Stability Analysis of Nonlinear Inviscid Traffic Flow Models of Bidirectional Cruise Controlled Vehicles

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***Abstract*— The paper introduces a new bidirectional microscopic inviscid Adaptive Cruise Control (ACC) model that uses only spacing information from the preceding and following vehicles in order to select the proper control action to avoid collisions and maintain a desired speed. Class KL estimates that guarantee uniform convergence of the ACC model to the set of equilibria are provided. Moreover, the corresponding macroscopic model is derived, consisting of a conservation equation and a momentum equation that contains a nonlinear relaxation term. It is shown that, if the density is sufficiently small, then the macroscopic model has a solution that approaches exponentially the equilibrium speed (in the sup norm) while the density converges exponentially to a traveling wave.**

I. INTRODUCTION

Microscopic traffic models describe the longitudinal (car-following) and lateral (lane-changing) movement of each single vehicle in the traffic stream, see [29]. Microscopic models based on Adaptive Cruise Control (ACC) and Cooperative Adaptive Cruise Control (CACC) systems are widely regarded as the basis of future generations of automated vehicles since they have the potential of increasing safety, reduce traffic accidents, and improve traffic flow on highways [13], [22]. Both ACC and CACC systems have been extensively studied in the literature (see for instance [13], [17], [22], [26]). The simplest form of interaction between vehicles, that gives more flexibility, than the typical Follow-the-Leader scheme [26], [29], is the bidirectional scheme which monitors the behavior of both the preceding and the following vehicles, see for instance [2], [7], [11], [21], [31].

Contrary to the microscopic models, macroscopic traffic models describe the traffic flow as a fluid that is characterized by macroscopic quantities, such as flow, density, and mean speed of vehicles. Several first-order models and second-order traffic flow models have appeared in the related, see for instance [1], [3], [6], [9], [12], [23], [25], [30], [32], [34]. Both first-order and second-order models have been extended to adjust the vehicle speed based on a perception of downstream density, see for instance [4], [5], [8] and references therein. In the era of connected and automated vehicles, it is possible for vehicles to use backward sensors or to communicate their presence to other vehicles; hence, vehicles may adjust their speed based also on upstream density, in addition to downstream density. In [15], [24] the effect of the upstream density on the speed adjustment was termed as "nudging".

This paper presents a novel bidirectional, microscopic, inviscid ACC model and its corresponding second-order macroscopic model. We call the model "inviscid" because it gives rise to a macroscopic model that contains no viscosity term. The proposed bidirectional microscopic model is based on the two-dimensional cruise controller for autonomous vehicles, recently proposed in [16], and we prove in this work that it has the following main features: (i) Each vehicle uses only the distance from its preceding and following vehicles to select the proper control action (vehicle acceleration); (ii) the vehicles do not collide with each other; (iii) the speeds of all vehicles are always non-negative and remain below an a priori given speed limit; (iv) the ultimate distance between two consecutive vehicles is guaranteed to be greater than a pre-specified constant; (v) all vehicle speeds converge to a given longitudinal speed set-point; and (vi) all the above features are valid globally.

In addition to the above features, it is shown, by exploiting LaSalle's Invariance Principle (see [19]) that, the solutions of the microscopic inviscid model converge asymptotically to a set of equilibrium points from any (arbitrary) physically relevant initial condition (Theorem 2.1). However, since LaSalle's Invariance Principle does not guarantee a uniform attractivity property to the set of equilibria, we construct a strict Lyapunov function for the closed-loop system (Theorem 2.2). Using the constructed Lyapunov function, we establish a KL estimate for the solutions of the microscopic model that guarantees uniformity of the convergence rate to the set of equilibrium points (Theorem 2.3). We also show that, for specific initial conditions, the convergence is exponential (Proposition 3.1). The main theoretical challenges stem from the fact that the control system studied in the paper evolves on a specific set (which is neither open nor closed), and, in addition, various objectives and constraints must be satisfied simultaneously and globally (positive speeds within road speed limits that converge to a specific speed set-point). Moreover, we prove that the vehicles reach a set of equilibrium configurations, where the distance between two consecutive vehicles is guaranteed to be bounded and be greater than a pre-specified constant. Finally, we (formally) derive the macroscopic inviscid model that corresponds to the

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bidirectional microscopic model, and show that, if the initial density is sufficiently small, then the macroscopic model has a solution that approaches exponentially the equilibrium speed (in the sup norm) while the density converges exponentially to a traveling wave (Theorem 3.2).

The structure of the paper is as follows. Section 2 is devoted to the presentation of the bidirectional microscopic inviscid ACC model and its stability properties. Section 3 presents the corresponding macroscopic inviscid model and its analogy to the microscopic model. Section 4, is devoted to simulation examples. Finally, some concluding remarks are given in Section 5. Due to space constraints, the proofs of all results and the formal derivation of the macroscopic model can be found in [18].

Notation. Throughout this paper, we adopt the following notation. $\Re_+ := [0,+\infty)$denotes the set of non-negative real numbers. By $|x|$ we denote both the Euclidean norm of a vector $x \in \Re^n$ and the absolute value of a scalar $x \in \Re$. By K we denote the class of increasing C^0 functions $a : \Re_+ \to \Re_+$ with $a(0) = 0$. By K_∞ we denote the class of increasing C^0 functions $a : \Re_+ \to \Re_+$ with $a(0) = 0$ and $\lim\limits_{s\to+\infty} a(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma : \Re_+ \times \Re_+ \to \Re_+$ with the properties: (i) for each $t \geq 0$ the mapping $\sigma(\cdot, t)$ is of class K; (ii) for each $s \geq 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim\limits_{t\to+\infty} \sigma(s, t) = 0$. By $C^0(A, \Omega)$, we denote the class of continuous functions on $A \subseteq \Re^n$, which take values in $\Omega \subseteq \Re^m$. By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \Re^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \Re^m$. For $f \in C^k(\Re)$, we denote by $f'(x)$, $f''(x), ..., f^{(k)}(x)$ its derivatives When $\Omega = \Re$ the we write $C^0(A)$ or $C^k(A)$. Let $I \subseteq \Re$ be a given interval. $L^\infty(I)$ denotes the set of equivalence classes of measurable functions $f : I \to \Re$ for which $\|f\|_\infty = ess\sup\limits_{x\in I}(|f(x)|) < +\infty$. By $W^{k,\infty}(I)$, where $k \geq 1$ is an integer, we denote the Sobolev spaces of functions $f \in L^\infty(I)$ which have weak derivatives of order $\leq k$, all of which belong to $L^\infty(I)$. For a set $S \subseteq \Re^n$, $\bar{S}$ denotes the closure of S. We denote by $dist(x, A)$ the Euclidean distance of the point $x \in \Re^n$ from the set $A \subset \Re^n$, i.e., $dist(x, A) = \inf\{|x - y| : y \in A\}$. Let $u : \Re_+ \times \Re \to \Re$, $(t, x) \to u(t, x)$ be any function differentiable with respect to its arguments. We use the notation $u[t]$ to denote the profile at certain $t \geq 0$, $(u[t])[x] := u(t, x)$, for all $x \in \Re$.

II. The microscopic inviscid ACC model

A. Description of the model

The movement of n identical vehicles on a straight road under the cruise controller that was proposed in [16], when the vehicles are constrained to move on a line (longitudinal motion), is described by the following set of ODEs:

$$\begin{aligned}
&\dot{x}_i = v_i \quad , \quad i = 1, 2, ..., n\\
&\dot{v}_1 = -k_1(s_2)(v_1 - v^*) - V'(s_2)\\
&\dot{v}_i = -k_i(s_i, s_{i+1})(v_i - v^*) + V'(s_i) - V'(s_{i+1}),\\
&\qquad\qquad i = 2, ..., n-1\\
&\dot{v}_n = -k_n(s_n)(v_n - v^*) + V'(s_n)
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
&s_i = x_{i-1} - x_i \quad , \quad i = 2, ..., n\\
&k_1(s_2) = \mu + g(-V'(s_2))\\
&k_i(s_i, s_{i+1}) = \mu + g(V'(s_i) - V'(s_{i+1})), i = 2, ..., n-1\\
&k_n(s_n) = \mu + g(V'(s_n))
\end{aligned} \tag{2}$$

$\mu, v^* > 0$ are constants, $V \in C^2((L, +\infty); \Re_+)$ is a potential function that satisfies

$$\begin{aligned}
&\lim_{s\to L^+}(V(s)) = +\infty,\\
&V''(s) \geq 0,\\
&V(s) = 0, \quad \text{for } s \geq \lambda\\
&V'(s) < 0, \quad \text{for } L < s < \lambda
\end{aligned} \tag{3}$$

where $\lambda > L > 0$ are constants, and

$$g(s) = \frac{v_{\max} f(s)}{v^*(v_{\max} - v^*)} - \frac{s}{v^*}, \tag{4}$$

$v_{\max} > v^*$ is a constant (the road speed limit), and $f \in C^1(\Re)$ is a non-decreasing function that satisfies

$$\max(x, 0) \leq f(x), \text{ for all } x \in \Re \tag{5}$$

Using (2), the model can be written in the following form

$$\begin{aligned}
&\dot{s}_i = v_{i-1} - v_i \quad , \quad i = 2, ..., n\\
&\dot{v}_1 = -k_1(s_2)(v_1 - v^*) - V'(s_2)\\
&\dot{v}_i = -k_i(s_i, s_{i+1})(v_i - v^*) + V'(s_i) - V'(s_{i+1}),\\
&\qquad\qquad i = 2, ..., n-1\\
&\dot{v}_n = -k_n(s_n)(v_n - v^*) + V'(s_n)
\end{aligned} \tag{6}$$

where s_i, $i = 1, ..., n$ is the back-to-back distance of the i-th vehicle from the (i-1)-th vehicle and v_i, $i = 1, ..., n$, is the speed of the i-th vehicle. The terms $k_1(s_2)$, $k_i(s_i, s_{i+1})$, and $k_n(s_n)$ in (6), are state-dependent gains which guarantee that the speed of each vehicle will remain positive and less than the speed limit $v_{\max}$. The functions $V'(s_i)$ are potential functions that repel vehicles based on their distance with the force of repulsion being stronger as the distance between two vehicles becomes smaller, while there is little or no repulsion when the vehicles are distant, see (3). Since V in (3) is decreasing, then, the term $-V'(s_{i+1})$ is positive and this term represents the effect of nudging, since vehicles that are close and behind vehicle will exert a “pushing” force towards it that will increase its acceleration.

Due to various constraints, such as minimum inter-vehicle distance and speeds within certain speed limits above, the state space of model (6) is

$$\Omega = \left\{ (s_2, ..., s_n, v_1, ..., v_n) \in \Re^{2n-1} : \begin{array}{l} \max\limits_{i=1,...,n}(v_i) \leq v_{\max}, \\ \min\limits_{i=1,...,n}(v_i) \geq 0 \end{array} \right\} \tag{7}$$

where L is a given positive constant (the minimum distance between two vehicles for which the vehicles do not collide with each other). In what follows, we refer to model (6) as the “microscopic, inviscid ACC model”, since the macroscopic analogue of (6) does not contain a viscosity term (see Section

3). Clearly, model (6) is nonlinear not only because of the nonlinearities appearing in the right-hand sides of (6) but also due to the fact that the state space Ω is not a linear subspace of $\Re^{2n-1}$. It should be also noticed that the state space is not an open set (see the recent paper [[27] for the extension of the Input-to-State Stability property to systems defined on open sets) and it is not a closed set.

B. Stability Analysis

Since $v^* \in (0, v_{\max})$, it follows by (3), (4), (6), that the set

$$S = \left\{ (s_2, ..., s_n, v_1, ..., v_n) \in \Re^{2n-1} : \begin{array}{c} \min\limits_{i=2,...,n} (s_i) \geq \lambda, \\ v_i = v^*, \\ i = 1, ..., n \end{array} \right\} \subset \Omega \tag{8}$$

is the set of equilibrium points for the model. In what follows, we use the notation

$$s = (s_2, ..., s_n) \in \Re^{n-1}, v = (v_1, ..., v_n) \in \Re^n \tag{9}$$

Moreover, in what follows we omit the arguments of the functions $k_i, i = 1, ..., n$, defined by (2) (for simplicity).

In this section we show the stability properties of the invariant set S. However, before we proceed, it is necessary to state clearly the characteristics of the problem that indicate the challenge (from a mathematical point of view) of the performed stability analysis:

1) system (6) is nonlinear,
2) the state space of system (6) is neither a closed set, nor an open set, and
3) the invariant set whose stability properties are to be investigated, is not a bounded set (and consequently not a compact set).

We define the function $H : \Omega \to \Re_+$ by the formula

$$H(s, v) = \frac{1}{2} \sum_{i=1}^{n} (v_i - v^*)^2 + \sum_{i=2}^{n} V(s_i) \tag{10}$$

The function H is nothing else but the mechanical energy of the system of n vehicles relative to an observer that moves with constant speed v^*. Using (6) and (10), we obtain the following equation for all $(s, v) \in \Omega$ for the time derivative of the function H along the solutions of (6):

$$\dot{H}(s, v) = -\sum_{i=1}^{n} k_i (v_i - v^*)^2 . \tag{11}$$

It should be noticed that the function H is not a strict Lyapunov function, because (11) shows that the derivative of H can be zero for points out of the invariant set S. However, using (10) and (11) we have the following theorem whose proof can be found in [18].

Theorem 2.1: *For every initial condition $(s(0), v(0)) \in \Omega$, the solution $(s(t), v(t)) \in \Omega$ of (6) is defined for all $t \geq 0$ and satisfies* for $i = 2, ..., n$

$$s_i(t) \leq \max(\lambda, s_i(0)) + \mu^{-1} v_{\max}, \text{ for all } t \geq 0 \tag{12}$$

Moreover, $\lim\limits_{t\to+\infty} (v_i(t)) = v^*$ *for all* $i = 1, ..., n$ *and* $\lim\limits_{t\to+\infty} (V(s_i(t))) = 0$ *for all* $i = 2, ..., n$.

It should be noticed that Theorem 2.1 is not a specialization of Theorem 1 in [16] for vehicles moving on a straight line. Indeed, an application of Theorem 1 in [16] does not guarantee that the distance of two consecutive vehicles (i.e., s_i) is bounded from above. In other words, an application of Theorem 1 in [14] does not give us estimates (12). Notice that since $\lim\limits_{t\to+\infty} (V(s_i(t))) = 0$ for all $i = 2, ..., n$, properties (3) guarantee that $\liminf\limits_{t\to+\infty} (s_i(t)) \geq \lambda$ for all $i = 2, ..., n$ and that $\lim\limits_{t\to+\infty} (dist\,((s(t), v(t)), S)) = 0$ (recall definition (8)). The proof of Theorem 2.1 relies on LaSalle's principle. However, LaSalle's principle does not guarantee *uniform* attraction to the set S. In order to be able to show uniform global attractivity properties for the set S we need to provide a strict Lyapunov function for system (6). This is done by the following theorem.

Theorem 2.2: *For every $\beta > 0$ there exist non-decreasing functions $R \in C^1(\Re_+; (0, +\infty))$, $\kappa \in C^0(\Re_+; (0, +\infty))$ such that the following inequalities hold for all $(s, v) \in \Omega$:*

$$H(s, v) \leq W(s, v) \leq \kappa(H(s, v)) H(s, v) \tag{13}$$

$$\dot{W}(s, v) \leq -\beta\mu \sum_{i=1}^{n} (v_i - v^*)^2 - \frac{1}{8} \sum_{i=2}^{n} 4^i (V'(s_i))^2 \tag{14}$$

where $W : \Omega \to \Re_+$ is defined by the equation

$$W(s, v) := R(H(s, v)) H(s, v) - \sum_{i=2}^{n} 4^i V'(s_i)(v_i - v^*), \quad \text{for all } (s, v) \in \Omega \tag{15}$$

and $\dot{W}(s, v)$ denotes the time derivative of W along the solutions of (6).

Remark: Notice that $W(s, v) > 0$, $\dot{W}(s, v) < 0$ when $(s, v) \in \Omega \backslash S$ and $W(s, v) = \dot{W}(s, v) = 0$ when $(s, v) \in S$. Thus, the function W defined by (15) is a strict Lyapunov function for the microscopic inviscid ACC model (6).

Using Theorem 2.2 we are in a position to prove that a KL estimate holds for the solutions of (6). This estimate is important, because it guarantees uniformity of the convergence rate to the set S and useful robustness properties (see the discussion in [28]; uniform global asymptotic stability with respect to two measures).

Theorem 2.3: *There exists a function $\sigma \in KL$ and a function $a \in K_\infty$ such that for every initial condition $(s(0), v(0)) \in \Omega$ the solution $(s(t), v(t)) \in \Omega$ of (6) is defined for all $t \geq 0$ and satisfies*

$$\begin{aligned} &a(dist((s(t), v(t)), S)) \leq W(s(t), v(t)) \\ &\quad \leq \sigma(W(s(0), v(0)), t), \text{ for all } t \geq 0 \end{aligned} \tag{16}$$

Remarks: (i) While estimate (16) implies a uniform rate of convergence to the invariant set S, it does not imply an exponential rate of convergence. A special case for exponential convergence to the set S and its relation to the macroscopic inviscid model that corresponds to model (6) will be discussed in the following section (Proposition 3.1).

(ii) It should be noticed that Theorem 2.1, Theorem 2.2 and Theorem 2.3 show global convergence of the solutions of the model (6) to the invariant set of equilibrium points S. However, this does not mean that every solution of (6) converges to an equilibrium point. In fact, we cannot conclude that the limits $\lim_{t\to+\infty}(s_i(t))$ exist for $i=2,...,n$. However, Theorem 2.1 shows that every solution of (6) satisfies the following estimates for $i=2,...,n$:

$$\begin{aligned}\lambda \le \liminf_{t\to+\infty}(s_i(t)) &\le \limsup_{t\to+\infty}(s_i(t)) \\ &\le \max(\lambda, s_i(0)) + \mu^{-1}v_{\max}\end{aligned} \tag{17}$$

(iii) It should be noticed that the stability estimate (16) does not establish Uniform Global Asymptotic Stability of the invariant set S, i.e., we do not show an estimate of the form $dist((s(t),v(t)),S) \le \bar{\sigma}(\,dist((s(0),v(0)),S)\,,\,t\,)$, for all $t\ge 0$ for a KL function $\bar{\sigma}$. Instead, the stability estimate (16) establishes Uniform Global Asymptotic Stability with respect to the measures $\omega_1(s,v)=dist((s,v),S)$ and $\omega_2(s,v)=W(s,v)$, (see [28]).

III. The Macroscopic Inviscid ACC Model

In this section we focus on the macroscopic traffic model that corresponds to the microscopic model (1).Various approaches have been suggested to derive macroscopic models for conventional traffic from microscopic models, see for instance [5], [9], [10], [25], [35], and references therein.

A. The PDE model

Let $\rho_{\max}, v_{\max} > 0$ and $v^* \in (0, v_{\max})$, $\bar{\rho}\in(0,\rho_{\max})$ be constants and let $\Phi:(0,\rho_{\max})\to\Re_+$ be a $C^3((0,\rho_{\max}))$ non-negative function that satisfies:

$$\lim_{\rho\to\rho_{\max}^-}(\Phi(\rho)) = +\infty,\ \Phi(\rho)=0,\ \text{for all } \rho\in(0,\bar{\rho}] \tag{18}$$

$$\Phi'(\rho) > 0,\ \text{for all } \rho\in(\bar{\rho},\rho_{\max}) \tag{19}$$

$$\Phi''(\rho)\ge 0,\ \text{for all } \rho\in(0,\rho_{\max}) \tag{20}$$

The macroscopic model that corresponds to the microscopic model (1), as the number of vehicles n tends to infinity and the potential function is given by $V(s)=\Phi\left(\frac{m}{ns}\right)$ with $\frac{m}{n}$ being the mass of every single vehicle, is the following nonlinear system of PDEs for $(t,x)\in(0,+\infty)\times\Re$

$$\begin{array}{l}\rho_t(t,x)+v(t,x)\rho_x(t,x)+\rho(t,x)v_x(t,x)=0\\ v_t(t,x)+v(t,x)v_x(t,x)-\Xi(t,x)= \\ \qquad -(\mu+g(\Xi(t,x)))(v(t,x)-v^*)\end{array}. \tag{21}$$

where $\rho(t,x)$ is the traffic density, $v(t,x)$ is the mean speed, and

$$\Xi(t,x) := -\frac{1}{\rho(t,x)}\left(\rho^2(t,x)\Phi'(\rho(t,x))\right)_x \tag{22}$$

with constrained values

$$0<\rho(t,x)<\rho_{\max},\ 0\le v(t,x)\le v_{\max} \tag{23}$$

for all $(t,x)\in\Re_+\times\Re$

Four things should be noticed about the nonlinear model (21)-(22):

1) there are no non-local terms in the model, despite the fact that the cruise controller proposed in [16] induces "nudging",
2) there are infinite equilibrium points for the model, namely the points where $v\equiv v^*$ and $\rho(x)\le\bar{\rho}$ for all $x\in\Re$,
3) it is a second-order model,
4) the model is highly nonlinear (it is not semilinear as the ARZ model [1], [35] or the PW model [25], [30]), due to the presence of a highly nonlinear relaxation term $-(\mu+g(\Xi(t,x)))(v(t,x)-v^*)$ in the speed PDE.

B. An analogy between the microscopic and the macroscopic model

For the microscopic model (6), the following statement holds: When the vehicles have large initial distances between them, then, they simply adjust their speeds without affecting each other. This is shown by the following proposition, whose proof is very simple and is omitted.

Proposition 3.1: *Suppose that for each* $i=2,...,n$, $s_i(0)\ge\max\left(\lambda-\omega^{-1}(v_{i-1}(0)-v_i(0)),\lambda\right)$ *where* $\omega=\mu+g(0)$. *Then the solution of the model (6) is given by the equations:*

$$\begin{array}{c}v_i(t)=v^*+\exp(-\omega t)(v_i(0)-v^*),\quad i=1,...,n\\ s_i(t)=s_i(0)+\omega^{-1}(v_{i-1}(0)-v_i(0))(1-\exp(-\omega t)),\\ i=2,...,n\end{array} \tag{24}$$

In this case we have exponential convergence to the set S.

Similarly with Proposition 3.1, if the initial density is sufficiently small then the macroscopic model (21)-(22) has a solution that approaches the equilibrium speed (in the sup norm) while the density remains small. The following theorem guarantees this fact.

Theorem 3.2: *Consider the initial-value problem*

$$\begin{array}{l}\rho_t+v\rho_x+\rho v_x=0\\ v_t+vv_x=-\omega(v-v^*)\end{array}\quad \text{for } t\ge0,\ x\in\Re \tag{25}$$

$$\begin{array}{l}\rho(0,x)=\rho_0(x)\\ v(0,x)=v_0(x)\end{array}\quad \text{for } x\in\Re \tag{26}$$

where $\rho_0\in C^1(\Re)\cap W^{1,\infty}(\Re), v_0\in C^2(\Re)\cap W^{2,\infty}(\Re)$ $\omega=\mu+g(0)$, *with* $\inf_{x\in\Re}(v_0'(x))>-\omega$ *and* $\rho_0(x)>0$ *for all* $x\in\Re$. *Then, the initial-value problem* (25), (26) *has a unique solution that satisfies the estimates:*

$$\sup_{x\in\Re}(\rho(t,x)) \le \frac{\omega\sup_{x\in\Re}(\rho_0(x))}{\omega+(1-\exp(-\omega t))\inf_{x\in\Re}(v_0'(x))} \quad \text{for all } t\ge0 \tag{27}$$

$$\sup_{x\in\Re}(|v(t,x)-v^*|)\le\exp(-\omega t)\sup_{x\in\Re}(|v_0(x)-v^*|) \quad \text{for all } t\ge0 \tag{28}$$

$$\rho(t,x)>0\quad \text{for all } t\ge0,\ x\in\Re \tag{29}$$

Moreover, there exists a function $f:\Re\to(0,+\infty)$ *of class* $C^1(\Re)\cap L^\infty(\Re)$ *for which the following estimate holds:*

$$\sup_{x\in\Re,t\ge0}(|\rho(t,x)-f(x-v^*t)|\exp(\omega t))<+\infty. \tag{30}$$

By virtue of (18) and (22), it follows that $\Xi(t,x)\equiv 0$ when $\sup_{x\in\Re}(\rho(t,x))\le\bar{\rho}$ for all $t\ge 0$. Therefore, in this case the PDE model (21)-(22) becomes identical to the PDE model (25). Consequently, Theorem 3.2 guarantees that if the initial conditions satisfy the requirements

$$\inf_{x\in\Re}\left(v_0'(x)\right) > -\omega \tag{31}$$

$$\sup_{x\in\Re}\left(\rho_0\left(x\right)\right)\le\bar{\rho}\left(1+\omega^{-1}\min\left(0,\inf_{x\in\Re}\left(v_0'\left(x\right)\right)\right)\right) \tag{32}$$

then the macroscopic model (21)-(22) has a solution that satisfies the estimates:

$$\sup_{x\in\Re}\left(\rho\left(t,x\right)\right)\le\bar{\rho}\text{ for all }t\ge 0 \tag{33}$$

$$\sup_{x\in\Re}\left(\left|v(t,x)-v^*\right|\right)\le\exp(-\omega t)\sup_{x\in\Re}\left(\left|v_0\left(x\right)-v^*\right|\right)\quad\text{for all }t\ge 0 \tag{34}$$

$$\rho(t,x)>0\quad\text{for all }t\ge 0,\ x\in\Re \tag{35}$$

and there exists a function $f:\Re\to(0,\bar{\rho}]$ of class $C^1(\Re)\cap L^\infty(\Re)$ for which the following estimate holds:

$$\sup_{x\in\Re,t\ge 0}\left(\left|\rho(t,x)-f(x-v^*t)\right|\exp(\omega t)\right)<+\infty.$$

Notice that the solution converges exponentially (in the sup norm) to the set of equilibrium points of the macroscopic model (but not necessarily to one equilibrium point). This is due to the fact that the whole profile ultimately moves with speed v^* and thus we have a traveling wave.

IV. Simulation Examples

Example 1: (Exponential convergence to the set S) As indicated by Proposition 3.1, for the exponential convergence to the set of equilibrium points S, the initial spacing needs to satisfy the condition $s_i(0)\ge\max\left(\lambda-\omega^{-1}\left(v_{i-1}(0)-v_i(0)\right),\lambda\right)$ for each $i=1,...,n$, where $\omega=\mu+g(0)$ and g is given by (4). Values of $L=5m,\lambda=20m,\ v^*=30m/s,\ \ v_{\max}=35m/s$, and $\mu=0.5$ were used. The speeds and inter-vehicle distances are shown in Figure 1. The exponential convergence to the set S is demonstrated by Figure 2, which shows the evolution of the Lyapunov function $H(s,v)$ and its logarithm $\ln(H(s,v))$.

Example 2: (Travelling waves of the macroscopic model (25) To illustrate the results of Theorem 3.1, we consider a road with initial density and initial speed given by

$$\rho_0(x)=0.1+\left\{\begin{array}{cc}5x^2(x-1)^2, & x\in(0,1)\\ 0 & else\end{array}\right.$$

$$v_0(x)=1+\left\{\begin{array}{cc}8x^3(x-1)^3, & x\in(0,1)\\ 0 & else\end{array}\right.$$

These initial conditions indicate that there is a congestion belt on the interval $x\in(0,1)$ where vehicles are moving at lower speed and and accelerate again to a speed of $v^*=1$ as the density decreases to a constant value. Values of $v^*=1$ and $\omega=1.2$ were used. In Figure 3 displays the speed profiles $v[t]$ (left) and the density profiles $\rho[t]$ (right) at different time instants $t=0,1,...,5$. This example illustrates the

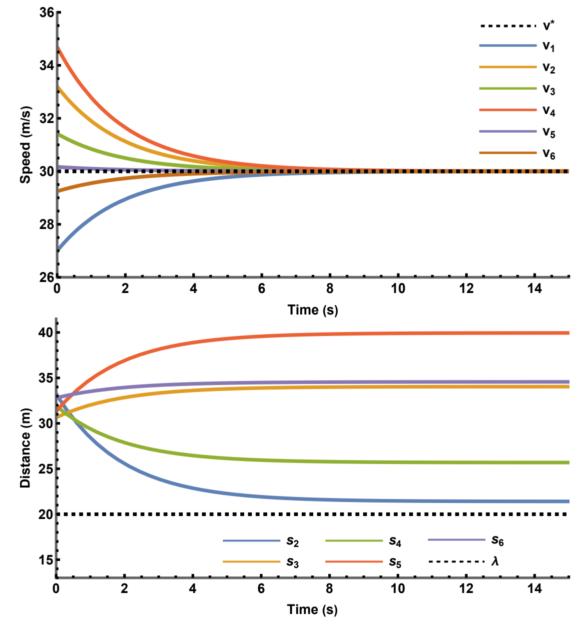

Fig. 1. Exponential convergence to the set S.

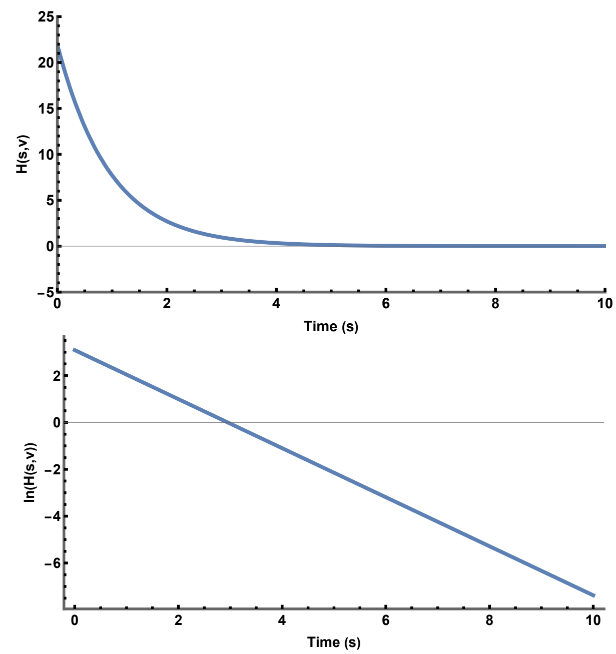

Fig. 2. Convergence of the function $H(s,v)$ (top) and the graph of $\ln(H(s,v))$ (bottom) indicating exponential convergence.

results of Theorem 3.1 which show that the speed converges exponentially to the speed set-point v^*; while the density converges to a travelling wave.

V. Conclusion

The paper introduces a new bidirectional microscopic inviscid Adaptive Cruise Control (ACC) model that uses only spacing information from the preceding and following vehicles. KL estimates that guarantee uniform convergence properties of the ACC model to the set of equilibria are provided. Moreover, the corresponding macroscopic model is derived, consisting of the continuity equation and a mo-

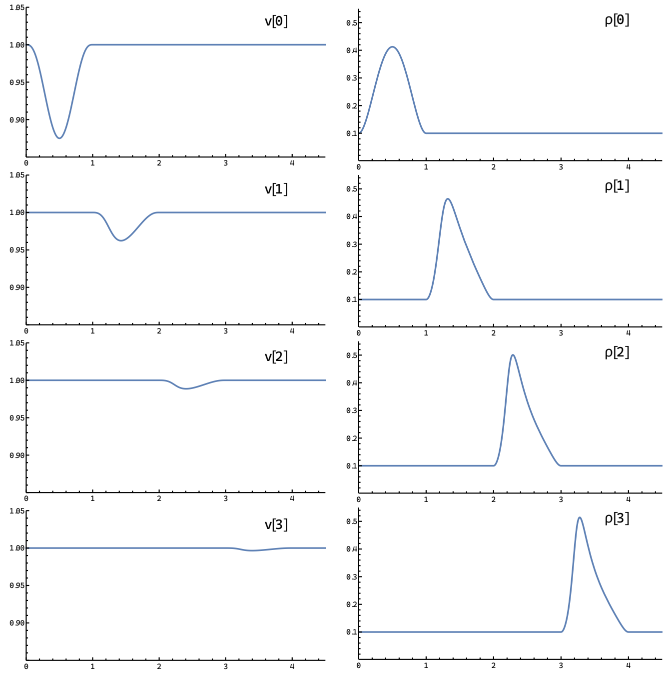

Fig. 3. Exponential convergence of speed profiles $v[t]$ (left) and convergence of density profiles $\rho[t]$ to a traveling wave (right).

mentum equation that contains a highly nonlinear relaxation term. It is shown that, if the density is sufficiently small then the solution of the macroscopic model approaches the equilibrium speed (in the sup norm) while the density converges exponentially to a traveling wave.

References

[1] Aw, A. and M. Rascle, "Resurrection of Second-Order Models of Traffic Flow", *SIAM Journal on Applied Mathematics,* 60, 2000, 916–938.

[2] Baldi, S., D. Liu, V. Jain and W. Yu, "Establishing Platoons of Bidirectional Cooperative Vehicles with Engine Limits and Uncertain Dynamics", *IEEE Transactions on Intelligent Transportation Systems*, 22, 2021, 2679-2691.

[3] Bekiaris-Liberis N. and A. I. Delis, "PDE-Based Feedback Control of Freeway Traffic Flow via Time-Gap Manipulation of ACC-Equipped Vehicles", *IEEE Transactions on Control Systems Technology*, 29, 2021, 461-469.

[4] Chiarello, F. A., "An Overview of Non-local Traffic Flow Models". In: Puppo G., Tosin A. (eds) *Mathematical Descriptions of Traffic Flow: Micro, Macro and Kinetic Models*, SEMA SIMAI Springer Series, vol. 12, 2021.

[5] Chiarello, F. A., J. Friedrich, P. Goatin and S. Göttlich, "Micro-Macro Limit of a Nonlocal Generalized Aw-Rascle Type Model", *SIAM Journal on Applied Mathematics*, 80, 2020, 1841-1861.

[6] Di Francesco, M., S. Fagioli and M. D. Rosini. "Many Particle Approximation of the Aw-Rascle-Zhang Second Order Model for Vehicular Traffic", *Mathematical Biosciences and Engineering*, 14, 2017, 127-141.

[7] Ghasemi, A., R. Kazemi and S. Azadi, "Stable Decentralized Control of a Platoon of Vehicles With Heterogeneous Information Feedback", *IEEE Transactions on Vehicular Technology*, 62, 2013, 4299-4308.

[8] Goatin, P., and S. Scialanga, "Well-Posedness and Finite Volume Approximations of the LWR Traffic Flow Model with Non-Local Velocity", *Networks and Hetereogeneous Media*, 11, 2016, 107–121.

[9] Helbing, D., "Derivation of Non-Local Macroscopic Traffic Equations and Consistent Traffic Pressures from Microscopic Car-Following Models", *The European Physical Journal B,* 69**,** 2009, 539–548.

[10] Helbing, D., "Traffic and Related Self-Driven Many-Particle Systems", *Reviews of Modern Physics,* 73, 2001, 1067–1141.

[11] Herman, I., D. Martinec, Z. Hurák and M. Sebek, "Scaling in Bidirectional Platoons With Dynamic Controllers and Proportional Asymmetry", *IEEE Transactions on Automatic Control*, 62, 2017, 2034-2040.

[12] Huang K., and Q. Du, "Stability of a Nonlocal Traffic Flow Model for Connected Vehicles", *SIAM Journal on Applied Mathematics*. (2021) 82, 2022, 221-243.

[13] Ioannou, P. A. and C. C. Chien, "Autonomous Intelligent Cruise Control", *IEEE Transactions on Vehicular Technology*, 42, 1993, 657-672.

[14] Karafyllis, I. and Z. P. Jiang, *Stability and Stabilization of Nonlinear Systems*, Springer-Verlag, New York, 2011.

[15] Karafyllis, I., D. Theodosis and M. Papageorgiou, "Analysis and Control of a Non-Local PDE Traffic Flow Model", *International Journal of Control,* 95, 2022, 660-678.

[16] Karafyllis, I., D. Theodosis and M. Papageorgiou, "Two-Dimensional Cruise Control of Autonomous Vehicles on Lane-Free Roads", submitted to *Automatica* (see also arXiv:2103.12205 [math.OC]).

[17] Karafyllis, I., D. Theodosis, and M. Papageorgiou, "Nonlinear Adaptive Cruise Control of Vehicular Platoons", *International Journal of Control*, , DOI: 10.1080/00207179.2021.1982015

[18] Karafyllis, I., D. Theodosis, and M. Papageorgiou, "Stability Analysis of Nonlinear Inviscid Microscopic and Macroscopic Traffic Flow Models of Bidirectional Cruise-Controlled Vehicles", arXiv:2109.09622 [math.OC]

[19] Khalil, H. K., *Nonlinear Systems*, Prentice Hall, 2002.

[20] Khatir, M. E. and E. J. Davison, "Decentralized Control of a Large Platoon of Vehicles Using Non-Identical Controllers", *Proceedings of the 2004 American Control Conference*, 2004, 2769-2776.

[21] Kwon, J. and D. Chwa, "Adaptive Bidirectional Platoon Control Using a Coupled Sliding Mode Control Method", *IEEE Transactions on Intelligent Transportation Systems*, 15, 2014, 2040-2048.

[22] Milanés, V., S. E. Shladover, J. Spring, C. Nowakowski, H. Kawazoe and M. Nakamura, "Cooperative Adaptive Cruise Control in Real Traffic Situations," *IEEE Transactions on Intelligent Transportation Systems*, 15, 2014, 296-305.

[23] Ngoduy D. and D. Jia, "Multi Anticipative Bidirectional Macroscopic Traffic Model Considering Cooperative Driving Strategy", *Transportmetrica B: Transport Dynamics*, 5, 2017, 96-110.

[24] Papageorgiou, M., Mountakis, K.-S., Karafyllis, I., Papamichail, I. and Y. Wang, "Lane-Free Artificial-Fluid Concept for Vehicular Traffic". *Proceedings of the IEEE* 109, 2021, 114-121.

[25] Payne, H. J. "Models for Freeway Traffic and Control". In: Bekey, G.A. (Ed.), *Mathematical Models of Public Systems*, 1, 1971, 51–61.

[26] Rajamani, R., *Vehicle Dynamics and Control*, Springer-Verlag, New York, 2012.

[27] Sontag, E. D., "Remarks on input to state stability of perturbed gradient flows, motivated by model-free feedback control learning", *Systems & Control Letters*, 161, 2022, 105138.

[28] Teel, A. R. and L. Praly, "A Smooth Lyapunov Function from a Class-KL Estimate Involving Two Positive Semidefinite Functions", *ESAIM: Control, Optimization and Calculus of Variation,* 5, 2000, 313-336.

[29] Treiber, M., Kesting, A., *Traffic Flow Dynamics: Data, Models and Simulation*, Springer, 2013.

[30] Whitham, G. B., *Linear and Nonlinear Waves,* John Wiley, New York, 1974.

[31] Zegers, C., E. Semsar-Kazerooni, J. Ploeg, N. van de Wouw and H. Nijmeijer, "Consensus Control for Vehicular Platooning With Velocity Constraints", *IEEE Transactions on Control Systems Technology*, 26, 2018, 1592-1605.

[32] Yu H., S. Amin and M. Krstic, "Stability Analysis of Mixed-Autonomy Traffic with CAV Platoons using Two-class Aw-Rascle Model," *2020 59th IEEE Conference on Decision and Control (CDC)*, 2020, 5659-5664.

[33] Yu H., and Krstic M., "Traffic Congestion Control for Aw–Rascle–Zhang Model", *Automatica,* 100, 2019, 38-51.

[34] Zhang, H. M., "A Non-Equilibrium Traffic Model Devoid of Gas-like Behavior", *Transportation Research Part B*, 36, 2002, 275–290.

[35] Zhang, Y., B. Kosmatopoulos, P. A. Ioannou and C. C. Chien, "Using Front and Back Information for Tight Vehicle Following Maneuvers", *IEEE Transactions on Vehicular Technology*, 48, 1999, 319-328.