# Global Exponential Stabilization of Freeway Models*

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***Abstract*— This work is devoted to the construction of feedback laws which guarantee the robust global exponential stability of the uncongested equilibrium point for general discrete-time freeway models. The feedback construction is based on a control Lyapunov function approach and exploits certain important properties of freeway models. The developed feedback laws are tested in simulation. A comparison with existing feedback laws in the literature is demonstrated as well.**

## I. INTRODUCTION

Freeway traffic congestion during peak periods and incidents has become a significant problem for modern societies, which leads to excessive delays, reduced traffic safety, increased fuel consumption and environmental pollution. The main traffic control measures employed (albeit not always in appropriate ways) to tackle traffic congestion, are ramp metering (RM) and variable speed limits (VSL). RM is implemented by use of traffic lights positioned at on-ramps to control the entering traffic flow [24]. VSL are used for speed harmonization, it may be used as a mainstream metering device as well [4]. To achieve their goal, these control measures must be driven by appropriate control strategies. A branch of related research has considered nonlinear optimal control and MPC (Model Predictive Control) as a network-wide freeway traffic control approach, see, e.g. [2, 3, 9, 11]. However, possibly due to the involved control strategy complexity, none of the proposed methods has advanced to a field-operational tool. Another significant branch of freeway traffic control research has considered explicit feedback control approaches to tackle congestion problems. A pioneering development in this direction was the I-type local feedback ramp metering regulator ALINEA [22], which has been used in hundreds of successful field implementations around the world, see, e.g. [23, 25]. ALINEA controls the traffic entering from an on-ramp and targets a critical density in the mainstream merging segment so as to maximize the freeway throughput. Most relevant extensions and modifications of ALINEA in the present context is the extension to a PI-type regulator so as to efficiently address bottlenecks which are located far downstream of the merge area [27]; and the parallel deployment of PI-type regulators to address multiple potential bottlenecks downstream of the metered on-ramp [28]. On the other hand, feedback control approaches for mainstream traffic control by use of VSL have been rather sparse, see [5]; see also [12] for a recent extension to the multiple bottleneck case.

To adequately address the increasing freeway traffic congestion problems, it is essential to investigate, develop and deploy the potentially most efficient methods; and recent control theory advances should be appropriately exploited to this end. In this work, we provide a rigorous methodology for the construction of explicit feedback laws that guarantee the robust global exponential stability of the uncongested equilibrium point for general nonlinear discrete-time freeway models. We focus on discrete-time freeway models which are generalized versions of the known first-order discrete Godunov approximations to the kinematic-wave partial differential equation of the LWR-model (see [21, 26]) with nonlinear ([17]) or piecewise linear (Cell Transmission Model - CTM, [7]) outflow functions. The constructed freeway models allow all possible cases for the relative priorities of the inflows to be taken into account and even allow time-varying (and unknown) priority rules. The construction of the robust global exponential feedback stabilizer is based on the Control Lyapunov Function (CLF) approach (see [13]) as well as on certain important properties of freeway models. The formulae for the Lyapunov function are explicit and can be used in a straightforward way for various purposes. A parameterized family of global exponential feedback stabilizers for the uncongested equilibrium point of freeway models is constructed. The achieved stabilization is robust with respect to all priority rules that can be used for the inflows.

A comparison is made, by means of simulation, with existing feedback laws proposed in the literature and employed in practice. More specifically, we focus on the Random Located Bottleneck (RLB) PI-type regulator which was proposed in [28] and is the most sophisticated of the very few comparable feedback regulators that have been employed in field operations [25]. The simulations, presented in Section IV of the present work, reveal that the performance guaranteed by the implementation of the proposed feedback law is better than the performance induced by the RLB PI regulator.

Due to space limitations all proofs are omitted and can be found in [15].

*Definitions and Notation* Throughout this manuscript, we adopt the following notation and terminology:

* $\Re_+ := [0,+\infty)$. For every set $S$, $S^n = \underbrace{S \times \ldots \times S}_{n\ times}$ for every positive integer $n$. For a set $S \subseteq \Re^n$, $\mathrm{int}(S)$ denotes the interior of $S \subseteq \Re^n$. Let $x \in \Re^n$. By $|x|$ we denote the Euclidean norm of $x$.

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* By $C^0(A;\Omega)$, we denote the class of continuous functions on $A \subseteq \Re^n$, which take values in $\Omega \subseteq \Re^m$. By $C^k(A;\Omega)$, where $k \ge 1$ is an integer, we denote the class of functions on $A \subseteq \Re^n$ with continuous derivatives of order $k$, which take values in $\Omega \subseteq \Re^m$.

Let $S \subseteq \Re^n$, $D \subseteq \Re^l$ be non-empty sets and consider the uncertain, discrete-time, dynamical system

$$x^+ = F(d,x), x \in S, d \in D \tag{1}$$

where $F : D \times S \to S$ is a mapping. Let $x^* \in S$ be an equilibrium point for the dynamical system, i.e., $x^* \in S$ satisfies $x^* = F(d,x^*)$ for all $d \in D$. Notice that $x \in S$ denotes the state of the dynamical system while $d \in D$ denotes a vanishing perturbation. We use the following definitions throughout the paper.

**Definition 1.1:** *We say that $x^* \in S$ is Robustly Globally Exponentially Stable (RGES) for system (1) if there exist constants $M,\sigma > 0$ such that for every $x_0 \in S$ and for every sequence $\{d(t) \in D\}_{t=0}^{\infty}$ the solution $x(t)$ of (1) with initial condition $x(0) = x_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ (i.e., the solution that satisfies $x(t+1) = F(d(t), x(t))$ and $x(0) = x_0$) satisfies the inequality $\left|x(t) - x^*\right| \le M \exp(-\sigma t)\left|x_0 - x^*\right|$ for all $t \ge 0$.*

**Definition 1.2:** *A function $V : S \to \Re_+$ for which there exist constants $K_2 \ge K_1 > 0$, $p > 0$ and $\lambda \in [0,1)$ such that the inequalities $K_1\left|x - x^*\right|^p \le V(x) \le K_2\left|x - x^*\right|^p$ and $V(F(d,x)) \le \lambda V(x)$ for all $(d,x) \in D \times S$, is called a Lyapunov function with exponent $p > 0$ for (1).*

**Remark 1.3:** If a Lyapunov function with exponent $p > 0$ exists for (1) then $x^* \in S$ is RGES. Indeed, if the state space were $\Re^n$ and not $S \subseteq \Re^n$ and if no disturbances were present then we would be able to use Theorem 13.2 on pages 765-766 in [10]. However, since the uncertain dynamical system (1) is defined on $S \subseteq \Re^n$ with disturbances $d \in D$, we cannot use Theorem 13.2 on pages 765-766 in [10]. On the other hand, we can use the inequality $V(F(d,x)) \le \lambda V(x)$ inductively and obtain the estimate $V(x(t)) \le \lambda^t V(x(0))$ for every solution of (1) for every sequence $\{d(t) \in [0,1]^{n-1}\}_{t=0}^{\infty}$ and for every integer $t \ge 0$. The required exponential estimate of the solution is obtained by combining the previous estimate with the inequality $K_1\left|x - x^*\right|^p \le V(x) \le K_2\left|x - x^*\right|^p$.

## II. Freeway Models and Their Properties

We consider a freeway which consists of $n \ge 3$ components or cells; typical cell lengths may be 200-500 m. Each cell may have an external controllable inflow (on-ramp flow), located near the cell's upstream boundary; and an external outflow (off-ramp flow), located near the cell's downstream boundary (Figure 1). The number of vehicles at time $t \ge 0$ in component $i \in \{1,...,n\}$ is denoted by $x_i(t)$. The total outflow and the total inflow of vehicles of the component $i \in \{1,...,n\}$ at time $t \ge 0$ are denoted by $F_{i,out}(t) \ge 0$ and $F_{i,in}(t) \ge 0$, respectively. All densities and flows during a time interval are measured in [veh]. The balance of vehicles for each component $i \in \{1,...,n\}$ gives:

$$x_i(t+1) = x_i(t) - F_{i,out}(t) + F_{i,in}(t)\,, \tag{2}$$

for $i = 1,...,n$ and $t \ge 0$.

Each component of the network has storage capacity $a_i > 0$ (i.e. $x_i \in [0,a_i]$ for each $i = 1,...,n$). Based on (2) and the assumption that the outflows of every cell are constant percentages of the total outflow from the same cell as proposed in [8], we obtain the freeway model:

$$\begin{aligned} x_1^+ &= x_1 - s_2 f_1(x_1) + \min\left(q_1, c_1(a_1 - x_1), u_1\right) \\ &= x_1 - s_2 f_1(x_1) + w_1 u_1 \end{aligned} \tag{3}$$

$$\begin{aligned} x_i^+ &= x_i - s_{i+1} f_i(x_i) + \min\left(q_i, c_i(a_i - x_i), (1-p_{i-1}) f_{i-1}(x_{i-1}) + u_i\right), \\ &= x_i - s_{i+1} f_i(x_i) + s_i (1-p_{i-1}) f_{i-1}(x_{i-1}) + w_i u_i \\ &\quad \text{for } i = 2,...,n-1 \end{aligned} \tag{4}$$

$$\begin{aligned} x_n^+ &= x_n - f_n(x_n) + \min\left(q_n, c_n(a_n - x_n), (1-p_{n-1}) f_{n-1}(x_{n-1}) + u_n\right) \\ &= x_n - f_n(x_n) + s_n (1-p_{n-1}) f_{n-1}(x_{n-1}) + w_n u_n \end{aligned} \tag{5}$$

where $f_i$, denote the attempted outflow from cell $i$ to cell $i+1$, illustrating what in the specialized literature of Traffic Engineering (see, e.g., [17]) is called the demand-part of the fundamental diagram of the $i$-th cell. Moreover, $q_i \in (0,+\infty)$ denotes the maximum flow that the $i$-th cell can receive (or the capacity flow of the $i$-th cell) and $c_i \in (0,1]$ ($i = 1,...,n$) is the jam velocity of the *i*-th cell. The variables $u_i(t) > 0$ denote the attempted external inflow to component $i \in \{1,...,n\}$ from regions out of the freeway and the variables $w_i(t) \in [0,1]$ indicate the percentage of the attempted external inflow to component $i \in \{1,...,n\}$ that becomes actual inflow. The variables $s_i(t) \in [0,1]$, for each $i = 2,...,n$, indicate the percentage of the attempted outflow $f_{i-1}(x_{i-1})$, that becomes actual outflow and they are given by the following formula:

$$\begin{aligned} s_i(t) &= \left(1 - d_i(t)\right)\min\left(1, \max\left(0, \frac{\min\left(q_i, c_i\left(a_i - x_i(t)\right)\right) - u_i(t)}{\left(1 - p_{i-1}\right) f_{i-1}\left(x_{i-1}(t)\right)}\right)\right) \\ &+ d_i(t)\min\left(1, \frac{\min\left(q_i, c_i\left(a_i - x_i(t)\right)\right)}{\left(1 - p_{i-1}\right) f_{i-1}\left(x_{i-1}(t)\right)}\right) \end{aligned} \tag{6}$$

where $d_i(t) \in [0,1]$, $i = 2,...,n$, $t \geq 0$ are time-varying parameters and the constants $p_i$ are the well-known exit rates of the freeway. Since the $n$-th cell is the last downstream cell of the considered freeway, we may assume that $p_n = 1$. We also assume that $p_i < 1$ for $i = 1,...,n-1$, and that all exits to regions out of the network (i.e. all off-ramps, as well as the main exit) can accommodate the respective exit flows. Although the parameters $d_i(t) \in [0,1]$ can be estimated by use of empirical or infrastructure-related (see, e.g. [1]) data, when they are constant or when they are slowly varying, we will treat them as unknown time-varying parameters (disturbances). The reader should notice that by introducing the parameters $d_i(t) \in [0,1]$ (and by allowing them to be time-varying), we have taken into account all possible cases for the relative priorities of the inflows (and we also allow the priority rules to be time-varying); see [6, 14, 16, 19, 20] for freeway models with specific priority rules, which are special cases of our general approach.

Furthermore, notice that $u_i$, $i = 2,...,n$, correspond to external on-ramp flows which may be determined by a ramp metering control strategy. For the very first cell 1, we assume, for convenience, that there is just one inflow, $u_1$.

Taking all the above into account, we can say that the freeway model (3), (4), (5), (6) is an uncertain control system on $S = (0,a_1] \times (0,a_2] \times \ldots \times (0,a_n]$ (i.e., $x = (x_1,...,x_n)' \in S$) with inputs $u = (u_1,...,u_n)' \in (0,+\infty) \times \Re_+^{n-1}$ and disturbances $d = (d_2,...,d_n) \in [0,1]^{n-1}$. Notice also that the uncertainty $d = (d_2,...,d_n) \in [0,1]^{n-1}$ appears in the equations (3), (4) and (5) only when congestion phenomena are present after the first cell, i.e., only when $u_i(t) + (1-p_{i-1}) f_{i-1}(x_{i-1}(t)) > c_i(a_i - x_i(t))$ for certain $i \in \{2,...,n\}$.

We next make the following assumption for the functions $f_i : [0,a_i] \to \Re_+$ ($i = 1,...,n$):

**(H)** *The function* $f_i \in C^0([0,a_i];\Re_+)$ satisfies $0 < f_i(z) \leq z$ for all $z \in (0,a_i]$. There exists $\delta_i \in (0,a_i]$ *such that* $f_i$ *is increasing on* $[0,\delta_i]$ *and non-increasing on* $[\delta_i, a_i]$. *Moreover, there exist constants* $L_i \in (0,1)$, $\tilde{\delta}_i \in (0,\delta_i]$ *such that* $f_i : [0,a_i] \to \Re_+$ *is* $C^1$ *on* $(0,\delta_i)$ *and* $1 - L_i \leq f_i'(z)$ *for all* $s \in (0,\tilde{\delta}_i)$ *and* $f_i'(z) \leq 1$ *for all* $z \in (0,\delta_i)$.

Assumption (H) reflects the basic properties of the so-called "demand function" [17] in the Godunov discretization; whereby $\delta_i$ is the critical density, where $f_i(x_i)$ achieves a maximum value. Note, however, that Assumption (H) includes the possibility of reduced demand flow for overcritical densities (i.e., when $x_i(t) \geq \delta_i$), since $f_i(x_i)$ is allowed to be decreasing for $x_i \in [\delta_i, a_i]$; this could be used to reflect the capacity drop phenomenon as proposed in [18]. Assumption (H), has non-trivial consequences. A list of the most important consequences can be found in [15]. These consequences play a crucial role at the proof of Theorem 3.1, which is the main result of this work.

In conclusion, the model (3)-(6) is a generalized version of the known first-order discrete Godunov approximation to the kinematic-wave partial differential equation of the LWR-model (see [21, 26]) with nonlinear ([17]) or piecewise linear (Cell Transmission Model - CTM, [7]) outflow functions. However, the presented framework can also accommodate recent modifications of the LWR-model as in [18] to reflect the so-called capacity drop phenomenon.

## III. Robust Global Exponential Stabilization of Freeways

Define the vector field $\tilde{F} : [0,1]^{n-1} \times S \times (0,+\infty) \times \Re_+^{n-1} \to S$, for all $x \in S := (0,a_1] \times \ldots (0,a_n]$, $d = (d_2,...,d_n) \in [0,1]^{n-1}$ and $u = (u_1,...,u_n) \in (0,+\infty) \times \Re_+^{n-1}$:

$$\tilde{F}(d,x,u) = (\tilde{F}_1(d,x,u),...,\tilde{F}_n(d,x,u))' \in \Re^n \text{ with}$$

$$\tilde{F}_1(d,x,u) := x_1 - s_2 f_1(x_1) + \min\left(q_1, c_1(a_1 - x_1), u_1\right),$$

$$\tilde{F}_i(d,x,u) = x_i - s_{i+1} f_i(x_i) + $$
$$+ \min\left(q_i, c_i(a_i - x_i), (1-p_{i-1}) f_{i-1}(x_{i-1}) + u_i\right),$$

for $i = 2,...,n-1$, and

$$\tilde{F}_n(d,x,u) = x_n - f_n(x_n) + $$
$$+ \min\left(q_n, c_n(a_n - x_n), (1-p_{n-1}) f_{n-1}(x_{n-1}) + u_n\right)$$

and

$$\begin{aligned} s_i &= (1-d_i)\min\left(1, \max\left(0, \frac{\min\left(q_i, c_i(a_i - x_i)\right) - u_i}{(1-p_{i-1}) f_{i-1}(x_{i-1})}\right)\right) \\ &+ d_i \min\left(1, \frac{\min\left(q_i, c_i(a_i - x_i)\right)}{(1-p_{i-1}) f_{i-1}(x_{i-1})}\right) \end{aligned} \tag{7}$$

for $i = 2,...,n$.

Notice that, using definition (7), the control system (3)-(6) can be written in the following vector form:

$$\begin{aligned} & x^+ = \tilde{F}(d,x,u) \\ & x \in S, d \in D, u \in (0,+\infty) \times \Re_+^{n-1} \end{aligned} \tag{8}$$

Consider the freeway model (8) under Assumption (H). We suppose that there exist $u_1^* > 0$, $u_i^* \geq 0$ ($i = 2,...,n$) and a vector $x^* = (x_1^*,...,x_n^*) \in (0,\tilde{\delta}_1) \times \ldots \times (0,\tilde{\delta}_n)$ with:

$$f_1(x_1^*) = u_1^*,$$

$$f_i(x_i^*) = u_i^* + (1-p_{i-1}) f_{i-1}(x_{i-1}^*) = u_i^* + \sum_{j=1}^{i-1}\left(\prod_{k=j}^{i-1}(1-p_k)\right) u_j^*,$$

for $i = 2,...,n$ and

$$u_1^* < \min\left(q_1, c_1(a_1 - x_1^*)\right),$$

$$u_i^* + (1-p_{i-1}) f_{i-1}(x_{i-1}^*) < \min\left(q_i, c_i(a_i - x_i^*)\right),$$

for $i = 2,...,n$. This is the uncongested equilibrium point

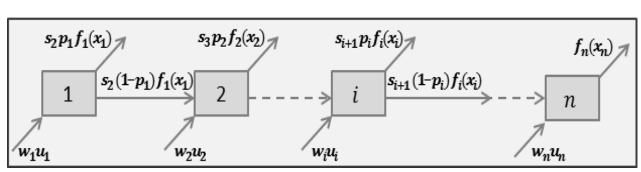


Figure 1. The Freeway Model (schematically).

(UEP) of the freeway model (8). Notice that, Assumption (H) guarantees that an UEP always exists for the freeway model (8) when $u_i^*$ are sufficient small. The UEP is not globally exponentially stable for arbitrary $u_i^*$; indeed for relatively large values of external demands $u_i^*$, there exist other equilibria for model (8) (congested equilibria) for which the cell densities are large and can attract the solution of (8) (see the numerical example (Figure 3) in section IV).

The following result is our main result in feedback design. The result shows that a continuous, robust, global exponential stabilizer exists for every freeway model of the form (8) under assumptions (H). The formula for the feedback law is explicit.

**Theorem 3.1:** *Consider system (8) with* $n \geq 3$ *under Assumption (H). Then, there exist a subset* $R \subseteq \{1,...,n\}$ *of the set of all indices* $i \in \{1,...,n\}$ *with* $u_i^* > 0$, *constants* $\sigma \in (0,1]$, $b_i \in (0, u_i^*)$ *for* $i \in R$ *and a constant* $\tau^* > 0$ *such that for every* $\tau \in (0, \tau^*)$ *the feedback law* $k : S \to \Re_+^n$ *defined by:*

$$k(x) := (k_1(x),...,k_n(x))' \in \Re^n \text{ with}$$

$$k_i(x) := \max\left(u_i^* - \gamma_i\, \Xi(x), b_i\right), \text{ for all } x \in S,\ i \in R \text{ and}$$

$$k_i(x) := u_i^*, \text{ for all } x \in S,\ i \notin R \tag{9}$$

*where* $\gamma_i := \tau^{-1}(u_i^* - b_i)$ *and*

$$\Xi(x) := \sum_{i=1}^{n} \sigma^i \max\left(0, x_i - x_i^*\right), \text{ for all } x \in S \tag{10}$$

*achieves robust global exponential stabilization of the uncongested equilibrium point* $x^*$ *of system (8), i.e.,* $x^*$ *is RGES for the closed-loop system (8) with* $u = k(x)$. *Moreover, for every* $\tau \in (0, \tau^*)$ *there exist constants* $Q, h, \theta, A, K > 0$ *so that the function* $V : S \to \Re_+$ *defined by:*

$$V(x) := \sum_{i=1}^{n} \sigma^i \left| x_i - x_i^* \right| + A\Xi(x) + K \max\left(0, \sum_{i=1}^{n} \beta_i I_i(x) - P(x)\right) \tag{11}$$

*for all* $x \in S$, *where* $I_j(x) := \sum_{i=1}^{j} x_i$, $j = 1,...,n$ *and*

$$P(x) := Q - \theta \min\left(h, \Xi(x)\right) \tag{12}$$

*is a Lyapunov function with exponent 1 for the closed-loop system (8) with* $u = k(x)$.

**Remark 3.2:** The importance of Theorem 3.1 lies on the facts that: (i) it provides a family of robust global exponential stabilizers (parameterized by the parameter $\tau \in (0, \tau^*)$) and an explicit formula for the feedback law (formula (9)); (ii) the achieved stabilization result is robust for all possible (and even time-varying) priority rules for the junctions that may apply at specific freeways; thus, there is no need to know or estimate the applied priority rules; and (iii) it provides an explicit formula for the Lyapunov function of the closed-loop system; this is important, because the knowledge of the CLF can allow the study of the robustness of the closed-loop system to various disturbances (measurement/modeling errors, etc.) as well as the study of the effect of interconnections of freeways (by means of the small gain theorem; see [13]).

## IV. SIMULATIONS

In this section, we consider a freeway model of the form (3)-(6) with $n = 5$ cells. Each cell has the same critical density $\delta_i = 55$ and the same jam density $a_i = 170$. The considered freeway stretch has no intermediate on/off-ramps (i.e. $u_i(t) \equiv u_i^* = 0$ for $i = 2,3,4,5$, $p_i = 0$ for $i = 1,...,4$). Thus, the only control possibility is the inflow $u_1$ of the first cell. We also suppose that the cell flow capacities are $q_i = 25$ for $i = 1,2,3,4$ and $q_5 = 20$, i.e. the last cell has 20% lower flow capacity than the first four cells and is therefore a potential bottleneck for the freeway. Finally, we suppose that each cell is described by a triangular FD (Figure 2). Then, the demand functions are given by the following formula:

$$f_i(z) = \begin{cases} (5/11)z & z \in [0,55] \\ (25/115)(170 - z) & z \in (55, 87.2] \\ 18 & z \in (87.2, 170] \end{cases} \quad (i = 1,...,4)$$

$$f_5(z) = \begin{cases} (4/11)z & z \in [0,55] \\ (20/115)(170 - z) & z \in (55, 72.25] \\ 17 & z \in (72.25, 170] \end{cases}$$

Assumption (H) holds with $\delta_i = \tilde{\delta}_i = 55$ ($i = 1,...,5$), $L_i = 6/11$ ($i = 1,...,4$), $L_5 = 7/11$. The uncongested equilibrium point $x_i^* = 11u_1^*/5$ ($i = 1,...,4$), $x_5^* = 11u_1^*/4$ exists for $u_1^* < 20$. Simulations showed that the open-loop system converges to an UEP for main inflow $u_1^* < 17$. For higher values of the main inflow, the UEP is not globally exponentially stable due to the existence of additional (congested) equilibria. Therefore, if the objective is the operation of the freeway with large flows, then a control strategy will be needed.

We are in a position to achieve global exponential stabilization of the UEP for the above model by using Theorem 3.1. Indeed, Theorem 3.1 guarantees that for every $\sigma \in (0,1]$ there exist a constant $b_1 \in (0, u_1^*)$ and a constant

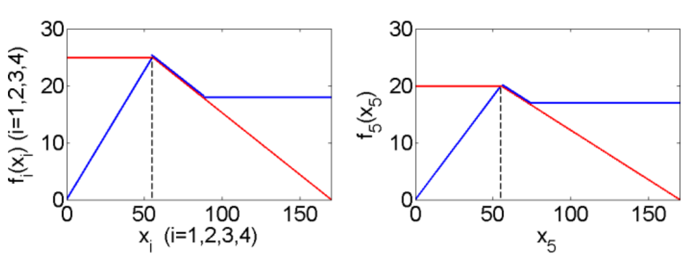

Figure 2 Fundamental diagram of every cell.

$\gamma^* > 0$ such that for every $\gamma > \gamma^*$ the feedback law $k:(0,10]^5 \to \Re_+$ defined by:

$$u_1 = \max\left( u_1^* - \gamma \sum_{i=1}^{5} \sigma^i \max\left(0, x_i - x_i^*\right), b_1 \right) \tag{13}$$

achieves robust global exponential stabilization of the UEP $x^* = (x_1^*,...,x_5^*) \in (0,55)\times\ldots\times(0,55)$ of system.

We selected $u_1^* = 19.99$, which is very close to 20, the capacity flow of cell 5. The corresponding equilibrium values are $x_i^* = 43.978$ for $i = 1,...,4$ and $x_5^* = 54.9725$; the value of the constant $b_1 \in (0, u_1^*)$ was chosen to be 0.2. Various values of the constants $\sigma \in (0,1]$ and $\gamma > 0$ were tested by performing a simulation study with respect to various initial conditions. Low values for $\sigma$ require large values for $\gamma$ in order to have global exponential stability for the closed-loop system. Moreover, in order to evaluate the performance of the controller, we used as a performance criterion, the total number of Vehicles Exiting the Freeway (*VEF*) on the interval $[0,T]$, i.e.,

$$VEF_T = \sum_{t=0}^{T} f_5(x_5(t)) \tag{14}$$

Notice that the freeway performs best (and total delays are minimised) if *VEF* is maximized; the maximum theoretical value for *VEF* is $20(T+1)$, which is achieved if cell 5 is operating at capacity 20 at all times (note that the maximum theoretical value of *VEF* for $T = 200$ is 4020).

The responses of the densities of every cell for the closed-loop system with the proposed feedback regulator (13) with $\sigma = 0.7$, $\gamma = 0.6$ and initial condition $x_0 = (60,57,58,6,62)$ are shown in Figure 4(a). For this case we had $VEF_{200} = 3979.8$. The feedback regulator is seen to respond very satisfactorily in this test and achieves an accordingly high performance. All following tests of the proposed regulator (13) were conducted with the same values $\sigma = 0.7$ and $\gamma = 0.6$.

A comparison of the proposed feedback regulator (13) was made with the Random Located Bottleneck (RLB) PI regulator, which was proposed in [28] and is one of the very few comparable feedback regulators that has been employed in field operations [25]. Essentially, the RLB PI regulator for system reflects the parallel operation of five bounded PI-type regulators, one for each cell (see [15] for a detailed representation of the exact equations and the selection of the parameters for RLB PI).

When applied to the same initial condition $x_0 = (60,57,58,6,62)$, the RLB PI regulator (Figure 4(b)) led to slower convergence compared with the proposed regulator (13). This is also reflected to the computed value of $VEF_{200} = 3785.9$ for the RLB PI regulator. In general, conducting a simulation study with various levels of initial conditions, the proposed regulator (13) exhibited faster performance than the RLB PI regulator. For example, Figure 5 depicts the evolution of the Euclidean norm $\left| x(t) - x^* \right|$ for the closed-loop system with the proposed feedback regulator (13) (blue curve) and with the RLB PI regulator (red curve) and initial condition $(170,170,...,170)$, reflecting a fully congested original state. The computed values of $VEF_{200}$ for this case are: $VEF_{200} = 3845.2$ for the proposed feedback regulator (13); and $VEF_{200} = 3007.8$ for the RLB PI regulator.

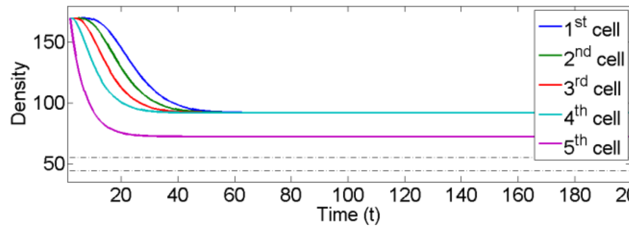


Figure 3. Open-loop system convergence (dashed lines correspond to the uncongested equilibrium point for the inflow $u_1^* = 19.99$ and initial condition (170, 170, 170, 170, 170).

## V. Conclusions

This work provided a rigorous methodology for the construction of a parameterized family of explicit feedback laws that guarantee the robust global exponential stability of the uncongested equilibrium point for general nonlinear and uncertain discrete-time freeway models. The construction of the global exponential feedback stabilizer was based on the CLF approach as well as on certain important properties of freeway models. We also compared by means of simulations, the performance of the closed-loop system under the effect of the proposed feedback law and under the effect of the Random Located Bottleneck (RLB) PI regulator [28]. It was found that the performance guaranteed by the implementation of the proposed feedback law was better than that of the RLB PI regulator.

Future research will address the robustness issues in a rigorous way: the knowledge of a Lyapunov function for the closed-loop system can be exploited to this purpose and

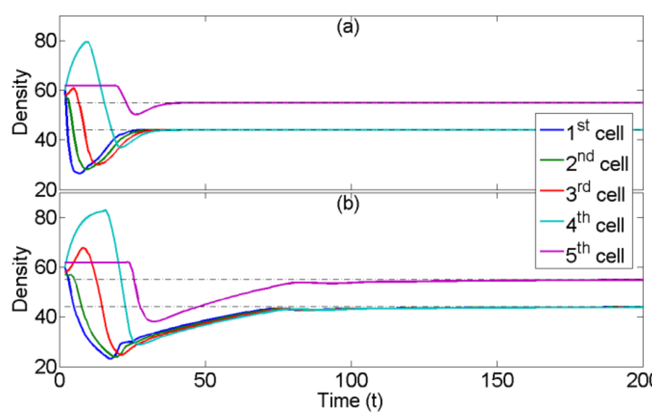


Figure 4. The responses of the densities of every cell for the close loop system with initial condition (60,57,58,60,62) using (a) the proposed feedback regulator; and (b) the RLB PI regulator.

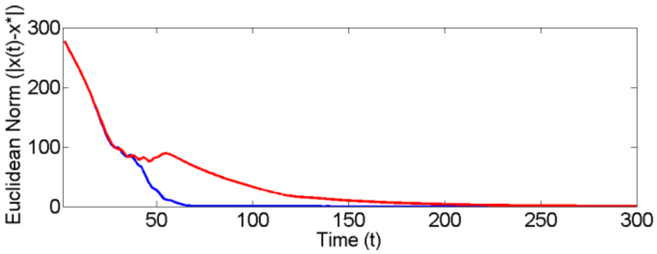


Figure 5. The evolution of the Euclidean norm for the closed-loop system and initial condition (170,170,170,170,170) for two cases: for the proposed feedback regulator (blue curve); and for the RLB PI regulator (red curve).

explicit formulas for the gains of various inputs (measurement or modelling errors) can be derived. Also, the estimation of the gains of various inputs can allow the study and control of interconnected freeways (traffic networks). Finally, the present approach does not consider the impact of inflow control on upstream traffic flow conditions (e.g. queue forming at on-ramps); future extensions will address these issues appropriately.

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