v^*, \dots, v_n(0) - v^*)|\,,\, t\,), \text{for all } t \geq 0. \end{aligned} \tag{6}$$

The stability requirement is a crucial requirement that guarantees the convergence of the vehicle states to the desired values.

While the stability requirement guarantees the desired asymptotic behavior, there is no guarantee for the transient behavior. A performance requirement which guarantees proper transient behavior is the requirement of string stability. Here we adopt a slightly stronger version of the L_p string stability notion given in [17]. As noted in [17], the L_p string stability notion is motivated by the requirement of energy dissipation along the string of vehicles for $p = 2$, whereas the case $p = \infty$ is related to maximum overshoot of the local error vector between the current speed and desired speed.

4) String Stability Requirement: *There exists $p \in [1, +\infty]$ with the following property:*
For every $q > 0$ there exists a continuous function $\beta_q : \mathbb{R}^2 \to \mathbb{R}_+$ with $\beta_q(0) = 0$, $\beta_q(s) > 0$ for $s \in \mathbb{R}^2 \setminus \{0\}$ such that every solution of (1), (2) *with $v_0 \in J$ satisfies the estimate*

$$\|v_i\|_{[0,t],p} \leq (1+q) \|v_{i-1}\|_{[0,t],p} \beta_q (s_i(0) - s^*, v_i(0) - v^*) \quad \text{for all } t \geq 0 \text{ and } i = 1, \dots, n \tag{7}$$

where $\|v_i\|_{[0,t],p} = \left(\int_0^t |v_i(l) - v^*|^p \, dl\right)^{1/p}$, $\|v_{i-1}\|_{[0,t],p} = \left(\int_0^t |v_{i-1}(l) - v^*|^p \, dl\right)^{1/p}$ *for* $p \in [1, +\infty)$, $\|v_i\|_{[0,t],\infty} = \sup_{0 \leq l \leq t} (|v_i(l) - v^*|)$, $\|v_{i-1}\|_{[0,t],\infty} = \sup_{0 \leq l \leq t} (|v_{i-1}(l) - v^*|)$, $v^* \in (0, v_{\max})$, $s^* \in (a, +\infty)$ *are constants with* $F(s^*, v^*, v^*) = 0$.

Another performance guarantee can be obtained by the existence of a globally exponentially stable manifold for the speed states. This requirement is described below.

5) Fundamental Diagram Requirement: *There exists a function $G \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ and constants $\bar{M}, \bar{\sigma} > 0$ such that every solution of* (1), (2) *with $v_0 \in J$ satisfies the estimate*

$$\sum_{i=1}^{n} |v_i(t) - G(s_i(t))| \leq \bar{M} \exp(-\bar{\sigma} t) \sum_{i=1}^{n} |v_i(0) - G(s_i(0))| , \quad \text{for all } t \geq 0. \tag{8}$$

The fundamental diagram requirement essentially demands that the vehicle speeds ultimately depend only on the local vehicle density. Since the vehicle density $\rho(t, x)$ is equal to $1/s_i(t)$ when x is a position between the i-th vehicle and the $(i-1)$-th vehicle, it is reasonable to say that ultimately the local speed of vehicles of the platoon obeys the equation

$$v = G\left(\rho^{-1}\right), \text{for } \rho \in (0, a^{-1}). \tag{9}$$

Even in the case that a globally exponentially manifold for the speed states is absent, it is reasonable to expect that all equilibrium points for (1), (2) satisfy a relation of the form $v_i = G(s_i)$ for $i = 1, \dots, n$ and an appropriate function $G \in C^1(\mathbb{R}_+; \mathbb{R}_+)$. The inverse of this relation, i.e., the equation $s_i = G^{-1}(v_i)$ when G is invertible, is called a spacing policy (see [24], [29]). A spacing policy allows the reduction of the system of n ODEs (1), (2) to the standard LWR model with speed given by (9) (although such a reduction is problematic in the absence of a fundamental diagram for the platoon). In this case, the following macroscopic stability condition arises.

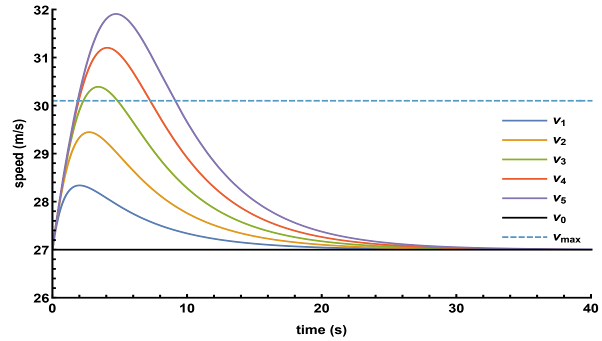

Fig. 1. CTG policy (11) with controller (2), (12) and speeds exceeding the road speed limit $v_{\max} = 30.1\, m/s$.

6) Macroscopic Stability Requirement: *There exist constants $0 < a < b$ such that a function*

$$\frac{\partial}{\partial \rho}\left(\rho G\left(\rho^{-1}\right)\right) > 0, \text{for all } \rho \in (a, b). \tag{10}$$

Inequality (10) was proposed in [24], [29] for the so-called "unconditional traffic-flow stability", i.e., the stability of the model to all possible boundary conditions. It was later used in [21] for the construction of macroscopically stable spacing policies.

A very common spacing policy used in ACC systems is the constant time-gap policy (CTG) in which the desired inter-vehicle spacing is proportional to speed:

$$s_d = r + T v_i \tag{11}$$

where $r \geq a$ is a safety or desired distance between vehicles and the constant $T > 0$ is referred to as the time-gap, i.e., the time required for the following vehicle to reach the current back side of the front vehicle while driving with its current speed v_i. For the CTG spacing policy (11), a typical control law (2) to regulate the spacing between vehicles is given by

$$F(s, w, v) = (k - g) g (s - r) + g w - k v \tag{12}$$

where $k > g > 0$, the time-gap being $T = 1/g$, see [12], [18]. The CTG policy (11), with the controller (2), (12) satisfies both the Stability Requirement and the String-Stability Requirement, see [18]. However, the Technical Requirement is not fulfilled since $F(s, w, v)$ in (12) grows linearly in s and, more importantly, there are cases where the Safe Operation Requirement on an open road may not be valid. To our knowledge, no researcher has ever shown what is the allowable set of inputs for an open road. This is illustrated in the following scenario.

For illustration of some of the above statements, consider a case of $n = 5$ vehicles of the same length $a = 5m$ moving on a road with speed limit $v_{\max} = 30.1\, m/s$ with all vehicles using the same CTG spacing policy (11) with controller (2), (12), initial speed $v_{i,0} = v_i(0) = 27\, m/s$ and initial spacing $s_{i,0} = s_i(0) = 70m$, $i = 1, \dots, 5$. Furthermore, suppose that the leading vehicle is also moving with constant speed $v_0 = 27\, m/s$, and let the time-gap be $T = 1/g = 1s$, and $k = 1.2\, s^{-1}$, $r = 33m$. Fig. 1 shows that in this setting, certain vehicles do not respect the speed limit $v_{\max} = 30.1\, m/s$.

As a second scenario, we consider a slowly moving leading vehicle $v_0 = 10m/s$ on a road with speed limit

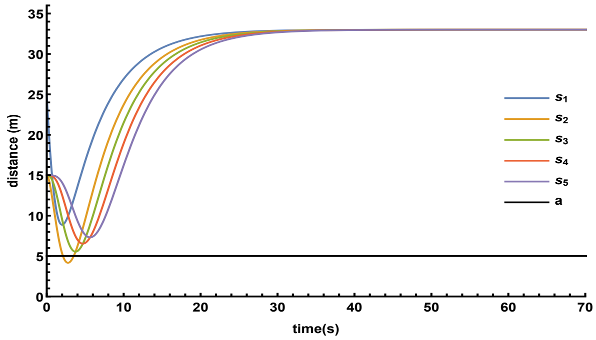

Fig. 2. Vehicle spacing for CTG policy (11) with controller (2), (12) with collision.

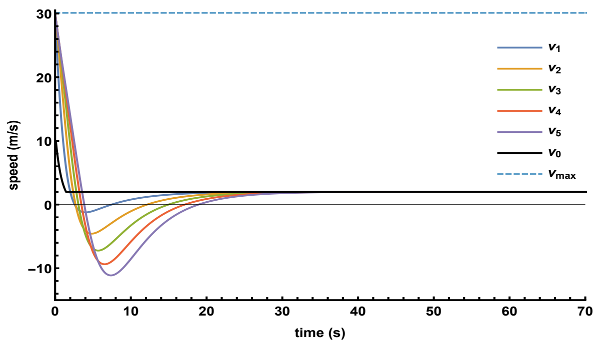

Fig. 3. Negative vehicle speed for the CTG policy (11) with controller (2), (12).

$v_{\max} = 30.1m/s$ and $n = 5$ vehicles moving with speed $v_{i,0} = 30m/s$, $i = 1, \ldots, 5$, and initial spacing $s_{1,0} = 25m$, $s_{i,0} = 15m$, $i = 2, \ldots, 5$. For this scenario, we let the time-gap $T = 1/g = 1s$, $k = 1.2s^{-1}$ and set $r = 33m$. Furthermore, suppose that the leading vehicle decelerates to a lower speed $v_0 = 1m/s$. Fig. 2 shows the back-to-back vehicle distances for this particular scenario. It can be seen that the safe operation requirement with $a = 5m$ (the vehicles' length) is not satisfied, since there exists time $T > 0$ with $s_2(T) < a$, which implies collision between the first and second vehicle. Notice also that the vehicles in the platoon attain negative speeds as shown in Fig. 3.

In addition to the above scenarios, there are certain macroscopic properties of the CTG policy, for a string of vehicles on a single-lane highway, that are of interest. More specifically, for the CTG policy (11), we can obtain from (9) with $G(s) = g(s-r)$, that the road speed in terms of density is

$$v = g\frac{1 - r\rho}{\rho} \tag{13}$$

and the traffic flow is

$$Q = \rho v = g(1 - r\rho). \tag{14}$$

Notice now that, as the density decreases, the speed grows unbounded. Conversely, larger values of r result in smaller traffic density with the speed being negative if $\rho \in (r^{-1}, a^{-1})$. Fig. 4 illustrates the density-flow relation (the so-called fundamental diagram) for different values of the minimum distance r and the time-gap $T = 1/g$. It can be seen that the fundamental diagram violates the maximum speed since it passes above the line $Q = v_{\max}\rho$. It is clear

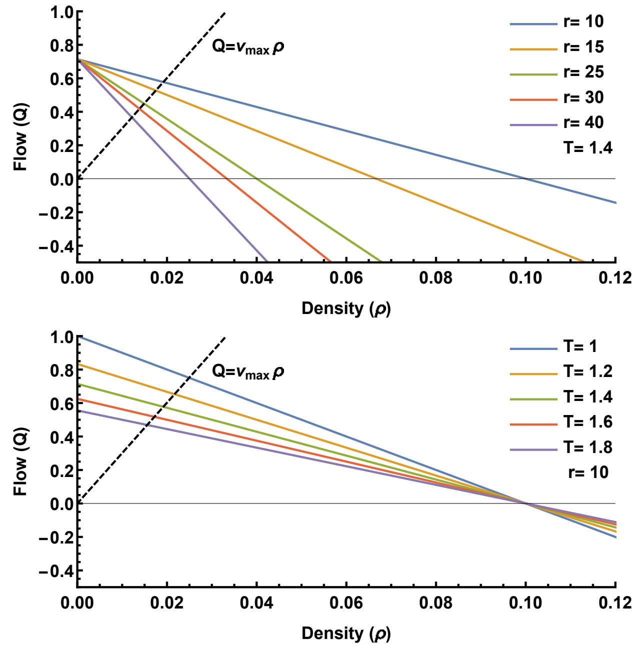

Fig. 4. Fundamental diagram of CTG policy (11) for several values of r and fixed $T = 1.4s$ on the top; on the bottom, the fundamental diagram for fixed $r = 10m$ and several values of T.

that the macroscopic stability requirement does not hold (as was already remarked in [24], [29]). Moreover, since the fundamental diagram is always a straight line, the CTG policy (11) has limited degrees of freedom for the optimal selection of the desired fundamental diagram.

It should be noted that practical ACC systems have two modes of operation: to maintain a desired speed as conventional cruise control; or switch to CTG car-following mode if the preceding vehicle is slower. These two modes are coupled with a transitional logic which determines when to switch from speed-control mode to spacing-control mode and vice versa, see [7]. Therefore, in practice, ACC systems would never increase the vehicle speed beyond the speed limit.

The previous scenarios show that the CTG policy (11) with the controller (12) fails to satisfy the safe requirement operation leading to negative speeds, collisions and speeds exceeding the road speed limits. Analogous behavior concerning safety can also been observed with the VTG policy

$$s_d = \frac{1}{\rho_m\,(1 - v_i/v_{\max})}$$

under the controller proposed in [29]:

$$F(s, w, v) = \frac{\rho_m}{v_{\max}}(v_{\max} - v)^2(w - v + \lambda s - \frac{v_{\max}\lambda}{\rho_m(v_{\max} - v)}) \tag{15}$$

where $\lambda > 0$ and ρ_m is the jam density. The VTG controller (15) is a nonlinear function of vehicle speed and satisfies the macroscopic stability requirement and the string stability requirement, see [29]. However, the safety requirement is not satisfied as shown next. We consider a scenario where all vehicles are moving with initial speed $v_{i,0} = 27m/s$, initial spacing $s_{i,0} = 30m$, $i = 1, ..., 5$, and the leading vehicle moving with speed $v_0 = 24m/s$ on a road with speed limit $v_{\max} = 30.1m/s$. Moreover, the leading vehicle strongly

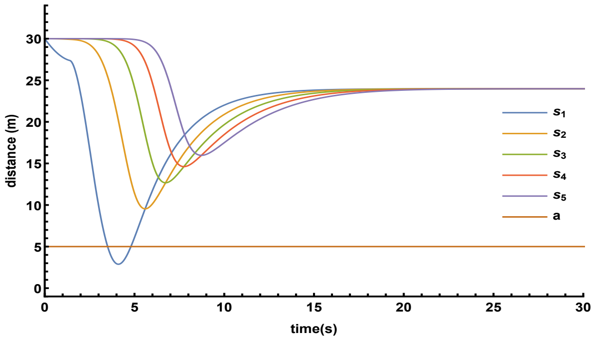

Fig. 5. VTG policy with controller (15) with collisions.

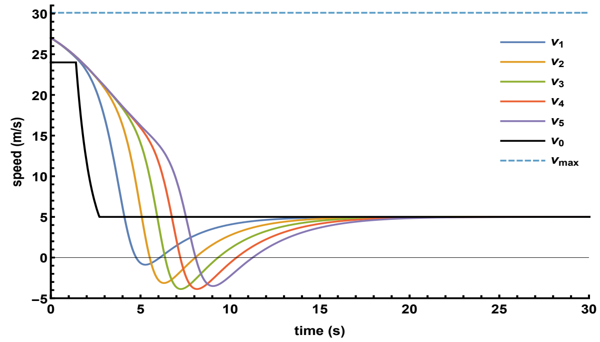

Fig. 6. VTG policy with controller (15) with negative speeds.

decelerates to a speed of $v_0 = 5m/s$. We set $\lambda = 0.5$ and $\rho_m = 0.05veh/m$. Fig. 5 shows that the first vehicle collides with the leading vehicle since there exists time $T > 0$ with $s_1(T) < a$. Finally, in Fig. 6 it is shown that the vehicles also attain negative speeds.

III. Safe Operation of Platoons

In this section, we provide sufficient conditions for the safe operation of a vehicular platoon. Our first result provides sufficient conditions for an open road and is given below.

Theorem 1: Let $f, g, \kappa : \mathbb{R} \to \mathbb{R}_+$ be locally Lipschitz functions and suppose that there exist constants $v_{\max} > 0$, $\lambda > a > 0$, $k > 0$ for which the functions $f, g, \kappa : \mathbb{R} \to \mathbb{R}_+$ satisfy the following properties:

$$0 \le g(s) < \kappa(s) \le k, \text{ for all } s \ge a \quad (16)$$

$$\frac{f(s)}{\kappa(s) - g(s)} \le v_{\max} < k(\lambda - a), \text{ for all } s \ge a \quad (17)$$

$$f(s) = g(s) = 0 \text{ and } \kappa(s) = k, \text{ for all } s \in [a, \lambda] \quad (18)$$

Given $v_0 \in (0, v_{\max})$, we define the set:

$$D(v_0) = \left\{ \begin{array}{c} (s_1, \dots, s_n, v_1, \dots, v_n) \in \mathbb{R}^{2n} : \\ 0 < v_i < v_{\max} \\ s_i > a + k^{-1} \max(0, v_i - v_{i-1}) \end{array}, i = 1, \dots, n \right\}. \quad (19)$$

Then, for every input $v_0 \in C^1(\mathbb{R}_+)$ satisfying

$$\dot{v}_0(t) \ge -k v_0(t), \ 0 < v_0(t) < v_{\max}, \text{ for all } t \ge 0 \quad (20)$$

and for every $(s_{1,0}, \dots, s_{n,0}, v_{1,0}, \dots, v_{n,0}) \in D(v_0(0))$, the initial-value problem (1), (2) with

$$F(s, w, v) = f(s) + g(s)w - \kappa(s)v, \text{ for all } s, v, w \in \mathbb{R} \quad (21)$$

with initial condition $(s_1(0), \dots, s_n(0), v_1(0), \dots, v_n(0)) = (s_{1,0}, \dots, s_{n,0}, v_{1,0}, \dots, v_{n,0})$ has a unique solution $(s_1(t), \dots, s_n(t), v_1(t), \dots, v_n(t))$ defined for all $t \ge 0$ that satisfies $(s_1(t), \dots, s_n(t), v_1(t), \dots, v_n(t)) \in D(v_0(t))$ for all $t \ge 0$.

Theorem 1 characterizes clearly the class of inputs that can be allowed for the safe operation of a vehicular platoon. Indeed, the speed of the leader v_0 must be a function of class $C^1(\mathbb{R}_+)$ which satisfies (20). When the speed of the leader satisfies this safety requirement, then all vehicles remain in a distance at least $a > 0$ from each other, and all vehicles' speeds are less than the speed limit $v_{\max}$. Thus, if the adaptive cruise controller has the form (21), where the functions $f, g, \kappa : \mathbb{R} \to \mathbb{R}_+$ satisfy (16), (17) and (18), then the safe operation requirement is satisfied. Notice that the sufficient conditions (16), (17) and (18), are not restrictive and depend on technical characteristics of the vehicles and the road. In particular, the constant k in (16) represents a friction term and condition (20) together with inequality $\kappa(s) \le k$, $s \ge a$, describe the maximum rate of deceleration of the leading and following vehicles in the platoon. Condition (18) describes the distance at which a following vehicle starts decelerating. Finally, conditions (16) and (17) are technical conditions that are required for the safe operation of the platoon.

Remark 1: If the adaptive cruise controller has the form (21), where the functions $f, g, \kappa : \mathbb{R} \to \mathbb{R}_+$ satisfy (16), (17) and (18), then the Technical Requirement holds for the function F defined by (21). Indeed, the fact that the functions $f, g, \kappa : \mathbb{R} \to \mathbb{R}_+$ are non-negative and inequality (17) show that

$$|F(s, w, v)| < k v_{\max}, \text{for all } s > a, \ v, w \in (0, v_{\max}). \quad (22)$$

Consequently, inequality (22) guarantees that inequality (4) holds with $A := k v_{\max}$.

IV. Stability, String Stability and Fundamental Diagram

If the adaptive cruise controller has the form (21), where the functions $f, g, \kappa : \mathbb{R} \to \mathbb{R}_+$ satisfy (16), (17) and (18), then the safe operation of a vehicular platoon is guaranteed. However, we have no guarantee for the string stability of the platoon or for the existence of a fundamental diagram. In order to achieve these objectives, we have to restrict the allowable form of the adaptive cruise controller, so that conditions (16), (17), (18) hold automatically, and additional sufficient conditions that guarantee string stability and the existence of a fundamental diagram for the platoon hold. This is shown by the following theorem:

Theorem 2: (String Stability and Fundamental Diagram) Let $g : \mathbb{R} \to \mathbb{R}_+$ be a locally Lipschitz function and suppose that there exist constants $k > g_{\max} > 0$, $\lambda > a > 0$ for which the following properties hold:

$$0 < g(s) \le g_{\max}, \text{ for all } s > \lambda \quad (23)$$

$$v_{\max} := \int_a^{+\infty} g(l)dl < k(\lambda - a) \tag{24}$$

$$g(s) = 0, \text{ for all } s \in [a, \lambda]. \tag{25}$$

Let $v^* \in (0, v_{\max})$ be a given constant and define $s^* \in (\lambda, +\infty)$ by means of the equation

$$v^* = G(s^*) \tag{26}$$

where

$$G(s) := \int_a^s g(l)dl, \text{ for all } s \in \mathbb{R}. \tag{27}$$

Also define

$$F(s, w, v) = (k - g(s))G(s) + g(s)w - kv, \text{ for all } s, v, w \in \mathbb{R}. \tag{28}$$

Given $v_0 \in (0, v_{\max})$, we define the set $D(v_0) \subset \mathbb{R}^{2n}$ by means of (19). Then, for every input $v_0 \in C^1(\mathbb{R}_+)$ satisfying (20) and for every $(s_{1,0}, \ldots, s_{n,0}, v_{1,0}, \ldots, v_{n,0}) \in D(v_0(0))$, the initial-value problem (1), (2) with (28), initial condition $(s_1(0), \ldots, s_n(0), v_1(0), \ldots, v_n(0)) = (s_{1,0}, \ldots, s_{n,0}, v_{1,0}, \ldots, v_{n,0})$ has a unique solution $(s_1(t), \ldots, s_n(t), v_1(t), \ldots, v_n(t))$ defined for all $t \geq 0$ that satisfies $(s_1(t), \ldots, s_n(t), v_1(t), \ldots, v_n(t)) \in D(v_0(t))$ for all $t \geq 0$. Moreover, the following inequalities hold for all $t \geq 0$, $i = 1, \ldots, n$ and $q > 0$:

$$\int_0^t (v_i(\tau) - v^*)^2 d\tau \leq (1+q) \int_0^t (v_{i-1}(\tau) - v^*)^2 d\tau + k^{-1} \left(W(s_i(0), v_i(0)) + \frac{1}{2q} (v_i(0) - G(s_i(0)))^2 \right) \tag{29}$$

$$\begin{aligned} &\int_0^t (G(s_i(\tau)) - v^*)^2 d\tau \leq \\ &(1+2q) \frac{2qk + k - g_{\max}}{k - g_{\max}} \int_0^t (v_{i-1}(\tau) - v^*)^2 d\tau \\ &+ \frac{2qk + k - g_{\max}}{k(k - g_{\max})} \Bigg(W(s_i(0), v_i(0)) \\ &\qquad + \frac{1}{2q} (v_i(0) - G(s_i(0)))^2 \Bigg) \end{aligned} \tag{30}$$

$$\begin{aligned} |v_i(t) - v^*| \leq & 2 |v_i(0) - v^*| + |G(s_i(0)) - v^*| \\ & + \sup_{0 \leq \tau \leq t} (|v_{i-1}(\tau) - v^*|) \end{aligned} \tag{31}$$

$$\sum_{i=1}^n |v_i(t) - G(s_i(t))| \leq e^{(-(k - g_{\max})t)} \sum_{i=1}^n |v_i(0) - G(s_i(0))| \tag{32}$$

where $W(s, v) := (v - v^*)^2 + 2 \int_{s^*}^s (k - g(z))(G(z) - v^*) dz$.

Due to (23), (24), (25), the function G, defined by (27), is strictly increasing on $[\lambda, +\infty)$. This feature guarantees that for every $v^* \in (0, v_{\max})$, the solution $s^* > \lambda$ of equation (26) is unique.

It should be noted that, if the adaptive cruise controller has the form (28), where $g : \mathbb{R} \to \mathbb{R}_+$ is a locally Lipschitz function that satisfies (23), (24), (25), then the conditions for the safe operation of the vehicular platoon hold. However, in this case we also have some additional properties shown by estimates (29), (30), (31) and (32). Estimate (29) shows that the L_2 string stability notion holds; and estimate (31) shows that the L_∞ string stability notion holds. The point $(s_1, \ldots, s_n, v_1, \ldots, v_n) = (s^*, \ldots, s^*, v^*, \ldots, v^*)$ is the desired equilibrium point for the vehicular platoon. Moreover, estimate (32) guarantees that the vehicular platoon under the cruise controller (28) has a fundamental diagram of the form (9), where G is defined by (27).

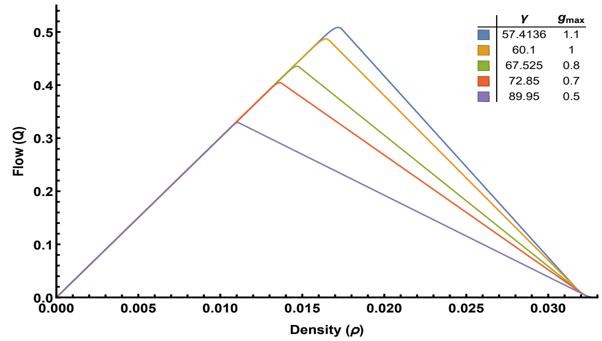

Fig. 7. Fundamental Diagram for the nonlinear adaptive cruise controller (2), with (28) and (33).

Theorem 2 allows the selection of the locally Lipschitz function $g : \mathbb{R} \to \mathbb{R}_+$ that satisfies (23), (24), (25) in order to have an appropriate fundamental diagram for the platoon. By changing $g : \mathbb{R} \to \mathbb{R}_+$, we are in a position to change the shape as well as the critical density and the capacity of the fundamental diagram. This feature is illustrated in Section V.

If the adaptive cruise controller has the form (28), where $g : \mathbb{R} \to \mathbb{R}_+$ is a locally Lipschitz function that satisfies (23), (24), (25) then the equilibrium point $(s^*, \ldots, s^*, v^*, \ldots, v^*) \in D(v^*)$ for a platoon on an open road is Globally Asymptotically Stable. This is guaranteed by the following theorem.

Theorem 3: Let $g : \mathbb{R} \to \mathbb{R}_+$ be a locally Lipschitz function for which there exist constants $k > g_{\max} > 0$, $\lambda > a > 0$ such that properties (23), (24), (25) hold. Consider a platoon of n vehicles on a open/straight road described by (1), (2) with (28), $v_0 = v^* \in (0, v_{\max})$ being the constant speed of the leading vehicle, defined on the set $\overline{D(v^*)}$, where $D(v^*)$ is given by (19) with $v_0 = v^* \in (0, v_{\max})$. Define also $s^* \in (\lambda, +\infty)$ by means of equation (26). Then, the equilibrium point $(s^*, \ldots, s^*, v^*, \ldots, v^*)$ is Globally Asymptotically Stable. Moreover, if in addition g is of class C^1 in a neighborhood of $s^* > \lambda$, then the equilibrium point $(s^*, \ldots, s^*, v^*, \ldots, v^*)$ is Locally Exponentially Stable.

V. Illustrative Examples

In the simulation results below, we compare the three scenarios of the CTG and VTG policies presented in Section II with the proposed controller (2) with (28) and the function g defined by

$$g(s) = \begin{cases} 0 & s \leq \lambda \\ (s - \lambda) & \lambda < s \leq g_{\max} + \lambda \\ g_{\max} & g_{\max} + \lambda < s \leq \gamma \\ g_{\max} \exp(\gamma - s) & s > \gamma \end{cases} \tag{33}$$

with $\gamma, \lambda > 0$ and $k > g_{\max} > 0$. From (33), (9), (27), and (24), we obtain the fundamental diagram shown in Fig. 7 for fixed values $\lambda = 32.5m$, $k = 1.1\, s^{-1}$ and different values of

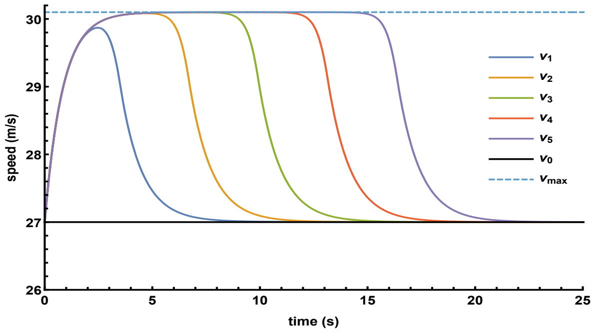

Fig. 8. The speed of all vehicles for the nonlinear adaptive cruise controller (2), (28) with (33) remain within the road speed limit range.

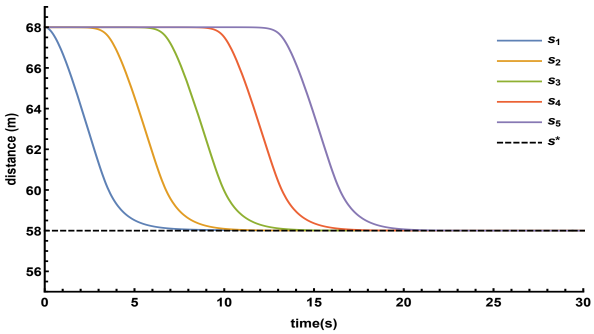

Fig. 9. Vehicle spacing for the nonlinear adaptive cruise controller (2), (28) with (33).

γ, $g_{\max}$, all of which satisfy $v_{\max} = 30.1\,m/s$ (recall (4.2)). Fig. 7 illustrates the macroscopic stability requirement and the freedom of controlling the capacity flow and the critical density via corresponding ACC settings. It should be noticed that $g(\cdot)$ in (33) was selected for its simplicity, and can in general be selected such that the emerging fundamental diagram may be any desired curve which satisfies necessary physical and technical requirements (for example it should satisfy $Q \leq v_{\max}\rho$).

For the following simulation scenarios, we consider the function $g(\cdot)$ in (33) with $\gamma = 62.1m$ and $g_{\max} = 1s^{-1}$. For this selection, all conditions (23), (24), and (25) are fulfilled and, in addition, both the CTG policy (11) with (2), (12) and the nonlinear controller (2), (28) with (33) have the same speed v^* and spacing equilibrium s^*.

Scenario 1. Recall that in this scenario the leading vehicle is moving with constant speed $v_0 = 27\,m/s$, $v_{\max} = 30.1m/s$ and $v_{i,0} = 27m/s$ for $i = 0, 1, \ldots, 5$ and $s_{i,0} = 70m$, $i = 1, \ldots, 5$. Notice that these initial conditions belong to the set $D(v_0)$ defined by (19) with $a = 5m$ for the Safe Operation requirement. Fig. 8 shows the speeds of all vehicles using the adaptive cruise controller (2), (28) with (33). Contrary to the CTG policy (11) with (2), (12) (see Fig. 1), the speeds of all vehicle with the nonlinear controller stay within the bounds $(0, v_{\max})$. Fig. 9 illustrates the vehicle spacing of the adaptive cruise controller (2), (28) with (33). Both Fig. 8 and Fig. 9 exhibit exponential convergence of the state to the equilibrium point.

Scenario 2: In this scenario, the leading vehicle has initial speed $v_0(0) = 10m/s$ on a road with $v_{\max} = 30.1m/s$ and decelerates to $1m/s$. Recall that the initial speed and

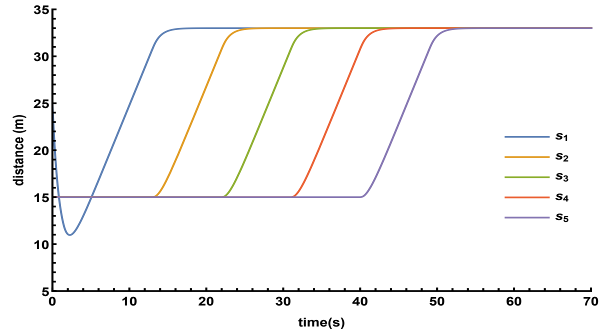

Fig. 10. Vehicle spacing for scenario 2 using the nonlinear adaptive cruise controller (2), (28) with (33).

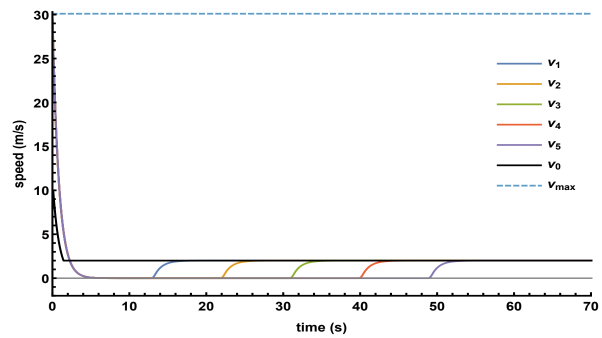

Fig. 11. Speed of vehicles for the nonlinear adaptive cruise controller (2), (28) with (33) for scenario 2.

initial spacing of the $n = 5$ vehicles are $v_{i,0} = 30m/s$, $i = 1, \ldots, 5$ and $s_{1,0} = 25m$, $s_{i,0} = 15m$, $i = 2, \ldots, 5$, respectively. Notice now that these initial conditions are in the safe operation set $D(v_0(0))$ given by (19). Indeed, $s_{1,0}(0) = 25 > a + k^{-1}\max(v_{1,0}(0) - v_0(0)) = 23.18m$ and $s_{i,0}(0) > 5m$ for $i = 2, \ldots, 5$. Under these initial conditions the Safe Operation requirement was not satisfied for the CTG policy (11) with cruise controller (2), (12) as was shown in Figure 2. On the other hand, using the proposed nonlinear adaptive cruise controller (2), (28) with (33), there are no collisions as shown in Fig. 10. Finally, Fig. 11 shows that the speeds of all vehicles are within the speed limits, verifying the Safe Operation requirement (compare with Fig. 3).

Scenario 3: We focus now on the third scenario where all vehicles have initially the same speed $v_{i,0} = 27m/s$ and the leading vehicle decelerates from the initial speed $v_0(0) = 24\,m/s$ to a speed of $5m/s$ with deceleration satisfying (22). Recall that the initial vehicle distances for this scenario are $s_{i,0} = 30m$, $i = 1, \ldots, 5$, which guarantees that the initial state is in the set $D(v_0(0))$ defined by (19) with $a = 5m$. Compared to the VTG controller (15), which is a nonlinear function of speed, the nonlinear adaptive cruise controller (2), (28) with (33), which is a nonlinear function of spacing, satisfies the safe operation requirement as shown in Fig. 12 and Fig. 13.

VI. Concluding Remarks

The present work proposed a novel nonlinear adaptive cruise controller for vehicular platoons which incorporates and integrates the car-following and cruise control tasks without any need for heuristic switching logic. Certain conditions were derived that guarantee safety in terms of

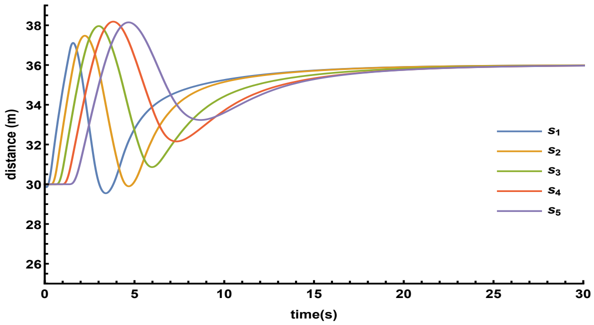

Fig. 12. Vehicle spacing of the nonlinear adaptive cruise controller (2), (28) with (33).

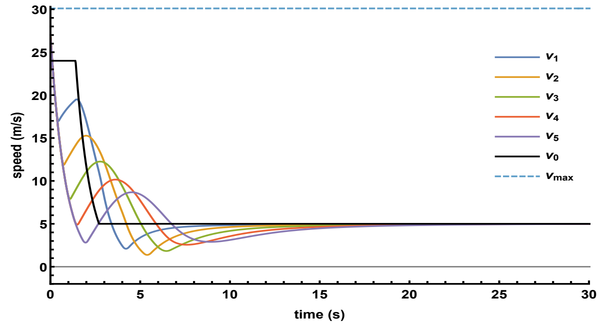

Fig. 13. Vehicle speeds of the nonlinear adaptive cruise controller (2), (28) with (33).

collision avoidance and bounded vehicle speeds by explicitly characterizing a set of admissible initial conditions and the set of allowable inputs. It is shown that a platoon of vehicles with this controller is L_p string stable, and all vehicles will converge to the desired speed/spacing configuration from any initial condition. Future work will address the impact of sensor and actuator delays, as well as the effects of nudging on the stability, string stability and safety of vehicular platoons.

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