Compensation of Input-Dependent Hydraulic Input Delay for a Model of a Microfluidic Process Under Zweifach-Fung Effect

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Abstract

We consider a model of a microfluidic process under Zweifach-Fung effect, which gives rise to a second-order nonlinear, non-affine system with control input that affects the plant both without delay and with an input-dependent delay defined implicitly through an integral of the past input values (that arises from a transport process with transport speed being the control input itself). We construct a predictor-feedback control law that exponentially stabilizes the output to a desired reference point. This is the first time that a predictor-feedback design is constructed that achieves *complete* input delay compensation for such a type of input delay and despite that control input affects the plant also without delay. This is attributed to the particular structure of the nonlinear system considered, which allows to deriving an implementable formula for the predictor state at the proper prediction horizon. We then identify a class of nonlinear systems with input-dependent input delay of hydraulic type for which complete delay compensation, through construction of an exact predictor state, is achievable.

Keywords: Predictor feedback; Hydraulic input-dependent delay; Nonlinear systems; Microfluidic process; Zweifach-Fung effect.

1 Introduction

Microfluidic processes are ubiquitous in lab-on-a-chip applications, see, for example, [18], [23]. An important phenomenon evident in such processes is the so-called Zweifach-Fung effect, which appears in microfluidic systems that involve separation of particles within a fluid at a bifurcation point, with a separation volume ratio that depends on the flow rates at the two daughter branches of the main channel. Fig. 1 illustrates an example of such a setup. This effect can be utilized in applications, such as blood purification [25], while it is studied within the framework of analysis of microcirculation dynamics, see, e.g., [7], [12], [13]. Regulating the volume fraction of particles in one of the reservoirs (corresponding to one of the daughter channels) is crucial for applications that involve, for example, filtering or enrichment of particles in a fluid, see, e.g., [19].

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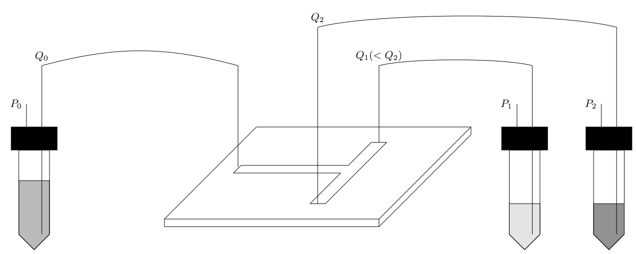

Fig. 1. Example of a microfluidic process [19]. Particles in a fluid are separated in the bifurcation point at a volume ratio that depends on flow rates Q_1, Q_2 of each daughter branch, which in turn can be manipulated via the respective pressures P_1, P_2 in the reservoirs.

A control-oriented model of such a phenomenon is presented in [19]. The main features of this model are the following. The control input is the flow ratio (with respect to total flow) in the first channel, while the output is the volume fraction of particles in the first reservoir. Owing to the transport of particles from the bifurcation point to the first reservoir there is a delay of hydraulic type (i.e., defined implicitly through an integral of past values of flow ratio), because the transport speed depends explicitly on the flow ratio itself. In addition, the Zweifach-Fung effect at the bifurcation point, gives rise to a nonlinear term in the dynamic equation for the volume ratio, which depends on the flow ratio at the delay time. Moreover, the flow ratio also affects directly the volume ratio of particles in the first reservoir, which gives rise to a term that depends on a non-delayed form of the flow ratio. Despite the practical importance of control of such processes and existence of a control-oriented model, there is no attempt to design a delay-compensating feedback law. As a result, the related literature for this problem can be categorized into results dealing with modeling and analysis of such processes; see, for example, [7], [12], [13], [19], [25], and into results dealing with predictor-based control of systems with input-dependent input delays; see, for example, [1], [2], [4], [5], [6], [8], [20], [22], and of systems with distributed input delay; see, e.g., [1], [17], [21], [24], [26].

In this paper, we develop a predictor-feedback law for a nonlinear model of a microfluidic process under Zweifach-Fung effect, which achieves exponential stabilization of a desired reference point. The design relies on two ingredients–the construction of an *exact* predictor state and the design of a nominal feedback law. Despite that the delay is defined implicitly through an integral of the input (over an interval from the delay time to the current time) and despite that the input enters the plant both in delayed and non-delayed form, the construction of the predictor state is made possible owing to the particular structure of the nonlinear system considered and the specific dependence of its vector field on the input variable (in fact, the predictor state is given in explicit form). The nominal feedback law is designed based on a particular delay-free system, which is not obtained in an obvious manner (e.g., considering that the input only appears in non-delayed form in the system's dynamic equation). It is rather derived constructing a stabilizing feedback law for the system in a new time variable, which allows, in fact, to recasting the problem of design of the nominal controller as a problem of design of a feedback law for a delay-free, time-varying, nonlinear non-affine system.

For guaranteeing the delay properties required for design of a predictor state and for well-posedness of the system a feasibility condition that the input is lower and upper bounded by positive constants needs to be satisfied (also making the model considered realistic from a practical viewpoint). This imposes derivation of a local stability result in the supremum norm of the delayed, actuator state. The proof of exponential stability of the closed-loop system relies on deriving estimates on solutions and on relating the norm of the overall, infinite-dimensional system to the norm of the predictor state. We then present an alternative proof that enables exact computation of the region of attraction of the control law. We also present simulation results of a microfluidic process with a sinusoidal nonlinearity, describing the Zweifach-Fung effect (see, for example, [10], [19]), which confirms the performance improvement of the closed-loop system under predictor feedback, as compared, for instance, to employment of an open-loop control strategy.

We then generalize the (exact) predictor-feedback design to a class of nonlinear systems. This class is characterized by a vector field that could be viewed as a product of a nominal vector field, which depends on the delayed input, with a scaling term (that depends on the non-delayed input), which is the function being integrated in the definition of the hydraulic delay (or, simply, the transport speed). Such systems may describe the dynamics of parallel microfluidic processes with a single outlet reservoir, actuated via a single pressure. Under an a priori assumption of lower boundedness, by positive constant, of the scaling function (and typical assumptions, imposed on the nominal vector field, which guarantee global stabilization under predictor feedback for long, input delays; see, for example, [16]), we establish global asymptotic stability of the closed-loop system under predictor feedback (otherwise, a local stability result would be achievable, as in the case of the microfluidic process model). The stability proof relies on derivation of estimates on solutions and introduction of a suitable change of the time variable.

2 Model of the Process and Open-Loop Behavior

2.1 Model of the Process

We consider the system

$$\dot{Y}(t) = \frac{f\left(U\left(t - D(t)\right)\right) - Y(t)}{X(t)} U(t) \qquad (1)$$

$$\dot{X}(t) = U(t) \qquad (2)$$

$$\int_{t-D(t)}^{t} U(s) ds = L, \qquad (3)$$

where $Y > 0$ denotes the ratio of particles volume with respect to the total volume in the first reservoir, $X > 0$ is normalized total volume in the first reservoir, $U > 0$ is flow ratio between flow in the first channel and total flow in the main (inlet) channel, which is the manipulated variable, $D > 0$ is delay, $L > 0$ is the ratio between total volume in the first channel and total flow in the inlet channel, and $t \geq 0$ is time variable. We also define the delay time ϕ as $\phi(t) = t - D(t)$, which is employed later on. The delay as defined in (3) is referred to as transport ([4]) or hydraulic ([6]) delay and expresses the conservation of total volume of particles along the first channel. Thus, it is defined such that the integral of the flow rate of particles in the first channel, from the time at which the particles were at the bifurcation point up to the current time at which the particles have been transported to the first reservoir, is equal to the total volume in the first reservoir, i.e., relation (3) holds. Further details on the model derivation can be found in [19]. We impose the following realistic (see, for example, [19]) assumption on f.

Assumption 1 The function $f : [c_1, c_2] \to [d_1, d_2]$, with $0 < c_1 < c_2 < 1$ and $0 < d_1 < d_2 < 1$, is Lipschitz with constant L_1, strictly increasing, and its inverse $f^{-1} : [d_1, d_2] \to [c_1, c_2]$ is Lipschitz with constant L_2.

To guarantee well-posedness of system (1)–(3) and for system (1)–(3) to be a realistic model of the process the following feasibility condition has to be satisfied

$$0 < c_1 \leq U(\theta) \leq c_2 < 1, \quad \text{for all } \theta \geq -D(0). \qquad (4)$$

Condition (4) guarantees that the delay D, defined implicitly via (3), satisfies all requirements of time-varying input delays that imply a uniquely defined delay that is positive and upper bounded, as well as that its rate is less than one and lower bounded [4]. These requirements also allow to guarantee well-posedness of a predictor state design [3], [4]. We summarize these properties for the delay in Proposition A.1 in Appendix A, together with presenting its proof.

The goal of the predictor-feedback law is regulation of the volume fraction Y to a desired reference value $\bar{Y}$, corresponding to a constant value $\bar{U}$ for the input (and thus, it also corresponds to a constant delay value). Under (4), system (2) does not have an equilibrium solution as its state X is a linearly increasing function of time (for this reason X can be viewed more as time variable) [2] . Thus, by an equilibrium of system (1), (3) we denote a scalar $\bar{Y}$ satisfying $\bar{Y} = f(c)$, which corresponds to a function $\bar{U} \in C\left(\left[-\frac{L}{c}, 0\right], (c_1, c_2)\right)$ with $\bar{U} \equiv c$, resulting in a zero right-hand side for (1), for all $X > 0$. For such an equilibrium we derive (in closed loop) direct stability estimates in an ad hoc manner, without necessarily invoking a definition of a specific type of stability [3] .

2.2 *Open-Loop Behavior*

Lemma 1 *Consider system (1)–(3), under a reference input $U(t) = c$, $t \geq 0$, for some $c_1 < c < c_2$. Under Assumption 1, for each $X(0) = X_0 > 0$ and all initial conditions $Y(0) = Y_0 \in (f(c_1), f(c_2))$, $U_0 \in C([-D(0), 0], (c_1, c_2))$, satisfying $U_0(0) = c$, the following holds*

$$Y(t) = \begin{cases} \dfrac{Y_0 X_0}{ct + X_0} + \dfrac{c\int_0^t f\left(U_0\left(s - D(s)\right)\right) ds}{ct + X_0}, \\ \hfill 0 \leq t \leq \dfrac{L}{c} \\ f(c) + \left(Y\left(\dfrac{L}{c}\right) - f(c)\right) \dfrac{L + X_0}{ct + X_0}, \quad t > \dfrac{L}{c}. \end{cases} \tag{5}$$

Proof The proof can be found in Appendix B.

Lemma 1 implies that, for constant input $U(t) = \bar{U} = c$, $t \geq 0$, Y remains bounded; while regulation to equilibrium $\bar{Y} = f(c)$, $\bar{U} = c \in C\left(\left[-\frac{L}{c}, 0\right], (c_1, c_2)\right)$ is achieved. In fact, if $X_0 \geq \bar{\delta} > 0$, then asymptotic stability, in the sense of satisfying $\Omega(t) \leq \beta_0\left(\rho(X_0)\Omega_0, t\right)$, $t \geq 0$, holds with $\beta_0\left(\rho(X_0)\Omega_0, t\right) = (2X_0 + L + L_1 L)\frac{\Omega_0}{ct+\bar{\delta}}$ (this is derived from (5)). To improve performance (e.g., convergence rate) and robustness we design next a predictor-feedback law.

3 Predictor-Feedback Control Design

Given a nominal (for the delay-free case), stabilizing feedback law κ, we construct the predictor-feedback law as

$$U(t) = \kappa\left(X(t) + L, P(t)\right) \tag{6}$$

$$P(t) = \frac{Y(t)X(t)}{X(t) + L} + \frac{\int_{\phi(t)}^t f\left(U(s)\right) U(s) ds}{X(t) + L}. \tag{7}$$

State P is the predictor of Y at the proper, for complete input delay compensation, prediction horizon, whereas $X + L$ is the predictor state of X. These facts are explained as follows.

3.1 *Predictor States Construction*

Denoting the delay time as $\phi(t) = t - D(t)$ and the prediction time as $\sigma(t) = \phi^{-1}(t)$ (that exists as long as (4) is satisfied, and thus, X is strictly increasing) we get for $t \geq 0$

$$\int_t^{\sigma(t)} U(s) ds = L. \tag{8}$$

Therefore, using (2) we get that

$$X\left(\sigma(t)\right) = X(t) + L, \tag{9}$$

showing that the predictor state of X, i.e., $X(\sigma)$, is $X + L$. The prediction horizon needed is $\sigma(t) = X^{-1}\left(X(t) + L\right)$ (respectively, integrating (2) from ϕ to t and using (3) we get for $\phi(t) \geq 0$ that $\phi(t) = X^{-1}\left(X(t) - L\right)$). To find the predictor state of Y we substitute $t = \sigma(\theta)$, for $\phi(t) \leq \theta \leq t$, in (1) to obtain

$$\frac{dY\left(\sigma(\theta)\right)}{d\theta} = \frac{d\sigma(\theta)}{d\theta} \frac{f\left(U(\theta)\right) - Y\left(\sigma(\theta)\right)}{X\left(\sigma(\theta)\right)} U\left(\sigma(\theta)\right). \tag{10}$$

Thus, defining $Y\left(\sigma(\theta)\right) = P(\theta)$ and since $U\left(\sigma(\theta)\right)\frac{d\sigma(\theta)}{d\theta} = U(\theta)$ (that follows differentiating (8) with respect to the time variable and which is the key for enabling construction of an implementable formula for the exact predictor state) we get

$$\frac{d\left(P(\theta)X\left(\sigma(\theta)\right)\right)}{d\theta} = f\left(U(\theta)\right) U\left(\theta\right). \tag{11}$$

Integrating (11) from $\theta = \phi(t)$ to $\theta = t$ and using (9) we get (7). Note that, according to (10), the Ordinary Differential Equation (ODE) satisfied by the predictor state is

$$\dot{P}(t) = \frac{f\left(U(t)\right) - P(t)}{X(t) + L} U(t). \tag{12}$$

[2] Because, in practice, convergence of Y is much faster (under the control law developed the rate is exponential) than the increase of X, it is expected that divergence of X would not have significant practical implications. In fact, in practice, it may take several hours for the reservoir to become full, while, in addition, the reservoir is usually replaced by an empty one, as soon as it becomes full.

[3] A general version of the type of asymptotic stability obtained could be stated as requiring that there exist a class $\mathcal{KL}$ function β_0, a continuous decreasing function $\epsilon : (0, +\infty) \to (0, +\infty)$, and a continuous function $\rho : (0, +\infty) \to (0, +\infty)$ such that, for each $X_0 > 0$, the solutions of system (1), (3) satisfy $\Omega(t) \leq \beta_0\left(\rho(X_0)\Omega_0, t\right)$, $t \geq 0$, where $\Omega(t) = |Y(t) - f(c)| + \sup_{t-D(t)\leq\theta\leq t} |U(\theta) - c|$, for all initial conditions $Y_0 > 0$ and $U_0 \in C([-D(0), 0], (c_1, c_2))$, which satisfy $\Omega_0 < \epsilon(X_0)$.

3.2 Nominal Feedback Law Design

We choose the following nominal feedback law function [4]

$$\kappa(\tau, H) = f^{-1}(H - k\tau(H - f(c))), \tag{13}$$

with some $k > 0$ and $c_1 < c < c_2$, which renders the equilibrium $\bar{H} = f(c)$ of system

$$\frac{dH(\tau)}{d\tau} = \frac{1}{\tau}(f(\kappa(\tau, H(\tau))) - H(\tau)), \tag{14}$$

asymptotically stable. The choice of the nominal feedback law such that it stabilizes system (14) is motivated by the requirement of achieving stabilization of the P system given in (12) and is explained as follows. With the change of variables $\tau = X(t) + L$ (with $X(0) = X_0 > 0$) for the time variable t, under (4) (implying that the change of variables is invertible) we get from (12) that

$$\frac{dH(\tau)}{d\tau} = \frac{1}{\tau}(f(W(\tau)) - H(\tau)), \tag{15}$$

where we defined $H(\tau) = P\left(X^{-1}(\tau - L)\right)$ and $W(\tau) = U\left(X^{-1}(\tau - L)\right)$, for $\tau \geq X_0 + L$. System (15) is a time-varying nonlinear system, which can be stabilized with the choice $W(\tau) = \kappa(\tau, H(\tau))$, with κ being defined in (13). The alternative representation (15) also reveals that X could be viewed more as time variable (rather than as state), and thus, as regards a nominal, delay-free design, one could seek a feedback law of the form $\kappa(\tau, H)$ that stabilizes (15), which is simpler than (12).

4 Stability Analysis

Theorem 1 *Consider the closed-loop system consisting of the plant (1)–(3) satisfying Assumption 1 and the control law (6), (7) with (13). For each $X(0) = X_0 > 0$, there exists a strictly decreasing function $\epsilon \in C((0, +\infty), (0, +\infty))$ such that for all initial conditions $Y(0) = Y_0 > 0$ and $U_0 \in C([-D(0), 0], (c_1, c_2))$, which satisfy*

$$\Omega_0 < \epsilon(X_0) \tag{16}$$

$$\Omega_0 = |Y_0 - f(c)| + \sup_{-D(0) \leq \theta \leq 0} |U_0(\theta) - c|, \tag{17}$$

and $U_0(0) = \kappa\left(X_0 + L, \frac{Y_0 X_0}{X_0 + L} + \frac{\int_{-D(0)}^{0} f(U_0(s)) U_0(s) ds}{X_0 + L}\right)$, *there exists a unique solution such that $Y(t) \in C^1[0, +\infty)$, $X(t) \in C^1[0, +\infty)$, $U(t)$ being locally Lipschitz on $[0, +\infty)$, and the following hold for $t \geq 0$*

$$|Y(t) - f(c)| \leq \Omega_0 e^{kL} \max\{1, L_1\} e^{-kc_1 t} \tag{18}$$

$$\sup_{t - D(t) \leq \theta \leq t} |U(\theta) - c| \leq \Omega_0 \max\{1, L_1\}(L_2 + 1) e^{kL} \times (1 + kL + kX_0 + kc_2 t) e^{-kc_1 t}. \tag{19}$$

Moreover, the feasibility condition (4) is satisfied.

Proof The proof can be found in Appendix C.

The proof of Theorem 1 provides a direct and compact manner for establishing local exponential stability of the closed-loop system. The region of attraction estimate (16) and the stability estimates (18), (19) may be, however, conservative. For this reason and for obtaining more exact/practical conditions we also provide an alternative proof for establishing the following theorem, in which we provide an exact computation of the region of attraction of the control law as well as exact stability estimates and control gain parametrization (with respect to initial conditions). The latter may be useful in applications, e.g., for performing gain scheduling.

Theorem 2 *Consider the closed-loop system consisting of the dynamics (1)–(3) satisfying Assumption 1 and the control law defined through* (6), (7), *and* (13). *Define the partition $(f(c_1), f(c_2))^2 = R_1 \cup R_2 \cup R_3$ with*

$$R_1 = \left\{(\Pi, f(c)) \in (f(c_1), f(c_2))^2 \mid f(c) - f(c_2) \leq \Pi - f(c) \leq f(c) - f(c_1)\right\} \tag{20}$$

$$R_2 = \left\{(\Pi, f(c)) \in (f(c_1), f(c_2))^2 \mid f(c) - f(c_1) < \Pi - f(c) < e^2(f(c) - f(c_1)) \text{ or } f(c_2) - f(c) < f(c) - \Pi < e^2(f(c_2) - f(c))\right\} \tag{21}$$

$$R_3 = (f(c_1), f(c_2))^2 \backslash (R_1 \cup R_2). \tag{22}$$

Denoting $\Psi_0 = (Y_0, X_0, U_0)$, let us define

$$P_0(\Psi_0) = \frac{Y_0 X_0}{X_0 + L} + \frac{1}{X_0 + L} \int_{-D(0)}^{0} U_0(s) f(U_0(s)) ds, \tag{23}$$

and

$$\xi(\Psi_0, c) = \begin{cases} \dfrac{f(c) - f(c_1)}{P_0(\Psi_0) - f(c)}, & \text{if } P_0(\Psi_0) > f(c) \\ \dfrac{f(c_2) - f(c)}{f(c) - P_0(\Psi_0)}, & \text{if } P_0(\Psi_0) < f(c) \\ +\infty, & \text{if } P_0(\Psi_0) = f(c) \end{cases}. \tag{24}$$

For each $X(0) = X_0 > 0$ and all initial conditions $Y(0) = Y_0 \in (f(c_1), f(c_2))$, $U_0 \in C([-D(0), 0), (c_1, c_2))$, it holds:

[4] Note that utilization of the inverse function (corresponding to the system's nonlinearity) in the nominal feedback law appears in [11] within the context of constructive control design for a class of non-affine, nonlinear systems.

- *if* $(P_0(\Psi_0), f(c)) \in R_1$*, then* (Y, U) *are bounded and exponential regulation is achieved, in particular,*

$$\begin{cases} |Y(t) - f(c)| = \mathrm{e}^{-k\int_{\sigma(0)}^{t} U(s)ds} |Y(\sigma(0)) - f(c)|, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \geq \sigma(0), \\ U(t) \in (c_1, c_2), \quad t \geq 0 \end{cases} \tag{25}$$

if and only if $k \in (0, k_1^\star(\Psi_0, c))$ *with*

$$k_1^\star(\Psi_0, c) = \frac{\xi(\Psi_0, c) + 1}{X_0 + L}; \tag{26}$$

- *if* $(P_0(\Psi_0), f(c)) \in R_2$*, then* (Y, U) *are bounded and exponential regulation is achieved, in particular, relation* (25) *holds, iff* $k \in (0, k_2^\star(\Psi_0, c))$ *with*

$$k_2^\star(\Psi_0, c) = \frac{2 + \ln(\xi(\Psi_0, c))}{X_0 + L}; \tag{27}$$

- *otherwise, if* $(P_0(\Psi_0), f(c)) \in R_3$*, there does not exist* $k > 0$ *such that relation* (25) *is satisfied.*

Proof The proof can be found in Appendix D.

Statements of Theorems 1 and 2 are complementary and consistent with each other. In particular, restricting Y_0 to $(f(c_1), f(c_2))$, which also implies (from (23), (25), (C.12)) that $Y(t) \in (f(c_1), f(c_2)), t \geq 0$ (that is a condition appearing in practice and stated in Theorem 2), is in agreement with the statement of Theorem 1 that requires Y_0 to be sufficiently close to $f(c)$, which eventually imposes $Y_0 \in (f(c_1), f(c_2))$ (see relation (C.11)) and it guarantees (from (C.14) and solving (C.15)) that $Y(t) \in (f(c_1), f(c_2))$, $t \geq 0$, as well (although stability estimate (18) is not tight). The fact that Theorem 1 does not restrict the size of $k > 0$, which guarantees stabilization and feasibility, provided that the initial conditions are sufficiently close to equilibrium, is also consistent with Theorem 2. This can be seen from (24) and (26), (27) (and (C.7)), which imply that the allowable range for k tends to infinity as the initial conditions tend to equilibrium.

5 Simulation Results

We consider the example from [19] in which $f : \left(\frac{1}{4}, \frac{3}{4}\right) \to \left(\frac{1}{4}, \frac{3}{4}\right)$ with $f(U) = \frac{1}{2} - \frac{1}{4}\sin(2\pi U)$ and $f^{-1}(U) = \frac{\pi - \arcsin(2-4U)}{2\pi}$. We choose the desired reference point as $\bar{Y} = f(c) = \frac{3}{5}$ with $\bar{U} = c = 0.566$ and a control gain $k = 1.5$. In Fig. 2 we compare the output, control effort, and delay in the cases of the open-loop system and for the closed-loop system under the proposed predictor-feedback law. One can observe that the predictor-feedback law stabilizes the desired equilibrium faster than the open-loop controller. Note that because the initial conditions for Y and U are at an equilibrium (although not at the desired one), there is a time interval in which Y

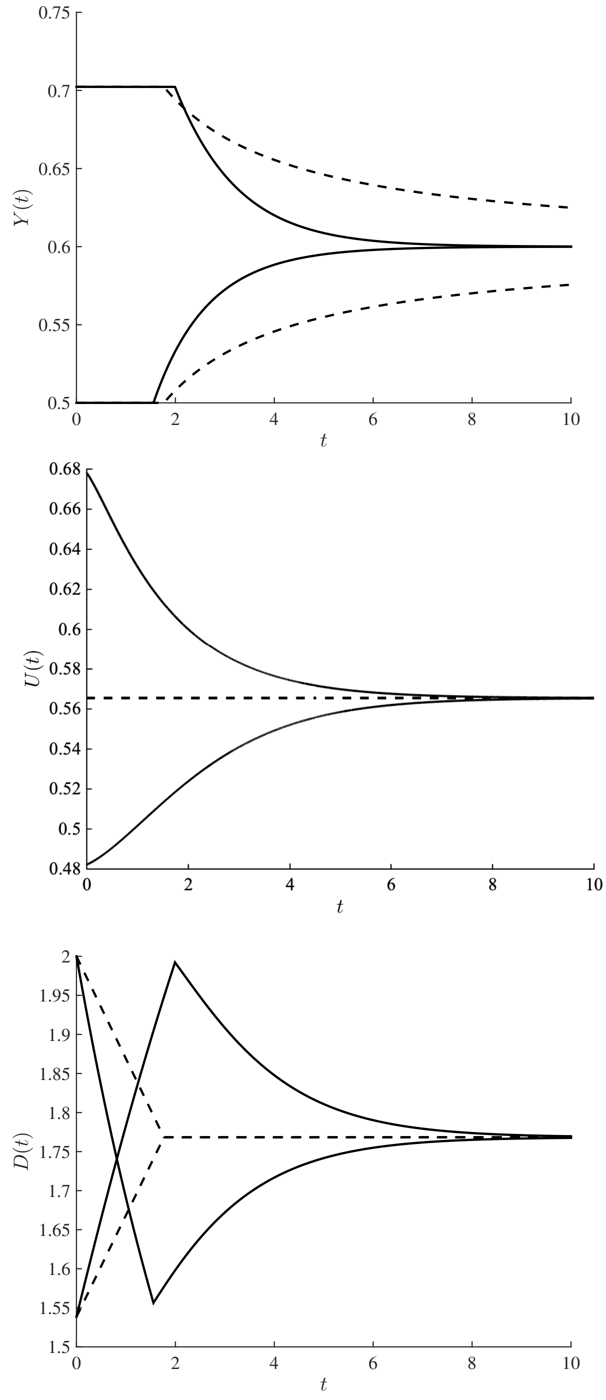

Fig. 2. Solid: Output $Y(t)$ (top), control input $U(t)$ (middle), and delay $D(t)$ (bottom) of system (1)–(3) for two different initial conditions, namely, $U_0 \equiv \frac{1}{2}$, $Y_0 = f\left(\frac{1}{2}\right) = \frac{1}{2}$ and $U_0 \equiv 0.65$, $Y_0 = f(0.65) = 0.7$, with $X_0 = \frac{1}{2}$, under the predictor-feedback control law (6), (7) with (13). Dashed: Output $Y(t)$ (top), control input $U(t)$ (middle), and delay $D(t)$ (bottom) of system (1)–(3) for two different initial conditions, namely, $U_0 \equiv \frac{1}{2}$, $Y_0 = f\left(\frac{1}{2}\right) = \frac{1}{2}$ and $U_0 \equiv 0.65$, $Y_0 = f(0.65) = 0.7$, with $X_0 = \frac{1}{2}$, under the open-loop control law $U(t) = \bar{U}$, for all $t \geq 0$.

remains constant. (This is consistent with equation (C.12); see also Lemma 1.) Note also that, in the open-loop control case, the respective delay functions are exact, linear functions of time. This follows from (3), which implies that for $0 \leq t \leq \sigma(0)$, it holds that $\int_{t-D(t)}^{0} U_0 ds + \int_0^t \bar{U} ds = L$, and thus, $D(t) = \frac{L}{U_0} + t\left(1 - \frac{\bar{U}}{U_0}\right)$; while for $t \geq \sigma(0)$, it holds that $D(t) = \frac{L}{\bar{U}}$, with $\sigma(0) = \frac{L}{\bar{U}}$.

The initial conditions considered result in $(P_0, f(c)) \in R_1$, as depicted in Fig. 3. The maximum allowable control gain values are $k_1^\star(\Psi_0) = 1.66$ and $k_1^\star(\Psi_0) = 2.99$, respectively. Thus, the chosen gain $k = 1.5$ lies in the feasibility interval provided in Theorem 2. Interestingly, simulations performed in the limiting case $k = k_1^\star$ led to numerical infeasibility resulting from U reaching the boundary of interval $(0.25, 0.75)$, which, in turn, implies non-invertibility of f.

In simulations we employ the delay model (1)–(3) with $L = 1$ and implement the predictor state (7). Thus, for control implementation one needs to measure the output Y and the state X. For computation of the finite integral in (7) we employ, at each time step, a simple, left-endpoint rule. The implicit relation (3) for computing the delay time ϕ (and thus, also the delay via equation $D = t - \phi$) is resolved at $t = 0$ by deriving the smallest value for the lower limit ϕ of the integral in (3), for which the integral does not exceed the value L, initializing the lower limit of the integral from a value equal to its upper limit (i.e., equal to the current time). In fact, in the simulation scenario considered, because the initial condition for the actuator state is chosen as constant, $\phi(0)$ is computed explicitly as $\phi(0) = -\frac{L}{U_0}$. Using the value for $\phi(0)$ obtained we consequently compute ϕ according to the ODE $\phi'(t) = \frac{U(t)}{U(\phi(t))}$ with initial condition $\phi(0)$. In general, performance of the design in actual implementations may be affected by model uncertainties and disturbances, as well as by errors due to numerical approximations and digital implementations. Study of robustness to such uncertainties and errors although important, it constitutes a different research topic itself that requires development of new results, which may be guided by the respective results in the constant-delay case; see, for example, [15], [26].

6 Generalization to a Class of Nonlinear Systems

We identify a class of nonlinear systems with input-dependent input delay of hydraulic type for which complete delay compensation, via construction of an exact predictor state, is achievable. The class of systems is described as

$$\dot{X}(t) = f(X(t), U(t - D(t)))\, g(U(t)) \qquad (28)$$

$$L = \int_{t-D(t)}^{t} g(U(s))\, ds, \qquad (29)$$

where $X \in \mathbb{R}^n$ is state, $U \in \mathbb{R}$ is control variable, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is locally Lipschitz vector field with $f(0,0) = 0$, and $L > 0$ is constant. We assume the following.

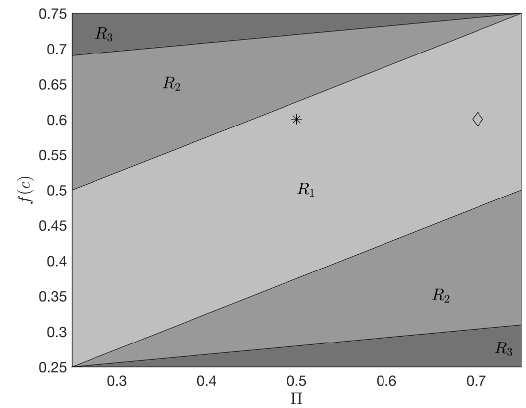

Fig. 3. Regions R_1, R_2, and R_3 in Theorem 2 for the numerical example. Point $(P_0, f(c))$ corresponds to the initial conditions under consideration and is depicted by the asterisk/diamond.

Assumption 2 Function $g : \mathbb{R} \to \mathbb{R}_+$ is locally Lipschitz and satisfies, for some positive constant c_1, the following

$$c_1 \leq g(U), \quad \text{for all } U \in \mathbb{R}. \qquad (30)$$

Assumption 3 System $\dot{X} = f(X, U)$ is forward complete.

Assumption 4 There exists a locally Lipschitz feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}$, with $\kappa(0) = 0$, which renders system $\dot{X} = f(X, \kappa(X))$ globally asymptotically stable.

Without Assumption 2 only a local stability result would be achievable (because the conditions on the delay could be, potentially, guaranteed restricting the size of initial conditions, as in Section 4; see also, for example, [3], [4]). Assumptions 3 and 4 are standard assumptions for predictor feedback-based control design of systems with long, input delays, achieving global stabilization (see, for example, [16]). We note here that Assumptions 3 and 4 are imposed on the system without the scaling term g. As shown within the proof of Theorem 3 (stated below) in Appendix E, this is adequate because, under Assumption 2, one can employ a suitable change of the time variable absorbing g and enabling the proof to be conducted under Assumptions 3 and 4.

The predictor-feedback law is given by

$$U(t) = \kappa(P(t)) \qquad (31)$$

$$P(\theta) = X(t) + \int_{\phi(t)}^{\theta} f(P(s), U(s))\, g(U(s))\, ds, \qquad (32)$$

for all $\phi(t) \leq \theta \leq t$, where $\phi(t) = t - D(t)$. The fact that P in (32) is the predictor state is shown employing change of variables $t = \sigma(\theta) = \phi^{-1}(\theta)$ in (28), where σ is defined via relation $\int_\theta^{\sigma(\theta)} g(U(s))\, ds = L$, $\phi(t) \leq \theta \leq t$. In more detail, from (28) with definition $X(\sigma(\theta)) = P(\theta)$ we get

$$\frac{dP(\theta)}{d\theta} = \frac{d\sigma(\theta)}{d\theta} g(U(\sigma(\theta)))\, f(P(\theta), U(\theta)), \qquad (33)$$

and hence, since $g\left(U\left(\sigma(\theta)\right)\right)\frac{d\sigma(\theta)}{d\theta} = g\left(U(\theta)\right)$, we arrive at (32) through integration. We have the following result.

Theorem 3 *Consider the closed-loop system consisting of the plant (28), (29) and feedback law (31) with (32). Under Assumptions 2–4 there exists a class $\mathcal{KL}$ function β such that for all $X_0 \in \mathbb{R}$ and $U_0 \in C[-D(0), 0]$, with $U_0(0) = \kappa\left(P(0)\right)$, there exists a unique solution with $X(t) \in C^1[0, +\infty)$, $U(t)$ locally Lipschitz on $[0, +\infty)$, and the following holds*

$$\Xi(t) \leq \beta\left(\Xi(0), t\right), \quad t \geq 0, \tag{34}$$

where

$$\Xi(t) = |X(t)| + \sup_{t-D(t)\leq\theta\leq t} |U(\theta)|. \tag{35}$$

Proof The proof can be found in Appendix E.

The proof strategy presented in Appendix E relies on estimates on solutions obtained under Assumptions 3, 4, with the aid of a suitable change of time variables, which is well-defined under Assumption 2. A Lyapunov-based proof is also possible, relying on the Lyapunov characterization of input-to-state stability and forward completeness, which can be established in a similar manner to, for example, [3], [16] (see also [9] for the case of an input delay defined implicitly through an integral of the state), utilizing the fact that the scaling function g, of the vector field f in (28), satisfies (30).

7 Conclusions

We constructed a predictor-feedback law for a second-order, nonlinear non-affine system with input-dependent input delay of hydraulic type arising in control of microfluidic processes under the Zweifach-Fung effect. We proved exponential stability of the reference point in closed loop utilizing estimates on solutions. The simulation results provided confirm the performance improvement of the closed-loop system under the developed design. We further generalized the predictor-feedback design to a class of nonlinear systems.

Although we impose the assumption on invertibility of the nonlinearity due to Zweifach-Fung effect, this is not restrictive, as, in certain applications, the operation region of interest lies in medium flow ratios. To operate over the whole spectrum of flow ratios, where f may not be increasing, one has to remove such an assumption (e.g., by constructing a different nominal feedback law). This is an issue that we currently investigate. As another topic of ongoing research, we aim at addressing the delay-compensating control design problem of a general network of microfluidic processes, featuring various bifurcation points, channels, and reservoirs. Although addressing the control design problem for the case of n channels is expected to be far from a pure replication of the approach for the one-channel case considered here, the key step in enabling to address this general case, is the control design and analysis step made here.

Appendix A

Proposition A.1 *For a delay time $\phi(t) = t - D(t)$ defined implicitly via $\int_{\phi(t)}^{t} g\left(s\right)ds = L$, with $L > 0$ and a continuous function $g : [\phi(0), +\infty) \to \mathbb{R}_+$, assume that relation $c_1 \leq g\left(s\right)$, for some positive constant c_1, is satisfied for all $s \geq \phi(0)$. Then the following holds for all $t \geq 0$*

$$1 - \dot{D}(t) > 0 \quad \textit{and} \quad 0 < D(t) \leq \frac{L}{c_1}. \tag{A.1}$$

In particular, there exists a unique time t^* such that $\phi\left(t^*\right) = 0$, with $t^* \leq \frac{L}{c_1}$. Furthermore, if $g \equiv c_1$, then $t^* = \frac{L}{c_1}$.

Proof Differentiating relation $\int_{\phi(t)}^{t} g\left(s\right)ds = L$ we get $\dot{\phi}(t) = \frac{g(t)}{g(\phi(t))}$. Thus, $\dot{\phi}(t) = 1 - \dot{D}(t) > 0$ is proved. Integrating relation $c_1 \leq g(s)$, we get $c_1 D(t) \leq \int_{\phi(t)}^{t} g\left(s\right)ds = L$, which proves that $D(t) \leq \frac{L}{c_1}$. With a contradiction argument (because g is positive) we show that $\phi(t) < t$, and hence, $D(t) > 0$, $t \geq 0$. Since ϕ is strictly increasing with $\phi(t) \geq t - \frac{L}{c_1}$ and $\phi(0) < 0$, there exists t^* such that $\phi\left(t^*\right) = 0$, with $0 = \phi\left(t^*\right) \geq t^* - \frac{L}{c_1}$. If the delay time is defined via $\int_{\phi(t)}^{t} c_1 ds = L$, it immediately follows that $D(t) = \frac{L}{c_1}$, and hence, $0 = \phi\left(t^*\right) = t^* - \frac{L}{c_1}$.

Appendix B: Proof of Lemma 1

Under the assumption for U_0 and the fact that $U(t) = c$, $t \geq 0$, from Proposition A.1, there exists a unique t_1 such that $\phi\left(t_1\right) = 0$. Using (3) we get $\int_{\phi(t)}^{0} U_0(s)ds = L - ct$. Hence, $t_1 = \frac{L}{c}$, since U_0 is continuous and strictly positive. Thus, for $t \geq \frac{L}{c}$ system (1), (2) evolves as $\dot{Y}(t) = \frac{f(c)-Y(t)}{X(t)}c$, $\dot{X}(t) = c$, which implies $Y(t)X(t) - Y\left(\frac{L}{c}\right)X\left(\frac{L}{c}\right) = cf(c)\left(t - \frac{L}{c}\right)$, $X(t) = ct + X_0$. Hence, we obtain (5) for $t \geq \frac{L}{c}$. For $0 \leq t \leq \frac{L}{c}$ we get from (1), (2) that $\dot{Y}(t) = c\frac{f(U_0(t-D(t)))-Y(t)}{ct+X_0}$, which implies (5) for $0 \leq t \leq \frac{L}{c}$.

Appendix C: Proof of Theorem 1

Feasibility condition satisfaction: [5] The feasibility condition (4) is satisfied for $-D(0) \leq \theta \leq 0$ by the assumption

[5] We study existence and uniqueness of solutions, separately, in the third part of Appendix C, to not distract the reader from the main contribution of the paper, which is the predictor-feedback control design and respective stability analysis. The proof relies on the autonomous, (P, X) ODE system in closed loop (C.21), (C.22), having a locally Lipschitz right-hand side. This enables to conclude existence/uniqueness utilizing known results for ODEs.

on the initial condition for U. In order to guarantee that the feasibility condition is satisfied for $t \geq 0$ we need to establish that the following holds for $t \geq 0$

$$f(c_1) \leq P(t) - k\left(X(t)+L\right)\left(P(t)-f(c)\right) \leq f(c_2), \tag{C.1}$$

which can be satisfied provided that the following holds

$$\left|\tilde{P}(t)\right| \left|1-k\left(X(t)+L\right)\right| < \delta, \quad t \geq 0, \tag{C.2}$$

with $\tilde{P} = P - f(c)$, $\delta = \min\left\{f(c_2)-f(c), f(c)-f(c_1)\right\}$. As long as U satisfies inequality (4), from (13), (6), (12) it follows that the predictor state P satisfies $\dot{\tilde{P}}(t) = -k\tilde{P}(t)\dot{X}(t)$, and thus,

$$\tilde{P}(t) = \tilde{P}(0)\mathrm{e}^{-k(X(t)-X_0)}. \tag{C.3}$$

Furthermore, as long as U satisfies inequality (4), it holds that $c_1 t + X_0 \leq X(t) \leq c_2 t + X_0$. Therefore,

$$\begin{aligned}\left|\tilde{P}(t)\right| \left|1-k\left(X(t)+L\right)\right| \leq &\left|\tilde{P}(0)\right| \mathrm{e}^{-kc_1 t}\left(1+kL\right.\\ &\left.+kX_0+kc_2 t\right).\end{aligned} \tag{C.4}$$

From (7) for $t=0$ it follows using (3) that

$$\begin{aligned}\left|\tilde{P}(0)\right| \leq &\frac{\left|Y_0-f(c)\right| X_0}{X_0+L}\\ &+\frac{\int_{\phi(0)}^{0}\left|f\left(U_0\left(s\right)\right)-f(c)\right| U_0(s)ds}{X_0+L}.\end{aligned} \tag{C.5}$$

Under Assumption 1 (f being Lipschitz) we get from (C.5) using (3) that

$$\begin{aligned}\left|\tilde{P}(0)\right| \leq &\frac{\left|Y_0-f(c)\right| X_0}{X_0+L}\\ &+\frac{LL_1\sup_{-D(0)\leq s\leq 0}\left|U_0\left(s\right)-c\right|}{X_0+L},\end{aligned} \tag{C.6}$$

and thus (since $X_0>0$ by assumption),

$$\left|\tilde{P}(0)\right| \leq \left|Y_0-f(c)\right| + L_1 \sup_{-D(0)\leq s\leq 0}\left|U_0\left(s\right)-c\right|. \tag{C.7}$$

Using (C.4), it follows that (C.2) is satisfied provided that

$$\left|\tilde{P}(0)\right| \left(1+kL+kX_0+kc_2 t\right)\mathrm{e}^{-kc_1 t} < \delta, \quad t \geq 0, \tag{C.8}$$

which is satisfied whenever

$$\left|\tilde{P}(0)\right| < \frac{\delta}{M\left(X_0\right)} \tag{C.9}$$

$$\begin{aligned}M\left(X_0\right) = \max &\left\{1+kL+kX_0,\right.\\ &\left.\frac{c_2}{c_1}\mathrm{e}^{-1+\frac{c_1}{c_2}\left(kL+1+kX_0\right)}\right\}.\end{aligned} \tag{C.10}$$

Using (C.7) we obtain that condition (C.9), and hence, also (C.2), is satisfied whenever (16) holds with

$$\epsilon\left(X_0\right) = \frac{\delta}{\max\left\{1, L_1\right\} M\left(X_0\right)}. \tag{C.11}$$

Derivation of stability estimate for Y: From Proposition A.1, under (4), there exists a unique finite time instant $\sigma(0) \geq 0$, with $\sigma(0) \leq \frac{L}{c_1}$, such that $\phi\left(\sigma(0)\right) = 0$. Hence, for all $0 \leq t \leq \sigma(0)$ we obtain from (1), (2) that $\frac{d(Y(t)X(t))}{dt} = f\left(U_0\left(t-D(t)\right)\right)U(t)$, and hence,

$$\begin{aligned}Y(t)-f(c) = &\frac{\int_0^t\left(f\left(U_0\left(s-D(s)\right)\right)-f(c)\right)U(s)ds}{X(t)}\\ &+\frac{\left(Y_0-f(c)\right)X_0}{X(t)}, \quad 0 \leq t \leq \sigma(0).\end{aligned} \tag{C.12}$$

Under Assumption 1 (f being Lipschitz) and the assumption on U_0 we get from (C.12) that

$$\begin{aligned}\left|Y(t)-f(c)\right| \leq &\frac{L_1\sup_{0\leq s\leq t}\left|U_0\left(s-D(s)\right)-c\right|}{X(t)}\\ &\times\int_0^t U(s)ds + \frac{\left|Y_0-f(c)\right| X_0}{X(t)}.\end{aligned} \tag{C.13}$$

Since $X(t) \geq c_1 t + X_0$ (under (4)), using (2) we obtain from (C.13) that for $0 \leq t \leq \sigma(0)$ it holds that

$$\begin{aligned}\left|Y(t)-f(c)\right| \leq &\left|Y_0-f(c)\right|\\ &+L_1\sup_{-D(0)\leq s\leq 0}\left|U_0\left(s\right)-c\right|.\end{aligned} \tag{C.14}$$

For $t \geq \sigma(0)$, which implies that $t-D(t) \geq 0$, since $X\left(\sigma(t)\right) = X(t)+L$ and $P(t) = Y\left(\sigma(t)\right)$, we obtain from (6) that $\dot{Y}(t) = \frac{f(\kappa(X(t),Y(t)))-Y(t)}{X(t)}\dot{X}(t)$, and hence, from (13) we get that

$$\dot{Y}(t) = -k\left(Y(t)-f(c)\right)\dot{X}(t). \tag{C.15}$$

Explicitly solving (C.15) and using (9) we get that $Y(t) = f(c) + \mathrm{e}^{-k(X(t)-X_0-L)}\left(Y\left(\sigma(0)\right)-f(c)\right)$, and hence,

$$\left|Y(t)-f(c)\right| \leq \mathrm{e}^{-kc_1 t}\mathrm{e}^{kL}\left|Y\left(\sigma(0)\right)-f(c)\right|. \tag{C.16}$$

Using (C.14) and the fact that for $t \leq \sigma(0)$ it holds that $X(t) \leq X_0 + L$ (since X is increasing), we obtain (18).

Derivation of stability estimate for U: Since (C.1) holds and since f^{-1} is Lipschitz (by assumption), it follows from (13), (6) that $\left|U(t)-c\right| \leq L_2\left|\tilde{P}(t)\right|\left|\left(1-k\left(X(t)+L\right)\right)\right|$, $t \geq 0$, and hence, using (C.4) it follows that

$$\begin{aligned}\left|U(t)-c\right| \leq &L_2\left|\tilde{P}(0)\right|\left(1+kL+kX_0+kc_2 t\right)\\ &\times\mathrm{e}^{-kc_1 t}, \quad t \geq 0.\end{aligned} \tag{C.17}$$

Using (C.7) we obtain from (C.17) that

$$|U(t)-c| \leq \max\{1,L_1\} L_2 \Omega_0 (1+kL+kX_0+kc_2 t) \times \mathrm{e}^{-kc_1 t}, \quad t \geq 0. \quad \text{(C.18)}$$

Thus, for $t \geq \sigma(0)$ we obtain from (C.18) that

$$\sup_{t-D(t)\leq\theta\leq t} |U(\theta)-c| \leq \Omega_0 \max\{1,L_1\} L_2 \mathrm{e}^{kL} \mathrm{e}^{-kc_1 t} \times (1+kL+kX_0+kc_2 t), \quad \text{(C.19)}$$

where we used the fact that $D(t) \leq \frac{L}{c_1}$, $t \geq 0$, which follows from (3), (4). Using the fact that $\sup_{t-D(t)\leq\theta\leq t} |U(\theta)-c| \leq \left(\sup_{-D(0)\leq\theta\leq 0} |U_0(\theta)-c| + \sup_{0\leq\theta\leq t} |U(\theta)-c|\right)$

$\times \mathrm{e}^{-kc_1(t-\sigma(0))}$, $0 \leq t \leq \sigma(0)$, we obtain using (C.18) that

$$\sup_{t-D(t)\leq\theta\leq t} |U(\theta)-c| \leq \Omega_0 (L_2+1) \max\{1,L_1\} \mathrm{e}^{kc_1\sigma(0)} \times (1+kL+kX_0+kc_2 t)\, \mathrm{e}^{-kc_1 t}, \quad \text{(C.20)}$$

for $0 \leq t \leq \sigma(0)$. Using (4), (8) it follows that $\sigma(0) \leq \frac{L}{c_1}$, and hence, using (C.19), (C.20) we obtain (19).

Existence, uniqueness, and regularity of solutions: We first note that from (2), (6), (12) it follows that

$$\dot{P}(t) = -k\left(P(t)-f(c)\right)\kappa\left(X(t)+L, P(t)\right) \quad \text{(C.21)}$$
$$\dot{X}(t) = \kappa\left(X(t)+L, P(t)\right), \quad \text{(C.22)}$$

and thus, since the right-hand side of the above ODE in (P,X) is locally Lipschitz in (P,X) we get (with (C.3), (C.7), and $c_1 t + X_0 \leq X(t) \leq c_2 t + X_0$) existence and uniqueness of a solution $(P(t), X(t)) \in C^1[0,+\infty)$. Thus, from (6), (13) it follows from Assumption 1 that $U(t)$ is locally Lipschitz on $[0,+\infty)$. Moreover, since mapping $F(\phi) = \int_0^{\phi} U(s)ds$ satisfies $F'(\phi) = U(\phi)$, we can uniquely define (with (4)) its inverse F^{-1}. As ϕ, for each t, satisfies (3), by the continuity of U on $[\phi(0),+\infty)$, we obtain existence of a unique solution $\phi(t) \in C^1[0,+\infty)$ defined via $\phi(t) = F^{-1}\left(\int_0^t U(s)ds - L\right) = F^{-1}(X(t)-X_0-L)$. Thus, from (C.12) it follows (with (18)) that there exists a unique solution $Y(t) \in C^1[0,\sigma(0))$. Similarly, from (C.15) it follows (with (18)) that there exists a unique solution $Y(t) \in C^1(\sigma(0),+\infty)$. Compatibility of U_0 with the feedback law guarantees that Y is continuously differentiable also at $t = \sigma(0)$.

Appendix D: Proof of Theorem 2

A necessary and sufficient condition for the controller to guarantee boundedness and exponential regulation is that $k > 0$ and $U(t) \in (c_1, c_2)$, $t \geq 0$, hold simultaneously. Under Assumption 1, this holds iff

$$\begin{cases} P(t) - k(X(t)+L)(P(t)-f(c)) \in (f(c_1), f(c_2)), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \geq 0\,. \\ k > 0 \end{cases} \quad \text{(D.1)}$$

Derivation of conditions on (Ψ_0, k, c) for satisfaction of (D.1): As long as $U \in (c_1, c_2)$, the closed-loop system gives $P(t) = f(c) + \mathrm{e}^{-k\int_0^t U(s)ds}(P_0 - f(c))$. Hence, the first equation in condition (D.1) is re-written as

$$f(c) + (1-k(X_0+L)-h(t))(P_0-f(c))e^{-h(t)} \in (f(c_1), f(c_2)), \quad t \geq 0, \quad \text{(D.2)}$$

with $h(t) = k\int_0^t U(s)ds$. For $k > 0$, as long as $U > c_1$, this function spans $\mathbb{R}_+$. Thus, introducing $g_1 : h \in \mathbb{R}_+ \mapsto f(c) + (1-k(X_0+L)-h))(P_0-f(c))e^{-h}$, the first equation in (D.1) is equivalent to $g_1(h) \in (f(c_1), f(c_2))$ for all $h \geq 0$. Let us observe that $\lim_{h\to+\infty} g_1(h) = f(c)$ belongs to $(f(c_1), f(c_2))$ from the definition of c and Assumption 1. Using (23) we get that $P_0 \in (f(c_1), f(c_2))$ for $Y_0 \in (f(c_1), f(c_2))$ and $U_0 \in C([-D(0), 0), (c_1, c_2))$. Thus, $g_1(0) \in (f(c_1), f(c_2))$ if and only if $k < k_1^\star(\Psi_0, c)$, with $k_1^\star$ defined in (26). Furthermore, one can obtain that, if $k \geq \frac{2}{X_0+L} \triangleq \eta(X_0)$, then g_1 is monotonic and, otherwise, g admits an extremum for $h = 2 - k(X_0+L)$, which is

$$g_1(2-k(X_0+L)) = f(c) - (P_0 - f(c))e^{-2+k(X_0+L)}, \quad \text{(D.3)}$$

and belongs to $(f(c_1), f(c_2))$ iff $k < k_2^\star(\Psi_0, c)$, with $k_2^\star$ defined in (27). Thus, stability and feasibility are established iff the following conditions on Ψ_0, k, and c are satisfied

$$\begin{cases} 0 < k < k_1^\star(\Psi_0, c) \\ 0 < k < k_2^\star(\Psi_0, c), \text{ if } k < \eta(X_0) \end{cases}. \quad \text{(D.4)}$$

Restatement as conditions on the gain dependent on (Ψ_0, c): To reformulate and simplify condition (D.4), let us observe that the following properties hold (from (26), (27)):
(P1) $k_1^\star \geq k_2^\star$;
(P2) $k_1^\star \leq \eta \Leftrightarrow \xi \leq 1 \Leftrightarrow k_2^\star \leq \eta$;
(P3) $k_2^\star > 0 \Leftrightarrow \xi > e^{-2}$.

Hence, using (P1) and (P2), condition (D.4) is satisfied iff one of the following two conditions holds

$$\begin{cases} k_2^\star \geq \eta \text{ and } 0 < k < k_1^\star \\ k_1^\star < \eta \text{ and } 0 < k < k_2^\star \end{cases}. \quad \text{(D.5)}$$

Equivalently, (D.4) holds iff one of the following holds

$$\begin{cases} \xi \geq 1 \text{ and } 0 < k < k_1^\star \\ \xi < 1 \text{ and } 0 < k < k_2^\star \end{cases}, \quad \text{(D.6)}$$

which, combining (20), (21), and (24), can be written as

$$\begin{cases} (P_0, f(c)) \in R_1 \text{ and } 0 < k < k_1^\star \\ (P_0, f(c)) \in R_2 \text{ and } 0 < k < k_2^\star \end{cases}. \tag{D.7}$$

The proof is completed noting from (22), (24) that $P_0 \in R_3$ is equivalent to $\xi \le \mathrm{e}^{-2}$, and thus, from (P3), to $k_2^\star \le 0$.

Appendix E: Proof of Theorem 3

Under Assumption 2, using Proposition A.1, we get from (29) that there exists t^* such that $t^* = \sigma(0) \le \frac{L}{c_1}$, where $\sigma(t) = \phi^{-1}(t)$ and $\phi(t) = t - D(t)$. We continue the proof treating separately the cases $0 \le t \le \sigma(0)$ and $t \ge \sigma(0)$.

Derivation of solutions estimate for X for $0 \le t \le \sigma(0)$: Define the change of variables

$$\tau = \int_0^t g(U(s))\, ds = \bar{g}(t), \tag{E.1}$$

which, under Assumption 2, is invertible as $\frac{d\tau}{dt} = g(U(t)) \ge c_1$, for $t \ge 0$, while $\bar{g} : [0, +\infty) \to [0, +\infty)$ is strictly increasing. In particular, since $\int_0^{\sigma(0)} g(U(s))\, ds = \bar{g}(\sigma(0)) = L$, we get from (28) in τ variable that for $0 \le t \le \sigma(0)$ it holds that $\frac{d\bar{X}(\tau)}{d\tau} = f\left(\bar{X}(\tau), \bar{U}(\tau - L)\right)$, $0 \le \tau \le L$, where $\bar{X}(\tau) = X\left(\bar{g}^{-1}(\tau)\right)$ and $\bar{U}(\tau - L) = U\left(\phi\left(\bar{g}^{-1}(\tau)\right)\right)$. Under Assumption 3 and [14] it holds that $\left|\bar{X}(\tau)\right| \le \nu(\tau)\, \psi\left(\left|\bar{X}(0)\right| + \sup_{0 \le s \le \tau} \left|\bar{U}(s - L)\right|\right)$, for a function $\psi \in \mathcal{K}$ and a continuous, positive, monotonically increasing function ν. Hence, there exists $\psi_1 \in \mathcal{K}$ such that $\left|\bar{X}(\tau)\right| \le \psi_1\left(\left|\bar{X}(0)\right| + \sup_{0 \le s \le L} \left|\bar{U}(s - L)\right|\right)$, for $0 \le \tau \le L$, with $\psi_1(s) = \nu(L)\, \psi(s)$. Therefore, since $\sigma(0) = \bar{g}^{-1}(L)$, we arrive at

$$|X(t)| \le \psi_1(\Xi(0)), \quad 0 \le t \le \sigma(0). \tag{E.2}$$

Derivation of solutions estimate for X for $t \ge \sigma(0)$: Since $U(t) = \kappa(P(t)) = \kappa(X(\sigma(t)))$, for all $t \ge 0$, we obtain that the closed-loop system for $t \ge \sigma(0)$ becomes

$$\dot{X}(t) = f(X(t), \kappa(X(t)))\, g(U(t)). \tag{E.3}$$

In τ variable we get $\frac{d\bar{X}(\tau)}{d\tau} = f\left(\bar{X}(\tau), \kappa\left(\bar{X}(\tau)\right)\right)$, $\tau \ge L$. Thus, under Assumption 4 we get $\left|\bar{X}(\tau)\right| \le \beta_1\left(\left|\bar{X}(L)\right|, \tau - L\right)$, $\tau \ge L$, with a $\beta_1 \in \mathcal{KL}$. Therefore,

$$|X(t)| \le \beta_1(|X(\sigma(0))|, \bar{g}(t) - L), \quad t \ge \sigma(0). \tag{E.4}$$

Since $\bar{g}(t) - L = \int_0^t g(U(s))\, ds - \int_{\phi(t)}^t g(U(s))\, ds = \int_0^{\phi(t)} g(U(s))\, ds \ge c_1(t - D(t))$, for all $t \ge \sigma(0)$, we obtain from (E.4) for $t \ge \sigma(0)$

$$|X(t)| \le \beta_1(|X(\sigma(0))|, c_1(t - D(t))). \tag{E.5}$$

Combining (E.5) with (E.2), we can get that $|X(t)| \le \beta_1(\psi_1(\Xi(0)), c_1 \max\{0, \phi(t)\}) + \psi_1(\Xi(0))\, \mathrm{e}^{-\max\{0, \phi(t)\}}$, $t \ge 0$, and hence, as $\phi(t) \ge t - \frac{L}{c_1}$, we arrive at

$$|X(t)| \le \beta_2(\Xi(0), t), \quad t \ge 0, \tag{E.6}$$

where $\beta_2(s, t) = \beta_1\left(\psi_1(s), c_1 \max\left\{0, t - \frac{L}{c_1}\right\}\right) + \psi_1(s)\mathrm{e}^{-\max\left\{0, t - \frac{L}{c_1}\right\}}$ is a class $\mathcal{KL}$ function.

Derivation of solutions estimate for U: Under Assumption 4 (local Lipschitzness of κ with $\kappa(0) = 0$), the fact that $U(t) = \kappa(P(t)) = \kappa(X(\sigma(t)))$, and (E.2), (E.5), there exists a class $\mathcal{K}_\infty$ function ρ such that

$$|U(t)| \le \rho(\beta_1(\psi_1(\Xi(0)), c_1 t)), \quad t \ge 0, \tag{E.7}$$

and hence, for $t \ge \sigma(0)$ it holds that

$$\sup_{t - D(t) \le \theta \le t} |U(\theta)| \le \rho(\beta_1(\psi_1(\Xi(0)), c_1 \phi(t))). \tag{E.8}$$

For $0 \le t \le \sigma(0)$ we have that $\sup_{t - D(t) \le \theta \le t} |U(\theta)| \le \sup_{-D(0) \le \theta \le 0} |U(\theta)| + \sup_{0 \le \theta \le \sigma(0)} |U(\theta)|$, and hence, using (E.7), we get for $0 \le t \le \sigma(0)$ that

$$\sup_{t - D(t) \le \theta \le t} |U(\theta)| \le \Xi(0) + \rho(\beta_1(\psi_1(\Xi(0)), 0)). \tag{E.9}$$

Combining (E.8), (E.9), we can get that

$$\sup_{t - D(t) \le \theta \le t} |U(\theta)| \le \beta_3(\Xi(0), t), \quad t \ge 0, \tag{E.10}$$

where $\beta_3(s, t) = \rho\left(\beta_1\left(\psi_1(s), c_1 \max\left\{0, t - \frac{L}{c_1}\right\}\right)\right) + (s + \rho(\beta_1(\psi_1(s), 0)))\, \mathrm{e}^{-\max\left\{0, t - \frac{L}{c_1}\right\}}$ is of class $\mathcal{KL}$. Combining (E.6), (E.10) we get (34) with $\beta = \beta_2 + \beta_3$.

Existence, uniqueness, and regularity of solutions follow in a similar manner to the respective part in Appendix C, starting from the ODE for P, $\dot{P}(t) = g(\kappa(P(t)))\, f(P(t), \kappa(P(t)))$, which has a locally Lipschitz right-hand side.

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