Compensation of Transport Actuator Dynamics with Input-Dependent Moving Controlled Boundary

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***Abstract*— We introduce and solve the stabilization problem of a transport PDE/nonlinear ODE cascade, in which the PDE state evolves on a domain whose length depends on the boundary values of the PDE state itself. In particular, we develop a predictor-feedback control design, which compensates such transport PDE dynamics. We prove local asymptotic stability of the closed-loop system in the C^1 norm of the PDE state employing a Lyapunov-like argument and introducing a backstepping transformation. We also highlight the relation of the PDE-ODE cascade to a nonlinear system with input delay that depends on past input values and present the predictor-feedback control design for this representation as well.**

I. INTRODUCTION

Nonlinear systems with input delays that depend on the input itself can describe the dynamics of numerous physical processes. Among several other applications, such systems may model the dynamics of automotive engines [16], [20], batch processes [8], [9], blending processes [14], water heating processes [32], production systems [17], chemical processes [19], [34], crushing mills [30], solar collectors [31], cooling systems [18] (where input-dependent delays appear due to the time required for the coolant to reach the consumers), and of vehicular traffic flow [21]. For this reason, it is of significant importance to develop control design methodologies for nonlinear systems with input-dependent input delays.

Prediction-based techniques have been successful in solving the stabilization problem of nonlinear systems with input delays that vary with time. In particular, prediction-based techniques are developed for the stabilization of systems with time-varying delays [3], [10], [25], [27], nonlinear systems with state-dependent delays [4], [5], [6], [7], [13], wave PDE/nonlinear ODE cascades with state-dependent moving boundaries [11], [12], and of nonlinear systems with input-dependent delays [8], [9], [17]. However, the problem of stabilization of a transport PDE/nonlinear ODE cascade in which the PDE state evolves on a domain whose length depends on the boundary values of the PDE state itself has never been addressed.

In this paper, we consider the stabilization problem of nonlinear systems with actuator dynamics governed by a transport PDE that evolves on a domain whose length depends on the boundary values of the PDE state itself. We develop a predictor-feedback control design methodology for the compensation of this type of actuator dynamics. The closed-loop system, under the predictor-feedback control law, it is shown to be locally asymptotically stable, in the C^1 norm of the PDE state, via the employment of a Lyapunov-like argument and the introduction of a backstepping transformation. Our stability result is local due to an inherent limitation of the class of transport PDEs under consideration, which ensures the well-posedness of the given transport PDE. More specifically, this restriction guarantees that, in an equivalent formulation of the transport PDE that employs a constant PDE domain and a transport speed that depends on the boundary values of the PDE state as well as its first-order spatial derivative, the transport speed remains always strictly positive as well as uniformly bounded from above and below by finite constants.

Furthermore, we demonstrate that a special case of the considered transport PDE/nonlinear ODE cascade may be viewed as a nonlinear system with an input delay that is defined implicitly through a nonlinear equation, which involves the input value at a time that depends on the delay itself. This class of systems is different than the classes of systems considered in [9] and [17], in which, the input delays are defined implicitly via an integral equation that involves past input values. Note that the latter form of input delay is the result of the explicit dependency of the transport speed (rather than of the controlled boundary) on the boundary values of the PDE state.

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [24]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. For scalar functions $u \in L^\infty[0,D(t)]$ or $v \in L^\infty[0,1]$ we denote by $\|u(t)\|_\infty$ or $\|v(t)\|_\infty$ their respective supremum norms i.e., $\|u(t)\|_\infty = \sup_{x\in[0,D(t)]}|u(x,t)|$ or $\|v(t)\|_\infty = \sup_{z\in[0,1]}|v(z,t)|$. For scalar functions $u_x \in L^\infty[0,D(t)]$ or $v_z \in L^\infty[0,1]$ we denote by $\|u_x(t)\|_\infty$ or $\|v_z(t)\|_\infty$ their respective supremum norms i.e., $\|u_x(t)\|_\infty = \sup_{x\in[0,D(t)]}|u_x(x,t)|$ or $\|v_z(t)\|_\infty = \sup_{z\in[0,1]}|v_z(z,t)|$. For vector valued functions $p \in L^\infty[0,D(t)]$ or $p_v \in L^\infty[0,1]$ we denote by $\|p(t)\|_\infty$ or $\|p_v(t)\|_\infty$ their respective supremum norms, i.e., $\|p(t)\|_\infty = \sup_{x\in[0,D(t)]}\sqrt{p_1(x,t)^2+\ldots+p_n(x,t)^2}$ or $\|p_v(t)\|_\infty = \sup_{z\in[0,1]}\sqrt{p_1(z,t)^2+\ldots+p_n(z,t)^2}$. For vector valued functions $p_x \in L^\infty[0,D(t)]$ or $p_{v_z} \in L^\infty[0,1]$ we denote by $\|p_x(t)\|_\infty$ or $\|p_{v_z}(t)\|_\infty$ their respective supremum norms, i.e., $\|p_x(t)\|_\infty = \sup_{x\in[0,D(t)]}\sqrt{p_{1_x}(x,t)^2+\ldots+p_{n_x}(x,t)^2}$ or

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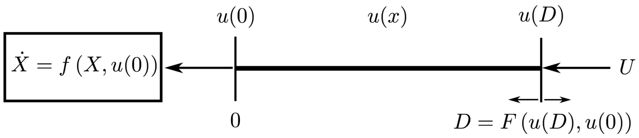

Fig. 1. Nonlinear system with actuator dynamics governed by a transport PDE, which evolves on a varying domain whose length depends on the boundary values of the PDE state itself.

$\|p_{v_z}(t)\|_\infty = \sup_{z\in[0,1]} \sqrt{p_{v1_z}(z,t)^2 + \ldots + p_{vn_z}(z,t)^2}$. We denote by $C^j(A;E)$ the space of functions that take values in E and have continuous derivatives of order j on A.

II. Problem Formulation and Predictor-Feedback Control Design

We consider the following system (see Fig. 1)

$$\dot{X}(t) = f(X(t), u(0,t)) \tag{1}$$

$$u_t(x,t) = u_x(x,t) \tag{2}$$

$$u(D(t), t) = U(t), \tag{3}$$

where $X \in \mathbb{R}^n$ is the ODE state, $t \geq 0$ is time, $x \in [0, D(t)]$ is spatial variable, U is a scalar control input, u is the PDE state of the actuator dynamics, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuously differentiable vector field that satisfies $f(0,0) = 0$, and D is a moving boundary that is defined as

$$D(t) = F(u(D(t),t), u(0,t)). \tag{4}$$

The following assumptions are imposed on system (1)–(4).

Assumption 1: Function $F : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable and satisfies

$$F(u_1, u_2) > 0, \quad \text{for all } (u_1, u_2) \in \mathbb{R}^2. \tag{5}$$

Assumption 2: System $\dot{X} = f(X, \omega)$ is strongly forward complete with respect to ω.

Assumption 3: There exists a twice continuously differentiable feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}$, with $\kappa(0) = 0$, which renders system $\dot{X} = f(X, \kappa(X) + \omega)$ input-to-state stable with respect to ω.

Assumption 1 is a mild assumption on the moving boundary function F, which ensures that the transport equation (2), (3) is meaningful. Assumption 2 (see, e.g., [1]) guarantees that for every initial condition and every locally bounded input signal, the corresponding solution of (1) is defined for all $t \geq 0$. Hence, it implies that the state X of system (1) doesn't escape to infinity before the control signal U reaches it, no matter the size of the delay (see, e.g., [5], [25], [26]). Assumption 3 (see, e.g., [33]) guarantees the existence of a nominal feedback law that renders system (1) input-to-state stable in the absence of the transport actuator dynamics (i.e., in the absence of the input delay). This assumption is a standard ingredient of the predictor-feedback control design methodology (see, e.g., [5], [25], [26]).

The predictor-feedback control law for system (1)–(3) is given by

$$U(t) = \kappa(p(D(t), t)), \tag{6}$$

where for all $x \in [0, D(t)]$ and $t \geq 0$

$$p(x,t) = X(t) + \int_0^x f(p(y,t), u(y,t))\, dy. \tag{7}$$

For the implementation of the predictor-feedback law (6), (7) it is required that the ODE state $X(t)$ and the PDE state $u(x,t)$, $x \in [0, D(t)]$, are measured for all $t \geq 0$. Note that the position of the moving boundary $D(t)$, for all $t \geq 0$, can be computed at each time instant t employing the right-hand side of expression (4) and the boundary measurements of the PDE state, unless it is directly measured. It is worth mentioning here that the implementation problem of predictor-feedback control laws is tackled in several works, such as, for example, [22], [23], [28], [35].

For the subsequent analysis it turns out that it is useful to transform the PDE (2), (3), which evolves on a varying domain, to a PDE that evolves on a constant domain. Defining

$$x = D(t)\, z \tag{8}$$

$$v(z,t) = u(D(t)\, z, t), \tag{9}$$

we re-write (1)–(3) as

$$\dot{X}(t) = f(X(t), v(0,t)) \tag{10}$$

$$v_t(z,t) = \frac{1 + z \frac{\nabla F(v(1,t), v(0,t)) \frac{(v_z(1,t), v_z(0,t))^{\mathrm{T}}}{F(v(1,t),v(0,t))}}{1 - F_{u_1}(v(1,t), v(0,t)) \frac{v_z(1,t)}{F(v(1,t),v(0,t))}}}{F(v(1,t), v(0,t))} v_z(z,t), \quad z \in [0,1] \tag{11}$$

$$v(1,t) = U(t). \tag{12}$$

In order to guarantee the well-posedness of the transport PDE (11), (12) the transport speed must be strictly positive as well as uniformly bounded from above and below. Since the transport speed depends on the PDE state itself the following conditions on the closed-loop solutions and the initial conditions it is needed to be satisfied for all $t \geq 0$

$$1 - \epsilon_1 < \frac{v_z(1,t) F_{u_1}(v(1,t), v(0,t))}{F(v(1,t), v(0,t))} < 1 - \epsilon_2 \tag{13}$$

$$\epsilon_3 - 1 < \frac{v_z(0,t) F_{u_2}(v(1,t), v(0,t))}{F(v(1,t), v(0,t))} < \epsilon_4 - 1 \tag{14}$$

$$\epsilon_5 < F(v(1,t), v(0,t)) < \epsilon_6, \tag{15}$$

for some positive constants ϵ_1, ϵ_2, ϵ_3, ϵ_4, ϵ_5, and ϵ_6.

III. Stability Analysis

Theorem 1: Consider the closed-loop system consisting of the plant (1)–(4) and the control law (6), (7). Under Assumptions 1, 2, and 3, there exist a positive constant δ_u and a class $\mathcal{KL}$ function β_u such that for all initial conditions $X(0) \in \mathbb{R}^n$ and $u(\cdot, 0) \in C^1[0, D(0)]$ which satisfy

$$|X(0)| + \|u(0)\|_\infty + \|u_x(0)\|_\infty < \delta_u, \tag{16}$$

as well as the compatibility conditions

$$u(D(0), 0) = \kappa(p(D(0), 0)) \tag{17}$$

$$u_x(D(0), 0) = \frac{\partial \kappa(p(D(0), 0))}{\partial p} \times f(p(D(0), 0), u(D(0), 0)), \tag{18}$$

the following holds

$$\begin{aligned} \Omega(t) &\leq \beta_u\left(\Omega(0), t\right), \quad \text{for all } t \geq 0 && (19)\\ \Omega(t) &= |X(t)| + \|u(t)\|_\infty + \|u_x(t)\|_\infty. && (20) \end{aligned}$$

The proof of Theorem 1 is based on the following lemmas. Sketches of the proofs of these lemmas are provided in Appendix A. The first two lemmas introduce a backstepping transformation, together with its inverse, which transforms the original closed-loop system (10)–(12), (6), (7) into a "target system" whose stability properties are established.

Lemma 1: The control law (6), (7) is expressed in terms of the v variable as

$$U(t) = \kappa\left(p_v\left(1, t\right)\right), \tag{21}$$

where, for all $z \in [0, 1]$,

$$\begin{aligned} p_v(z,t) &= X(t) + F\left(v(1,t), v(0,t)\right)\\ &\quad \times \int_0^z f\left(p_v(s,t), v(s,t)\right) ds. && (22) \end{aligned}$$

Lemma 2: Consider the backstepping transformation

$$w(z,t) = v(z,t) - \kappa\left(p_v(z,t)\right), \tag{23}$$

together with its inverse

$$v(z,t) = w(z,t) + \kappa\left(\pi_v(z,t)\right), \tag{24}$$

where

$$\begin{aligned} \pi_v(z,t) =& X(t) + F\left(\kappa\left(\pi_v(1,t)\right), w(0,t) + \kappa\left(X(t)\right)\right)\\ & \int_0^z f\left(\pi_v(y,t), w(y,t) + \kappa\left(\pi_v(y,t)\right)\right) dy. && (25) \end{aligned}$$

Transformation (23) together with the control law (21), (22) transform system (10)–(12) to the following target system

$$\begin{aligned} \dot{X}(t) &= f\left(X(t), \kappa\left(X(t)\right) + w(0,t)\right) && (26)\\ w_t(z,t) &= \xi(z,t) w_z(z,t) && (27)\\ w(1,t) &= 0, && (28) \end{aligned}$$

where

$$\xi(z,t) = \frac{1 + z\frac{\nabla F(v(1,t),v(0,t))\frac{(v_z(1,t),v_z(0,t))^{\mathrm{T}}}{F(v(1,t),v(0,t))}}{1-F_{u_1}(v(1,t),v(0,t))\frac{v_z(1,t)}{F(v(1,t),v(0,t))}}}{F\left(v(1,t), v(0,t)\right)}, \tag{29}$$

with $v(0,t)$ expressed in terms of $w(0,t)$ and $X(t)$ using (24), (25) for $z = 0$ and $v(1,t)$ expressed in terms of $\pi_v(1,t)$ using (24), (25) for $z = 1$ and (28).

The next lemma shows that the target system (26)–(28) is asymptotically stable.

Lemma 3: There exists a class $\mathcal{KL}$ function β_w such that for all solutions of the system satisfying (13)–(15) the following holds

$$\begin{aligned} \Omega_w(t) &\leq \beta_w\left(\Omega_w(0), t\right), \quad \text{for all } t \geq 0 && (30)\\ \Omega_w(t) &= |X(t)| + \|w(t)\|_\infty + \|w_z(t)\|_\infty. && (31) \end{aligned}$$

Lemmas 4–6, establish the norm equivalency between the original system (10)–(12) and the target system (26)–(28).

Lemma 4: There exists a class $\mathcal{K}_\infty$ function ρ_1 such that the following holds for all $t \geq 0$

$$\|p_v(t)\|_\infty + \|p_{v_z}(t)\|_\infty \leq \rho_1\left(|X(t)| + \|v(t)\|_\infty\right). \tag{32}$$

Lemma 5: There exists a class $\mathcal{K}_\infty$ function ρ_2 such that the following holds for all $t \geq 0$

$$\|\pi_v(t)\|_\infty + \|\pi_{v_z}(t)\|_\infty \leq \rho_2\left(|X(t)| + \|w(t)\|_\infty\right). \tag{33}$$

Lemma 6: There exist class $\mathcal{K}_\infty$ functions ρ_3 and ρ_4 such that the following hold for all $t \geq 0$

$$\begin{aligned} \Omega_w(t) &\leq \rho_3\left(\Omega_v(t)\right) && (34)\\ \Omega_v(t) &\leq \rho_4\left(\Omega_w(t)\right). && (35) \end{aligned}$$

where Ω_w is defined in (31) and

$$\Omega_v(t) = |X(t)| + \|v(t)\|_\infty + \|v_z(t)\|_\infty. \tag{36}$$

The next two lemmas show the equivalency of the C^1 norm of state $(X, u(x))$, $x \in [0, D(t)]$, to the state $(X, v(z))$, $z \in [0, 1]$.

Lemma 7: There exists a class $\mathcal{K}_\infty$ function ρ_5 such that the following holds for all $t \geq 0$

$$\Omega_v(t) \leq \rho_5\left(\Omega(t)\right), \tag{37}$$

where Ω_v and Ω are defined in (36) and (20), respectively.

Lemma 8: There exists a class $\mathcal{K}_\infty$ function ρ_6 such that for all solutions of the system satisfying (15) the following holds for all $t \geq 0$

$$\Omega(t) \leq \rho_6\left(\Omega_v\right), \tag{38}$$

where Ω and Ω_v are defined in (20) and (36), respectively.

In the last three lemmas an estimate of the region of attraction of the predictor-feedback control law (6), (7) is provided.

Lemma 9: There exists a positive constant δ_1 such that for all solutions of the system that satisfy

$$|X(t)| + \|v(t)\|_\infty + \|v_z(t)\|_\infty < \delta_1, \quad \text{for all } t \geq 0, \tag{39}$$

they also satisfy (13)–(15).

Lemma 10: There exists a positive constant δ_v such that for all initial conditions of the closed-loop system (10)–(12), (21), (22) that satisfy

$$|X(0)| + \|v(0)\|_\infty + \|v_z(0)\|_\infty < \delta_v, \tag{40}$$

the solutions of the system satisfy (39), and hence, satisfy (13)–(15).

Lemma 11: There exists a positive constant δ_u such that for all initial conditions of the closed-loop system (1)–(3), (6), (7) that satisfy (16) they also satisfy (40).

Proof of Theorem 1: Theorem 1 is proved combining Lemmas 3, 6, 7, and 8 with

$$\beta_u(s,t) = \rho_6\left(\rho_4\left(\beta_w\left(\rho_3\left(\rho_5(s)\right), t\right)\right)\right). \tag{41}$$

IV. Relation to a System with Delayed-Input-Dependent Input Delay

Consider the following system

$$\dot{X}(t) = f\left(X(t), U\left(\phi(t)\right)\right), \tag{42}$$

where the delayed time ϕ is defined implicitly through relation

$$\phi(t) = t - F\left(U\left(\phi(t)\right)\right), \tag{43}$$

and $F:\mathbb{R}\rightarrow\mathbb{R}_{+}$ is a delay. In fact, system (42), (43) is an equivalent delay system representation of system (1)–(4), where for simplicity of presentation we consider only the dependency of F from $U(\phi)$. To see this note that the solution to (2), (3) is given for all $x\in[0,D(t)]$ and $t\geq 0$ by

$$u(x,t)=U\left(\phi(t+x)\right), \tag{44}$$

where the prediction time σ, i.e., the inverse function of ϕ, is given by

$$\begin{aligned}\sigma(t) &= t+D(t)\\ &= t+F\left(U(t)\right). \end{aligned}\tag{45}$$

The predictor-feedback control law for system (42) with an input delay defined via (43) is given by

$$U(t)=\kappa\left(P(t)\right), \tag{46}$$

where the predictor P is given for all $t\geq 0$ by

$$\begin{aligned}P(\theta)=&X(t)+\int_{\phi(t)}^{\theta}\left(1+F'\left(U(s)\right)\dot{U}(s)\right)\\ &\times f\left(P(s),U(s)\right)ds,\quad \text{for all } \phi(t)\leq\theta\leq t. \end{aligned}\tag{47}$$

The predictor-feedback control law is implementable since it depends on the history of $U(s)$ and $\dot{U}(s)$, over the window $\phi(t)\leq s\leq t$, as well as on the ODE state $X(t)$, which are assumed to be measured for all $t\geq 0$. Moreover, the implementation of the predictor-feedback design requires the computation at each time step of the delayed time ϕ. This can either be performed by numerically solving relation (43) or by employing the following integral equation

$$\begin{aligned}\phi\left(\theta\right) &= t-\int_{\theta}^{\sigma(t)}\frac{ds}{1+F'\left(U\left(\phi(s)\right)\right)U'\left(\phi(s)\right)},\\ &\text{for all } t\leq\theta\leq\sigma(t), \end{aligned}\tag{48}$$

where σ is defined in (45). Note that the key condition for the well-posedness of system (42), (43) and the predictor-feedback control design (46), (47) is reflected by the need to keep the denominator in (48) positive.

V. CONCLUSIONS

We introduced a predictor-feedback control design methodology for nonlinear systems with transport actuator dynamics, which evolve on a varying domain whose length depends on the boundary values of the transport PDE sate. We proved local asymptotic stability of the closed-loop system under predictor-feedback in the C^1 norm of the actuator state, employing a Lyapunov-like argument and a novel backstepping transformation. The relation of the PDE-ODE cascade to a nonlinear system with input delay that depends on past input values was also highlighted and the predictor-feedback control design for this representation was also presented.

APPENDIX A

Proof of Lemma 1

Performing in the integral in (7) the change of variables $y=D(t)s$ and using the fact that F is positive (Assumption 1), with definitions (8), (9), and

$$p_v(z,t)=p\left(D(t)z,t\right),\quad \text{for all } z\in[0,1], \tag{A.1}$$

the proof is completed.

Proof of Lemma 2 (Sketch)

The function p satisfies

$$p_t(x,t)=p_x(x,t),\quad x\in[0,1], \tag{A.2}$$

which can be shown by noting that u satisfies (2). Therefore, using (2), (A.2) and definitions (9), (A.1) we get that

$$w_t(z,t) = \frac{z\dot{D}(t)+1}{D(t)}w_z(z,t), \tag{A.3}$$

where from (2), (4) it follows that

$$\frac{1+z\dot{D}(t)}{D(t)}=\frac{1+z\frac{\nabla F(v(1,t),v(0,t))\frac{(v_z(1,t),v_z(0,t))^{\mathrm{T}}}{F(v(1,t),v(0,t))}}{1-F_{u_1}(v(1,t),v(0,t))\frac{v_z(1,t)}{F(v(1,t),v(0,t))}}}{F\left(v(1,t),v(0,t)\right)}. \tag{A.4}$$

Proof of Lemma 3 (Sketch)

Under (13)–(15) it holds that

$$\frac{\min\left\{1,\frac{\epsilon_3}{\epsilon_1}\right\}}{\epsilon_6}\leq\xi(z,t)\leq\frac{\min\left\{1,\frac{\epsilon_4}{\epsilon_2}\right\}}{\epsilon_5}, \tag{A.5}$$

for all $z\in[0,1]$ and $t\geq 0$, where ξ is defined in (29). Moreover, from (27), (28), and (A.5) it follows that

$$w_{zt}(z,t) = \xi_z(z,t)w_z(z,t)+\xi(z,t)w_{zz} \tag{A.6}$$

$$w_z(1,t) = 0. \tag{A.7}$$

Consider now the following Lyapunov functional

$$\begin{aligned}L_{c,m}(t) &= \int_0^1 e^{2(c+\lambda)zm}w(z,t)^{2m}dz\\ &\quad+\int_0^1 e^{2(c+\lambda)zm}w_z(z,t)^{2m}dz, \end{aligned}\tag{A.8}$$

for any $c>0$ and any positive integer m. Taking the time derivative of (A.8) along the solutions of (26)–(28), (A.6), (A.7) and using definition (29) together with (13)–(15), we conclude that there exists $\lambda>0$ such that

$$\dot{L}_{c,m}(t) \leq -2mc\epsilon L_{c,m}(t), \tag{A.9}$$

where $\epsilon=\frac{\min\{\epsilon_2,\epsilon_3\}}{\epsilon_1\epsilon_6}$. From (A.8) this implies that

$$\Xi_{c,m}(t)\leq 2e^{-c\epsilon(t-s)}\Xi_{c,m}(s), \tag{A.10}$$

where

$$\begin{aligned}\Xi_{c,m}(t)=&\left(\int_0^1 e^{2(c+\lambda)zm}w(z,t)^{2m}dz\right)^{\frac{1}{2m}}\\ &+\left(\int_0^1 e^{2(c+\lambda)zm}w_z(z,t)^{2m}dz\right)^{\frac{1}{2m}}. \end{aligned}\tag{A.11}$$

Taking the limit of (A.11) as m goes to infinity, with the definition of the supremum norm, i.e., with relation $\|\theta(t)\|_\infty = \lim_{m\to\infty}\left(\int_0^1 |\theta(z,t)|^{2m}dz\right)^{\frac{1}{2m}}$, we obtain

$$\|w(t)\|_\infty + \|w_z(t)\|_\infty \le 2e^{-c\epsilon(t-s)}e^{(c+\lambda)}\left(\|w(s)\|_\infty + \|w_z(s)\|_\infty\right). \tag{A.12}$$

Under Assumption 3 (see, e.g., [33]) we obtain from (26)

$$|X(t)| \le \beta_1\left(|X(s)|, t-s\right) + \gamma_1\left(\sup_{s\le\tau\le t}\|w(\tau)\|\right), \tag{A.13}$$

for all $t \ge s \ge 0$, some class $\mathcal{KL}$ function β_1, and some class $\mathcal{K}$ function γ_1. Mimicking the arguments in the proof of Lemma 4.7 from [24] and using (A.12) we get (30) with $\beta_w(s,t) = \beta_1\left(\beta_1(s,0) + \gamma_1\left(2e^{(c+\lambda)}s\right), \frac{t}{2}\right) + 2e^{-c\epsilon t}e^{(c+\lambda)}s + \gamma_1\left(2e^{-c\epsilon\frac{t}{2}}e^{(c+\lambda)}s\right)$.

Proof of Lemma 4 (Sketch)

From (22) it follows that

$$p_{v_z}(z,t) = F\left(v(1,t), v(0,t)\right) f\left(p_v(z,t), v(z,t)\right). \tag{A.14}$$

Under Assumption 2, the ODE in z (A.14), and under Assumption 1 (continuity and positiveness of F), which allows us to conclude that

$$F\left(v(1,t), v(0,t)\right) \le F(0,0) + \hat{\alpha}\left(|v(1,t)| + |v(0,t)|\right), \tag{A.15}$$

for some class $\mathcal{K}_\infty$ function $\hat{\alpha}$, we get

$$\|p_v(t)\|_\infty \le \alpha_4\left(|X(t)| + \|v(t)\|_\infty\right), \tag{A.16}$$

for some class $\mathcal{K}_\infty$ function α_4. Since f is continuously differentiable with $f(0,0) = 0$, using (A.14) and (A.16), we arrive at

$$|p_{v_z}(z,t)| \le \alpha_6\left(|X(t)| + \|v(t)\|_\infty\right), \quad \forall z \in [0,1], \tag{A.17}$$

for some class $\mathcal{K}_\infty$ function α_6. The proof is completed by taking a supremum in both sides of (A.17).

Proof of Lemma 5 (Sketch)

From (25) it follows that

$$\pi_{v_z}(z,t) = F\left(\kappa\left(\pi_v(1,t)\right), w(0,t) + \kappa\left(X(t)\right)\right) \times f\left(\pi_v(z,t), w(z,t) + \kappa\left(\pi_v(z,t)\right)\right), \tag{A.18}$$

and thus, defining

$$\pi(x,t) = \pi\left(D(t)z,t\right) \equiv \pi_v\left(z,t\right), \tag{A.19}$$

and

$$w_u(x,t) = w_u\left(D(t)z,t\right) \equiv w\left(z,t\right), \tag{A.20}$$

we get using (28) and (24) that for all $x \in [0, D(t)]$

$$\pi_x(x,t) = f\left(\pi(x,t), w_u(x,t) + \kappa\left(\pi(x,t)\right)\right). \tag{A.21}$$

Under Assumption 3 (see, e.g., [33]), with definitions (A.19) and (A.20) we obtain

$$\|\pi_v(t)\|_\infty \le \hat{\beta}\left(|X(t)|, 0\right) + \zeta\left(\|w(t)\|_\infty\right), \tag{A.22}$$

for some class $\mathcal{KL}$ function $\hat{\beta}$ and some class $\mathcal{K}$ function ζ. Under Assumption 3 (continuity of κ and the fact that $\kappa(0) = 0$) and Assumption 1, from (A.15), (A.18), and (A.22) we obtain

$$|\pi_{v_z}(z,t)| \le \alpha_7\left(|X(t)| + \|w(t)\|_\infty\right), \tag{A.23}$$

for some class $\mathcal{K}_\infty$ function α_7, which completes the proof.

Proof of Lemma 6 (Sketch)

Under Assumption 3 (continuous differentiability of κ), using (32) we get from (23) estimate (34). Similarly, combining (24) with (33), estimate (35) follows.

Proof of Lemma 7 (Sketch)

From (A.15) we get that

$$\sup_{z\in[0,1]}|v_z(z,t)| = \sup_{x\in[0,D(t)]}|u_x(x,t)F\left(u(D(t),t), u(0,t)\right)| \le \|u_x(t)\|_\infty\left(F(0,0) + \hat{\alpha}\left(2\|u(t)\|_\infty\right)\right) \tag{A.24}$$

and thus, we also get (37).

Proof of Lemma 8

Utilizing the fact that

$$|u_x\left(D(t)z,t\right)| = \frac{|v_z(z,t)|}{F\left(v(1,t), v(0,t)\right)}, \tag{A.25}$$

from the left-hand side of inequality (15) we obtain

$$\|u_x(t)\|_\infty \le \frac{1}{\epsilon_5}\|v_z(t)\|_\infty. \tag{A.26}$$

Therefore, relation (38) is obtained with $\rho_6(s) = \left(1 + \frac{1}{\epsilon_5}\right)s$.

Proof of Lemma 9 (Sketch)

Under Assumption 1 and (A.15) one can conclude that

$$F(0,0) - \hat{\alpha}\left(|v(1,t)| + |v(0,t)|\right) \le F\left(v(1,t), v(0,t)\right). \tag{A.27}$$

Taking any constant δ_1 such that

$$\delta_1 < \hat{\alpha}_1^{-1}\left(F(0,0)\right), \tag{A.28}$$

where $\hat{\alpha}_1(s) = \hat{\alpha}\left(2s\right)$, it follows from (A.15) and (A.27) that (15) is satisfied with any choice of ϵ_5, ϵ_6 such that

$$0 < \epsilon_5 < F(0,0) - \hat{\alpha}\left(2\delta_1\right) \tag{A.29}$$

$$\epsilon_6 > F(0,0) + \hat{\alpha}\left(2\delta_1\right). \tag{A.30}$$

Moreover, under Assumption 1 (continuous differentiability of F) there exists a class $\mathcal{K}_\infty$ function $\hat{\rho}$ such that

$$|\nabla F\left(v(1,t), v(0,t)\right)| \le |\nabla F(0,0)| + \hat{\rho}\left(|v(1,t)| + |v(0,t)|\right). \tag{A.31}$$

Thus, from (A.27), (A.28), (A.31), and (39) we have that

$$\Gamma\left(\delta_1\right) \ge |\Gamma_1\left(v(1,t), v(0,t), v_z(0,t)\right)| + |\Gamma_2\left(v(1,t), v(0,t), v_z(1,t)\right)| \tag{A.32}$$

$$\Gamma = \frac{2\delta_1\left(|\nabla F(0,0)| + \hat{\rho}\left(2\delta_1\right)\right)}{F(0,0) - \hat{\alpha}_1\left(\delta_1\right)} \tag{A.33}$$

$$\Gamma_1 = \frac{v_z(0,t)F_{u_2}\left(v(1,t), v(0,t)\right)}{F\left(v(1,t), v(0,t)\right)} \tag{A.34}$$

$$\Gamma_2 = \frac{v_z(1,t)F_{u_1}\left(v(1,t), v(0,t)\right)}{F\left(v(1,t), v(0,t)\right)}. \tag{A.35}$$

Hence, (13), (14) hold with any choice $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ such that

$$\epsilon_1 = \epsilon_4 > 1 + \frac{2\delta_1 \left(|\nabla F(0,0)| + \hat{\rho}\left(2\delta_1\right) \right)}{F(0,0) - \hat{\alpha}_1\left(\delta_1\right)} \quad \text{(A.36)}$$

$$\epsilon_2 = \epsilon_3 < 1 - \frac{2\delta_1 \left(|\nabla F(0,0)| + \hat{\rho}\left(2\delta_1\right) \right)}{F(0,0) - \hat{\alpha}_1\left(\delta_1\right)}, \quad \text{(A.37)}$$

whenever,

$$\delta_1 < \bar{\alpha}^{-1}\left(F(0,0)\right), \quad \text{(A.38)}$$

where $\bar{\alpha} \in \mathcal{K}_\infty$ is $\bar{\alpha}(s) = \hat{\alpha}(2s) + 2s\left(|\nabla F(0,0)| + \hat{\rho}(2s)\right)$. Note that (A.28) holds if (A.38) holds.

Proof of Lemma 10

Combining (30) with (35) and (34) we arrive at $\Omega_v(t) \leq \rho_4\left(\beta_u\left(\rho_3\left(\Omega_v(0)\right), 0\right)\right)$, where Ω_v is defined in (36). The proof is completed choosing $\delta_v \leq \bar{\gamma}^{-1}\left(\delta_1\right)$, where $\bar{\gamma}$ is the class $\mathcal{K}$ function $\bar{\gamma}(s) = \rho_4\left(\beta_u\left(\rho_3(s, 0)\right)\right)$.

Proof of Lemma 11

From Lemma 7 the lemma is proved with $\delta_u \leq \rho_5^{-1}\left(\delta_v\right)$.

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References

[1] D. Angeli and E. D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *Systems & Control Letters*, vol. 38, pp. 209–217, 1999.

[2] Z. Artstein, "Linear systems with delayed controls: A reduction," *IEEE Transactions on Automatic Control*, vol. 27, pp. 869–879, 1982.

[3] N. Bekiaris-Liberis and M. Krstic, "Compensation of time-varying input and state delays for nonlinear systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 134, paper 011009, 2012.

[4] N. Bekiaris-Liberis and M. Krstic, "Compensation of state-dependent input delay for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 58, pp. 275–289, 2013.

[5] N. Bekiaris-Liberis and M. Krstic, *Nonlinear Control Under Nonconstant Delays*, SIAM, 2013.

[6] N. Bekiaris-Liberis and M. Krstic, "Robustness of nonlinear predictor feedback laws to time- and state-dependent delay perturbations," *Automatica*, 49 (2013) 1576–1590.

[7] N. Bekiaris-Liberis and M. Krstic, "Nonlinear control under delays that depend on delayed states," *European Journal of Control*, vol. 19, pp. 389–398, 2013.

[8] D. Bresch-Pietri, J. Chauvin, and N. Petit, "Sufficient conditions for the prediction-based stabilization of linear systems subject to input with input-varying delay," *American Control Conference*, 2013.

[9] D. Bresch-Pietri, J. Chauvin, & N. Petit, "Prediction-based stabilization of linear systems subject to input-dependent input delay of integral type," *IEEE Trans. Autom. Contr.*, vol. 59, pp. 2385–2399, 2014.

[10] D. Bresch-Pietri and N. Petit, "Robust compensation of a chattering time-varying input delay," *IEEE CDC*, Los Angeles, CA, 2014.

[11] X. Cai and M. Krstic, "Nonlinear control under wave actuator dynamics with time-and state-dependent moving boundary," *International Journal of Robust and Nonlinear Control*, vol 25, pp. 222–251, 2015.

[12] X. Cai and M. Krstic, "Nonlinear stabilization through wave PDE dynamics with a moving uncontrolled boundary," *Automatica*, vol. 68, pp. 27–38, 2016.

[13] X. Cai, L. Liao, Y. Liu, and C. Lin, "Predictor-based stabilization for discrete nonlinear systems with state-dependent input delays," *International Journal of Systems Science*, vol. 48, pp. 769–777, 2017.

[14] M. Chebre, Y. Creff, and N. Petit, "Feedback control and optimization for the production of commercial fuels by blending," *Journal of Process Control*, vol. 20, pp. 441–451, 2010.

[15] J-M. Coron, *Control and Nonlinearity*, Providence, RI: American Mathematical Society, 2007.

[16] C. Depcik and D. Assanis, "One-dimensional automotive catalyst modeling," *Progr. Ener. & Combust. Sci.*, vol. 31, pp. 308–369, 2005.

[17] M. Diagne, N. Bekiaris-Liberis, and M. Krstic, "Compensation of input delay that depends on delayed input," *American Control Conference*, Seattle, WA, 2017.

[18] M. Hansen, *Model Based Control of Single-Phase Marine Cooling Systems*, PhD thesis, Aalborg University, Aalborg, 2013.

[19] J. Harmand and D. Dochain, "The optimal design of two interconnected (bio) chemical reactors revisited," *Computers & Chemical Engineering*, vol. 30, pp. 70–82, 2005.

[20] J. P. Hathout, M. Fleifil, A. M. Annaswamy, and A. F. Ghoniem, "Active control using fuel-injection of time-delay induced combustion instability," *AIAA J. Prop. & Power*, vol. 18, pp. 390–399, 2002.

[21] M. Herty, J.-P. Lebacque, and S. Moutari, "A novel model for intersections of vehicular traffic flow," *Networks and Heterogenous Media*, vol. 4, pp. 813–826, 2009.

[22] I. Karafyllis and M. Krstic, *Predictor Feedback for Delay Systems: Implementations and Approximations*, Birkhauser, Basel, 2017.

[23] I. Karafyllis, "Stabilization by means of approximate predictors for systems with delayed input," *SIAM Journal on Control and Optimization*, vol. 49, pp. 1100–1123, 2011.

[24] H. Khalil, *Nonlinear Systems*, 3rd Edition, Prentice Hall, 2002.

[25] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, Basel, 2009.

[26] M. Krstic, "Input delay compensation for forward complete and feedforward nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, pp. 287–303, 2010.

[27] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Stability and control design for time-varying systems with time-varying delays using a trajectory based approach," *SIAM J. Contr. & Optim.*, to appear, 2017.

[28] S. Mondie and W. Michiels, "Finite spectrum assignment of unstable time-delay systems with a safe implementation," *IEEE Transactions on Automatic Control*, vol. 48, pp. 2207–2212, 2003.

[29] N. Petit, Y. Creff and P. Rouchon, "Motion planning for two classes of nonlinear systems with delays depending on the control," *IEEE Conference on Decision and Control*, Tampa, FL, 1998.

[30] J.-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667–1694, 2003.

[31] L. Roca, M. Berenguel, L. Yebra, and D. C. Alarcon-Padilla, "Solar field control for desalination plants," *Solar Energy*, vol. 82, pp. 772–786, 2008.

[32] M. Sbarciog, R. De Keyser, S. Cristea, and C. De Prada, "Nonlinear predictive control of processes with variable time delay. A temperature control case study," *IEEE International Conference on Control Applications*, San Antonio, TX, 2008.

[33] E. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Let.*, vol. 24, pp. 351–359, 1995.

[34] K. Zenger and A. J. Niemi, "Modeling and control of a class of time-varying continuous flow processes," *Journal of Process Control*, vol. 19, pp. 1511–1518, 2009.

[35] Q.-C. Zhong, "On distributed delay in linear control laws-part I: discrete-delay implementations," *IEEE Transactions on Automatic Control*, vol. 49, pp. 2074–2080, 2004.