Nonlinear Bilateral Full-State Feedback Trajectory Tracking for a Class of Viscous Hamilton-Jacobi PDEs

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***Abstract*— We tackle the boundary control problem for a class of viscous Hamilton-Jacobi PDEs, considering bilateral actuation, i.e., at the two boundaries of a 1-D spatial domain. First, we solve the nonlinear trajectory generation problem for this type of PDEs, providing the necessary feedforward actions at both boundaries. Second, in order to guarantee trajectory tracking with an arbitrary decay rate, we construct nonlinear, full-state feedback laws employed at the two boundary ends. All of our designs are explicit since they are constructed interlacing a feedback linearizing transformation (which we introduce) with backstepping. Due to the fact that the linearizing transformation is locally invertible, only regional stability results are established, which are, nevertheless, accompanied with region of attraction estimates. Our stability proofs are based on the utilization of the linearizing transformation together with the employment of backstepping transformations, suitably formulated to handle the case of bilateral actuation. We illustrate the developed methodologies via application to traffic flow control and we present consistent simulation results.**

I. Introduction

Contrary to linear parabolic Partial Differential Equations (PDEs), for which explicit boundary control designs are now largely available, see, for instance, [26], [29], in the nonlinear case, the design of explicit boundary control schemes is a more challenging problem. In addition, specific engineering applications, such as, for example, vehicular traffic [22], [37], plasma systems [10], fluids [4], [5], [11], chemical reactors [29], heat exchangers [29], and litium-ion batteries [36], to name only a few, call for the development of systematic control design methodologies that, besides being able to efficiently exploit the capabilities of the available actuators, they can also be made fault tolerant. Motivated by scalar, conservation law models for vehicular traffic flow that include a viscous term, in order to account for drivers' look-ahead ability [22], [37], we consider the problem of boundary control of a certain class of viscous Hamilton-Jacobi (HJ) PDEs, which constitutes an alternative macroscopic description of traffic flow dynamics [13], [32]. In particular, we consider the case in which actuation is available at both boundaries (which we refer to as "bilateral" in our control approach), aiming at constructing control schemes capable of utilizing efficiently the available actuators.

Arguably, the most relevant results to the ones presented here are those dealing with the controller design for viscous Burgers-type PDEs, which may be viewed as conservation law counterparts of the class of viscous HJ PDEs with quadratic Hamiltonian considered here. The trajectory generation problem for certain forms of viscous Burgers equations is considered in [25], [30], [34], whereas full-state boundary feedback laws are designed in [16], [21], [24], [27]. Explicit boundary control designs for other nonlinear parabolic PDEs also exist, see, e.g., [18], [38]. Although it is a different problem, for completeness, it should be mentioned that the control design problem of inviscid versions of Burgers or of specific HJ PDEs is considered in, e.g., [1], [8], [13]. Bilateral controllers for certain classes of linear parabolic and hyperbolic PDEs are recently developed in [2], [3], [39], [40]. We should also mention here that, in comparison to [6], in the present paper we consider, 1) a more general class of viscous HJ PDE systems, 2) the problems of trajectory generation and tracking, and 3) the problems of bilateral control.

Our contributions are summarized as follows. First, we solve the nonlinear trajectory generation problem for the considered viscous HJ PDE, providing explicit feedforward actions at both boundaries. The key ingredient in our approach is the employment of a feedback linearizing transformation (inspired by the Hopf-Cole transformation [14], [20]) that we introduce, which allows us to convert the original nonlinear problem to a motion planning problem for a linear heat equation. We then establish the well-posedness of the feedforward controllers for the original nonlinear PDE system, for reference outputs that belong to Gevrey class (of certain order) with sufficiently small magnitude.

Second, we design full-state feedback laws in order to achieve trajectory tracking, with an arbitrary decay rate, as the system is not, in general, asymptotically stable around a given reference trajectory. Modifying, in a suitable way, the introduced feedback linearizing transformation we recast the original nonlinear control problem to a problem of full-state feedback stabilization of a linear heat equation, with Neumann actuation at each of the two boundaries. The bilateral boundary controllers are designed using the recently introduced backstepping technique [39]. We then establish local asymptotic stability of the closed-loop system in H^1 norm, employing a Lyapunov functional and we provide an estimate of the region of attraction of the controller. Our stability result is local in H^1 norm due to the fact that the linearizing transformation is invertible only locally and, in particularly, the size of the supremum norm of the transformed PDE state should be appropriately restricted.

Finally, we apply the developed methodologies to a model of highway traffic flow and we illustrate, in simulation, the

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effectiveness of the proposed control design techniques.

We start presenting the class of viscous HJ PDEs under consideration and introducing the feedback linearizing transformation in Section II. In Section III we present the nonlinear feedforward control designs. In Section IV we present the nonlinear, full-state feedback controllers and in Section V we prove local asymptotic stability of the closed-loop system. We present an example of traffic flow control in Section VI. Concluding remarks are provided in Section VII.

Notation and Definitions: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [23]. For a function $u \in L^2(0,1)$ we denote by $\|u(t)\|_{L^2}$ the norm $\|u(t)\|_{L^2} = \sqrt{\int_0^1 u(x,t)^2 dx}$. For $u \in H^1(0,1)$ we denote by $\|u(t)\|_{H^1}$ the norm $\|u(t)\|_{H^1} = \sqrt{\int_0^1 u(x,t)^2 dx} + \sqrt{\int_0^1 u_x(x,t)^2 dx}$. We denote by $C^j(A)$ the space of functions that have continuous derivatives of order j on A. We denote an initial condition as $u_0(x) = u(x,t_0)$ with some $t_0 \geq 0$, for all $x \in [0,1]$. With $C\left([t_0,\infty); H^2(0,1)\right)$ we denote the class of continuous mappings on $[t_0,\infty)$ with values into $H^2(0,1)$. We denote by $C_T^{2,1}([0,1]\times(t_0,T))$ the space of functions that have continuous spatial derivatives of order 2 and continuous time derivatives of order 1 on $[0,1]\times(t_0,T)$, and define $C_\infty^{2,1} = C^{2,1}$.

Definition 1: The function $f(t)$ belongs to $G_{F,M,\gamma}(\mathbb{S})$, the Gevrey class of order γ in $\mathbb{S}$, if $f(t) \in C^\infty(\mathbb{S})$ and there exist positive constants F, M such that $\sup_{t\in\mathbb{S}}\left|f^{(n)}(t)\right| \leq FM^n(n!)^\gamma$, for all $n = 0,1,2,\ldots$.

II. Problem Formulation and Feedback Linearization

We consider the following viscous HJ PDE system

$$
\begin{aligned}
u_t(x,t) &= \epsilon u_{xx}(x,t) - a u_x(x,t)\left(b + u_x(x,t)\right) && (1)\\
u_x(0,t) &= U_0(t) && (2)\\
u_x(1,t) &= U_1(t), && (3)
\end{aligned}
$$

where u is the PDE state, $x \in [0,1]$ is the spatial variable, $t \geq t_0 \geq 0$ is time, $\epsilon > 0$ is a viscosity coefficient, $a \neq 0$ and $b \in \mathbb{R}$ are constant parameters, and U_0, U_1 are control variables. We introduce next a feedback linearizing transformation, which allows us to convert the problems of trajectory generation and tracking for the nonlinear HJ PDE (1)–(3) to the corresponding problems for a linear diffusion-advection PDE.

The following locally invertible transformation

$$
\bar{v}(x,t) = e^{-\frac{a}{\epsilon}u(x,t)} - 1, \qquad (4)
$$

and the control laws

$$
\begin{aligned}
U_0(t) &= -\frac{\epsilon}{a} e^{\frac{a}{\epsilon}u(0,t)} \bar{V}_0(t) && (5)\\
U_1(t) &= -\frac{\epsilon}{a} e^{\frac{a}{\epsilon}u(1,t)} \bar{V}_1(t), && (6)
\end{aligned}
$$

where $\bar{V}_0$, $\bar{V}_1$ are the new control variables yet to be chosen, transform system (1)–(3) to

$$
\begin{aligned}
\bar{v}_t(x,t) &= \epsilon \bar{v}_{xx}(x,t) - ab\bar{v}_x(x,t) && (7)\\
\bar{v}_x(0,t) &= \bar{V}_0(t) && (8)\\
\bar{v}_x(1,t) &= \bar{V}_1(t). && (9)
\end{aligned}
$$

It turns out that in the control design and analysis it is more convenient to perform an additional transformation, namely

$$
v(x,t) = \bar{v}(x,t) e^{-\frac{ab}{2\epsilon}x}, \qquad (10)
$$

in order to re-write (7)–(9) as

$$
\begin{aligned}
v_t(x,t) &= \epsilon v_{xx}(x,t) - \frac{a^2b^2}{4\epsilon} v(x,t) && (11)\\
v_x(0,t) &= V_0(t) && (12)\\
v_x(1,t) &= V_1(t), && (13)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{V}_0(t) &= V_0(t) + \frac{ab}{2\epsilon}\bar{v}(0,t) && (14)\\
\bar{V}_1(t) &= e^{\frac{ab}{2\epsilon}} V_1(t) + \frac{ab}{2\epsilon}\bar{v}(1,t), && (15)
\end{aligned}
$$

and V_0, V_1 are the new control variables.

III. Trajectory Generation

In this section we design the feedforward boundary control laws that generate the desired reference outputs. We solve the problem first for the linearized system (11)–(13) and we then provide the feedforward actions for the original system (1)–(3). We consider as outputs of the system the values $u(x_0,t)$ and $u_x(x_0,t)$, where x_0 is some fixed point within the interval $[0,1]$. The proof of the following theorem is omitted due to space limitations, but it can be found in [7] (Section 3).

Theorem 1: Let $y_1^{\mathrm{r}}(t)$ and $y_2^{\mathrm{r}}(t)$ be in $G_{F,M,\gamma}([0,+\infty))$ class with $1 \leq \gamma < 2$. There exists a positive constant μ_1 such that if $F \leq \mu_1$ then the functions

$$
\begin{aligned}
u^{\mathrm{r}}(x,t) &= -\frac{\epsilon}{a}\ln\left(e^{\frac{ab}{2\epsilon}x} v^{\mathrm{r}}(x,t) + 1\right) && (16)\\
U_0^{\mathrm{r}}(t) &= -\frac{\epsilon}{a}\frac{v_x^{\mathrm{r}}(0,t) + \frac{ab}{2\epsilon}v^{\mathrm{r}}(0,t)}{1 + v^{\mathrm{r}}(0,t)} && (17)\\
U_1^{\mathrm{r}}(t) &= -\frac{\epsilon e^{\frac{ab}{2\epsilon}}}{a}\frac{v_x^{\mathrm{r}}(1,t) + \frac{ab}{2\epsilon}v^{\mathrm{r}}(1,t)}{1 + e^{\frac{ab}{2\epsilon}} v^{\mathrm{r}}(1,t)}, && (18)
\end{aligned}
$$

where

$$
\begin{aligned}
v^{\mathrm{r}}(x,t) &= \sum_{k=0}^{\infty}\frac{1}{\epsilon^k}\frac{(x-x_0)^{2k}}{(2k)!}\sum_{m=0}^{k}\binom{k}{m}\left(\frac{a^2b^2}{4\epsilon}\right)^{k-m}\\
&\quad \times y_{1,v}^{\mathrm{r}\,(m)}(t) + \sum_{k=0}^{\infty}\frac{1}{\epsilon^k}\frac{(x-x_0)^{2k+1}}{(2k+1)!}\\
&\quad \times \sum_{m=0}^{k}\binom{k}{m}\left(\frac{a^2b^2}{4\epsilon}\right)^{k-m} y_{2,v}^{\mathrm{r}\,(m)}(t) && (19)\\
y_{1,v}^{\mathrm{r}}(t) &= e^{-\frac{ab}{2\epsilon}x_0}\left(e^{-\frac{a}{\epsilon}y_1^{\mathrm{r}}(t)} - 1\right) && (20)\\
y_{2,v}^{\mathrm{r}}(t) &= e^{-\frac{ab}{2\epsilon}x_0}\left(-\frac{a}{\epsilon}e^{-\frac{a}{\epsilon}y_1^{\mathrm{r}}(t)} y_2^{\mathrm{r}}(t)\right.\\
&\quad \left. -\frac{ab}{2\epsilon}\left(e^{-\frac{a}{\epsilon}y_1^{\mathrm{r}}(t)} - 1\right)\right), && (21)
\end{aligned}
$$

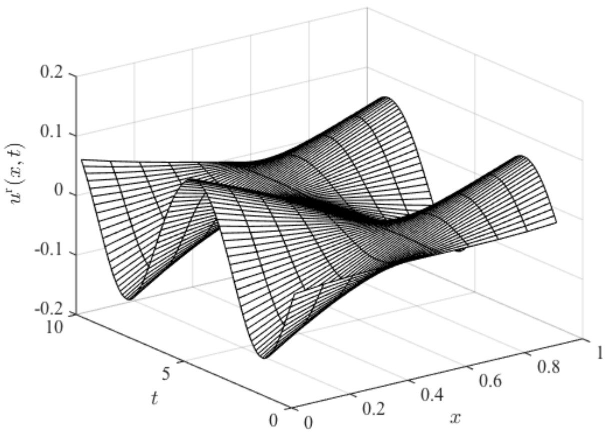

Fig. 1. Function (22) that solves the nonlinear trajectory generation problem for system (1)–(3) with $a = -1$, $b = 0$, and $\epsilon = 0.5$, with reference trajectories $y_1^{\mathrm{r}}(t) = 0$ and $y_2^{\mathrm{r}}(t) = 0.25\sin(t)$, for $x_0 = \frac{1}{2}$.

satisfy the boundary value problem (1)–(3) and, in particular, $u^{\mathrm{r}}(x_0,t) = y_1^{\mathrm{r}}(t)$ and $u_x^{\mathrm{r}}(x_0,t) = y_2^{\mathrm{r}}(t)$.

Example 1: Consider system (1)–(3) with $a=-1$, $b=0$ and assume that the desired reference trajectories are $y_1^{\mathrm{r}}(t) = 0$ and $y_2^{\mathrm{r}}(t) = d\sin(t)$, where $d > 0$. For sufficiently small d the conditions of Theorem 1 are satisfied. The reference trajectory as well as the reference inputs are given by

$$u^{\mathrm{r}}(x,t)=\epsilon\ln\left(1+g_1(x,t)\right) \tag{22}$$

$$\begin{aligned} g_1(x,t)=&\frac{d}{2\sqrt{\epsilon}}e^{\frac{x-x_0}{\sqrt{2\epsilon}}}\sin\left(t+\frac{x-x_0}{\sqrt{2\epsilon}}-\frac{\pi}{4}\right)\\ &-\frac{d}{2\sqrt{\epsilon}}e^{\frac{x_0-x}{\sqrt{2\epsilon}}}\sin\left(t+\frac{x_0-x}{\sqrt{2\epsilon}}-\frac{\pi}{4}\right) \end{aligned} \tag{23}$$

$$U_0^{\mathrm{r}}(t)=\frac{d}{2}\frac{e^{-\frac{x_0}{\sqrt{2\epsilon}}}\sin\left(t-\frac{x_0}{\sqrt{2\epsilon}}\right)+e^{\frac{x_0}{\sqrt{2\epsilon}}}\sin\left(t+\frac{x_0}{\sqrt{2\epsilon}}\right)}{1+g_1(0,t)} \tag{24}$$

$$\begin{aligned} U_1^{\mathrm{r}}(t)=&\frac{d}{2}\frac{e^{\frac{1-x_0}{\sqrt{2\epsilon}}}\sin\left(t+\frac{1-x_0}{\sqrt{2\epsilon}}\right)}{1+g_1(1,t)}\\ &+\frac{d}{2}\frac{e^{\frac{x_0-1}{\sqrt{2\epsilon}}}\sin\left(t+\frac{x_0-1}{\sqrt{2\epsilon}}\right)}{1+g_1(1,t)}, \end{aligned} \tag{25}$$

where we also used the fact that $\sin(y) - \cos(y) = \sqrt{2}\sin\left(y-\frac{\pi}{4}\right)$, for any $y \in \mathbb{R}$. In Fig. 1 we show the generated trajectory u^{r}.

IV. Bilateral Full-State Feedback Boundary Control Design

Having available the reference trajectory for system (1)–(3), in this section, we design the boundary feedback laws that stabilize the desired reference trajectory for any initial condition. We start deriving the dynamics of the error between the actual and the reference states. We then introduce a feedback linearizing transformation for the tracking error's dynamics, which, in turn, enables us to design full-state feedback, boundary control laws utilizing infinite-dimensional backstepping for linear systems.

A. Tracking error dynamics and motivation for control

We define the error variables

$$\begin{aligned} \tilde{u}(x,t) &= u(x,t) - u^{\mathrm{r}}(x,t) && (26)\\ \tilde{U}_0(t) &= U_0(t) - U_0^{\mathrm{r}}(t) && (27)\\ \tilde{U}_1(t) &= U_1(t) - U_1^{\mathrm{r}}(t). && (28) \end{aligned}$$

Differentiating (26) with respect to t and x, using the fact that $u^{\mathrm{r}}(x,t)$ satisfies system (1)–(3) we get that $\tilde{u}$ satisfies the following system

$$\begin{aligned} \tilde{u}_t(x,t) &= \epsilon\tilde{u}_{xx}(x,t) - a\tilde{u}_x(x,t)\left(b+\tilde{u}_x(x,t)\right)\\ &\quad -2au_x^{\mathrm{r}}(x,t)\tilde{u}_x(x,t) && (29)\\ \tilde{u}_x(0,t) &= \tilde{U}_0(t) && (30)\\ \tilde{u}_x(1,t) &= \tilde{U}_1(t). && (31) \end{aligned}$$

A feedback control design is needed to asymptotically stabilize the origin of (29)–(31). To see this note that the zero solution of (29)–(31) is not asymptotically stable since any constant could be an equilibrium of (29)–(31).

B. Feedback linearizing transformation for the tracking error dynamics

Guided from the feedback linearizing transformation (4) we define

$$\tilde{\tilde{v}}(x,t) = e^{-\frac{a}{\epsilon}\tilde{u}(x,t)} - 1, \tag{32}$$

which it is readily shown that satisfies the following PDE

$$\begin{aligned} \tilde{\tilde{v}}_t(x,t) &= \epsilon\tilde{\tilde{v}}_{xx}(x,t) - a\left(b+2u_x^{\mathrm{r}}(x,t)\right)\tilde{\tilde{v}}_x(x,t) && (33)\\ \tilde{\tilde{v}}_x(0,t) &= \tilde{\tilde{V}}_0(t) && (34)\\ \tilde{\tilde{v}}_x(1,t) &= \tilde{\tilde{V}}_1(t), && (35) \end{aligned}$$

where we choose

$$\begin{aligned} \tilde{U}_0(t) &= -\frac{\epsilon}{a}e^{\frac{a}{\epsilon}\tilde{u}(0,t)}\tilde{\tilde{V}}_0(t) && (36)\\ \tilde{U}_1(t) &= -\frac{\epsilon}{a}e^{\frac{a}{\epsilon}\tilde{u}(1,t)}\tilde{\tilde{V}}_1(t), && (37) \end{aligned}$$

and $\tilde{\tilde{V}}_0(t)$, $\tilde{\tilde{V}}_1(t)$ are new control variables. With the additional transformation

$$\tilde{v}(x,t) = \tilde{\tilde{v}}(x,t)e^{-\frac{ab}{2\epsilon}x-\frac{a}{\epsilon}u^{\mathrm{r}}(x,t)}, \tag{38}$$

and selecting the control variables $\tilde{\tilde{V}}_0(t)$, $\tilde{\tilde{V}}_1(t)$ as

$$\begin{aligned} \tilde{\tilde{V}}_0(t) &= e^{\frac{a}{\epsilon}u^{\mathrm{r}}(0,t)}\tilde{V}_0(t)\\ &\quad +\frac{a}{\epsilon}\left(\frac{b}{2}+u_x^{\mathrm{r}}(0,t)\right)\left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)}-1\right) && (39)\\ \tilde{\tilde{V}}_1(t) &= e^{\frac{ab}{2\epsilon}+\frac{a}{\epsilon}u^{\mathrm{r}}(1,t)}\tilde{V}_1(t)\\ &\quad +\frac{a}{\epsilon}\left(\frac{b}{2}+u_x^{\mathrm{r}}(1,t)\right)\left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)}-1\right), && (40) \end{aligned}$$

we arrive at the following system

$$\begin{aligned} \tilde{v}_t(x,t) &= \epsilon\tilde{v}_{xx}(x,t) - \frac{a^2b^2}{4\epsilon}\tilde{v}(x,t) && (41)\\ \tilde{v}_x(0,t) &= \tilde{V}_0(t) && (42)\\ \tilde{v}_x(1,t) &= \tilde{V}_1(t), && (43) \end{aligned}$$

where the control variables $\tilde{V}_0(t)$ and $\tilde{V}_1(t)$ are chosen later on (in Section IV-C) via the backstepping methodology.

Note that system (41)–(43), besides being linear, does not incorporate any spatially- or time-dependent terms, which may be the case when considering trajectory tracking problems for nonlinear systems. This is possible here because the overall feedback linearizing transformation (38) may be expressed as the difference of two nonlinear functions of u and u^{r}, which both satisfy the linear PDE (11) (or, equivalently, (41)) since both u and u^{r} satisfy (1). Moreover, relations (42), (43) are derived differentiating (38) with respect to x and using (34), (35) as well as defining the new control inputs $\tilde{V}_0$, $\tilde{V}_1$ according to (39), (40).

C. Bilateral boundary control design

Exploiting the fact that the $\tilde{v}$ variable satisfies the linear diffusion-advection PDE (41)–(43) we design the boundary feedback laws as [39]

$$\tilde{V}_0(t) = k(0,0)\,\tilde{v}(0,t) - \int_0^1 k_x(0,\xi)\,\tilde{v}(\xi,t)\,d\xi \quad (44)$$

$$\tilde{V}_1(t) = k(1,1)\,\tilde{v}(1,t) + \int_0^1 k_x(1,\xi)\,\tilde{v}(\xi,t)\,d\xi, \quad (45)$$

where the kernel $k(x,\xi)$ is given explicitly, for (x,ξ) in the domain $D = D_1 \cup D_2$, where $D_1 = \{(x,\xi) : \frac{1}{2} \leq x \leq 1, -x+1 \leq \xi \leq x\}$ and $D_2 = \{(x,\xi) : 0 \leq x \leq \frac{1}{2}, x \leq \xi \leq 1-x\}$, by

$$k(x,\xi) = -\frac{1}{2}\sqrt{\frac{c_1}{\epsilon}}\frac{\mathrm{I}_1\left(\sqrt{\frac{c_1}{\epsilon}\left(\left(x-\frac{1}{2}\right)^2 - \left(\xi-\frac{1}{2}\right)^2\right)}\right)}{\sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\xi-\frac{1}{2}\right)^2}} \times (x+\xi-1), \quad (46)$$

with I_1 denoting the modified Bessel function of the first kind of first order. Combining (36), (37) and (39), (40) with (44), (45) the boundary feedback laws in the original variables are written via (32), (38) as

$$U_0(t) = -\frac{\epsilon}{a}e^{\frac{a}{\epsilon}\tilde{u}(0,t)}\Bigg(\left(k(0,0)+\frac{ab}{2\epsilon}\right)\left(e^{-\frac{a}{\epsilon}\tilde{u}(0,t)}-1\right) - e^{\frac{a}{\epsilon}u^{\mathrm{r}}(0,t)}\int_0^1 k_x(0,\xi)\,e^{-\frac{ab}{2\epsilon}\xi-\frac{a}{\epsilon}u^{\mathrm{r}}(\xi,t)} \times \left(e^{-\frac{a}{\epsilon}\tilde{u}(\xi,t)}-1\right)d\xi\Bigg) + U_0^{\mathrm{r}}(t)e^{\frac{a}{\epsilon}\tilde{u}(0,t)} \quad (47)$$

$$U_1(t) = -\frac{\epsilon}{a}e^{\frac{a}{\epsilon}\tilde{u}(1,t)}\Bigg(\left(k(1,1)+\frac{ab}{2\epsilon}\right)\left(e^{-\frac{a}{\epsilon}\tilde{u}(1,t)}-1\right) + e^{\frac{ab}{2\epsilon}+\frac{a}{\epsilon}u^{\mathrm{r}}(1,t)}\int_0^1 k_x(1,\xi)\,e^{-\frac{ab}{2\epsilon}\xi-\frac{a}{\epsilon}u^{\mathrm{r}}(\xi,t)} \times \left(e^{-\frac{a}{\epsilon}\tilde{u}(\xi,t)}-1\right)d\xi\Bigg) + U_1^{\mathrm{r}}(t)e^{\frac{a}{\epsilon}\tilde{u}(1,t)}, \quad (48)$$

where $U_0^{\mathrm{r}}(t)$ and $U_1^{\mathrm{r}}(t)$ are defined in (17) and (18), respectively, with the error variable $\tilde{u}$ being defined in (26) and the reference trajectory u^{r} being defined in (16).

V. Trajectory Tracking Under Full-State Feedback

In order to show asymptotic stability of the closed-loop system, under the full-state feedback laws, in the original variable $\tilde{u}$ we have to ensure that the linearizing transformation (32) is invertible. The inverse of transformation (32) is given by

$$\tilde{u}(x,t) = -\frac{\epsilon}{a}\ln\left(\tilde{\bar{v}}(x,t)+1\right), \quad (49)$$

which is well-defined when the initial conditions and solutions of the system satisfy for some $c \in (0,1]$

$$\sup_{x\in[0,1]}|\tilde{\bar{v}}(x,t)| < c, \quad \text{for all } t \geq t_0. \quad (50)$$

Due to the feasibility condition (50), only a local stability result can be obtained, which is stated next.

Theorem 2: Consider a closed-loop system consisting of the plant (1)–(3) and the control laws (47), (48). Under the conditions of Theorem 1 for the reference outputs, there exist a positive constant μ and a class $\mathcal{KL}$ function β such that for all initial conditions $u_0 \in H^2(0,1)$ which are compatible with the feedback laws (47), (48) and which satisfy

$$\|\tilde{u}(t_0)\|_{H^1} < \mu, \quad (51)$$

the following holds

$$\|\tilde{u}(t)\|_{H^1} \leq \beta\left(\|\tilde{u}(t_0)\|_{H^1}, t-t_0\right), \quad \text{for all } t \geq t_0. \quad (52)$$

Moreover, the closed-loop system has a unique solution $u \in C\left([t_0,\infty); H^2(0,1)\right)$ with $u \in C^{2,1}\left([0,1]\times(t_0,\infty)\right)$.

The proof of Theorem 2 is based on the following three lemmas whose proofs can be found in [7] (Appendix B).

Lemma 1: There exists a class $\mathcal{K}_\infty$ function α_1 such that if $\tilde{u} \in H^1(0,1)$ then $\tilde{\bar{v}} \in H^1(0,1)$ and the following holds

$$\|\tilde{\bar{v}}(t)\|_{H^1} \leq \alpha_1\left(\|\tilde{u}(t)\|_{H^1}\right). \quad (53)$$

Lemma 2: For all solutions of the system that satisfy (50) for some $0 < c < 1$, if $\tilde{\bar{v}} \in H^1(0,1)$ then $\tilde{u} \in H^1(0,1)$ and the following holds

$$\|\tilde{u}(t)\|_{H^1} \leq \frac{\epsilon}{|a|(1-c)}\|\tilde{\bar{v}}(t)\|_{H^1}. \quad (54)$$

Lemma 3: Under the conditions of Theorem 1 for the reference outputs, if $\tilde{\bar{v}} \in H^1(0,1)$ then $\tilde{v} \in H^1(0,1)$ and there exists a positive constant ξ_1 such that the following holds

$$\|\tilde{v}(t)\|_{H^1} \leq \xi_1\|\tilde{\bar{v}}(t)\|_{H^1}. \quad (55)$$

In reverse, if $\tilde{v} \in H^1(0,1)$ then $\tilde{\bar{v}} \in H^1(0,1)$ and there exists a positive constant ξ_2 such that the following holds

$$\|\tilde{\bar{v}}(t)\|_{H^1} \leq \xi_2\|\tilde{v}(t)\|_{H^1}. \quad (56)$$

Proof of Theorem 2: See [7] (Section 5).

VI. Application to Traffic Flow Control

A. Model description

Consider a highway stretch with inlet at $x = 0$ and outlet at $x = 1$. We model the traffic density dynamics within the stretch with a conservation law PDE. In order to account for drivers' look-ahead ability, we incorporate in the expression for the traffic flow, in addition to the term that corresponds to a conventional fundamental diagram relation between speed and density of vehicles, an additional term that depends on the spatial derivative of the traffic density, giving rise to the following model, see, e.g., [22], [37]

$$\rho_t(x,t) + \left(\rho(x,t)V\left(\rho(x,t)\right) - \epsilon\rho_x(x,t)\right)_x = 0 \quad (57)$$

$$\rho(0,t) = -U_0(t) \quad (58)$$

$$\rho(1,t) = -U_1(t), \quad (59)$$

where, for Greenshield's fundamental diagram [17] we have

$$V(\rho) = a(b-\rho), \quad (60)$$

with a, b being free-flow speed and maximum density, respectively, whereas ρ denotes the traffic density. The density at the boundaries may be imposed manipulating either the flow or the speed of vehicles, via the employment of ramp-metering (RM) and variable speed limits (VSL), as well as exploiting the capabilities of connected and automated vehicles see, e.g., [12], [33].

In order to bring model (57)–(59) into the form (1)–(3) we define the following variable

$$u(x,t) = \int_x^1 \rho(y,t)dy + \int_0^t Q\left(\rho(1,s), \rho_x(1,s)\right)ds, \quad (61)$$

where

$$Q(\rho, \rho_x) = \rho V(\rho) - \epsilon\rho_x. \quad (62)$$

It can be shown, by direct differentiation of (61) with respect to t and x, and by employing (57), that the variable u satisfies (1)–(3). The state u represents the so-called Moskowitz function, which constitutes an alternative macroscopic description of the dynamics of traffic flow in a highway. In particular, the value of the Moskowitz function $M = u(x,t)$ is interpreted as the "label" of a given vehicle at position x at time t, along a road segment [13], [32].

B. Design and motivation of the feedforward/feedback control laws

A typical aim of a traffic control scheme is to regulate the outlet flow to a certain set-point, say q^*, which may be the point that achieves the maximum flow (capacity flow) [12]. In terms of the u variable this corresponds to $u(1,t)$ tracking the reference trajectory q^*t. This motivates the trajectory generation and tracking problems for the class of systems described by (1)–(3). Moreover, since the value $u_x(1,t)$ could be also assigned, one may choose for reference value of $-u_x(1,t)$ the value of the density that corresponds to the critical density (i.e., the density at which capacity flow is achieved) of the nominal fundamental diagram relation (i.e., when there is no ρ_x term in (62)) between flow and density at the outlet of the considered stretch, which in turn would guarantee that the obtained desired profile for u_x (or, for ρ) is uniform with respect to space. Setting $a=b=1$, we obtain $y_1^{\mathrm{r}}(t)=\frac{1}{4}t$ and $y_2^{\mathrm{r}}(t)=-\frac{1}{2}$. Using (19)–(21) for $x_0=1$ we get

$$v^{\mathrm{r}}(x,t) = e^{-\frac{1}{2\epsilon}}\left(e^{-\frac{1}{4\epsilon}t} - e^{\frac{1-x}{2\epsilon}}\right). \quad (63)$$

Therefore, employing (16)–(18) the reference trajectory and reference inputs are given explicitly as

$$u^{\mathrm{r}}(x,t) = \frac{1}{4}t + \frac{1-x}{2} \quad (64)$$

$$U_0^{\mathrm{r}}(t) = U_1^{\mathrm{r}}(t) = -\frac{1}{2}. \quad (65)$$

The feedback controllers are given in (47), (48) with $c_1 = 1$.

C. Trajectory tracking

We choose $\epsilon = 0.25$, whereas the initial condition is defined as $u(x,0) = u^{\mathrm{r}}(x,0) + 0.1\sin(\pi x) = \frac{1-x}{2} + 0.1\sin(\pi x)$. In Fig. 2 we show the output $u(1,t)$, from which it is evident that asymptotic trajectory tracking is achieved.

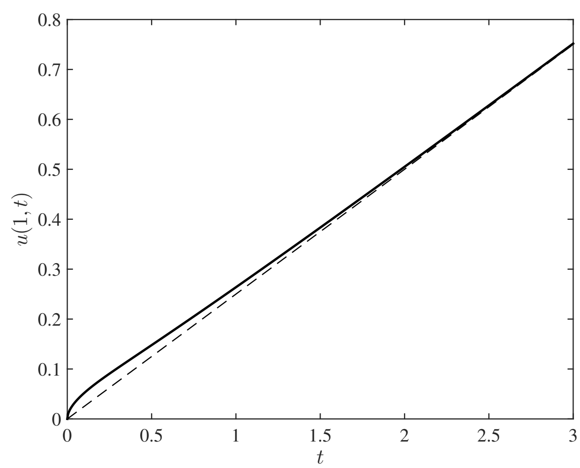

Fig. 2. Solid: Output $u(1,t)$ of system (1)–(3) with $a = b = 1$, $\epsilon = 0.25$, under the feedback laws (47), (48), (16)–(18) with $c_1 = 1$ for initial condition $u(x,0) = \frac{1-x}{2} + 0.1\sin(\pi x)$. Dashed: The reference output $u^{\mathrm{r}}(1,t) = \frac{1}{4}t$.

Furthermore, in Fig. 3, we show the density $\rho(x,t) = -u_x(x,t)$ in the highway. One can observe that the density converges to the desired reference profile, namely, to the uniform profile $\rho^{\mathrm{e}}(x) = \frac{1}{2}$, for all $x \in [0,1]$.

VII. Conclusions

For a class of viscous HJ PDEs with actuation at both boundaries we, 1) solved the nonlinear trajectory generation problem, 2) presented nonlinear, bilateral full-state feedback control designs, 3) established local asymptotic stability of the closed-loop system, and 4) illustrated our results in simulation via a traffic flow control example.

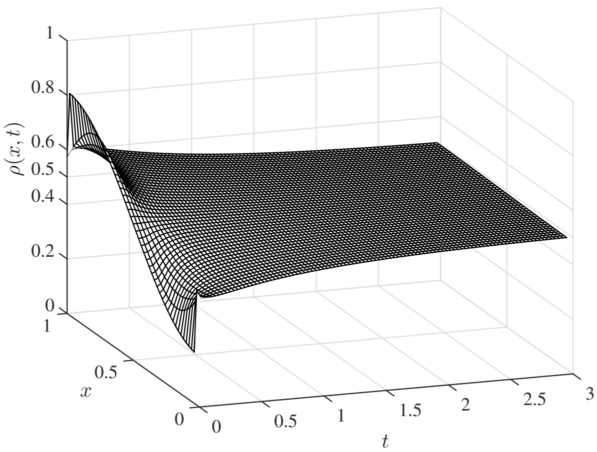

Fig. 3. The density evolution of the highway stretch.

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