Global Exponential Stability for Discrete-Time Networks with Applications to Traffic Networks

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Abstract—**This paper provides sufficient conditions for global asymptotic stability and global exponential stability, which can be applied to nonlinear, large-scale, uncertain discrete-time networks. The conditions are derived by means of vector Lyapunov functions. The obtained results are applied to traffic networks for the derivation of sufficient conditions of global exponential stability of the uncongested equilibrium point of the network. Specific results and algorithms are provided for freeway traffic models. Various examples illustrate the applicability of the obtained results.**

Index Terms- **Nonlinear systems, discrete-time systems, traffic networks.**

I. INTRODUCTION

Exponential stability is a very useful property for the equilibrium point of a given network. The purpose of the present paper is three-fold:

- to provide sufficient conditions for Global Asymptotic Stability (GAS) and Global Exponential Stability (GES), which can be easily applied to nonlinear, large-scale, uncertain discrete-time networks;
- to apply the aforementioned sufficient conditions to traffic networks and obtain conditions, which guarantee the GES of the uncongested equilibrium point;
- to study the stability properties of freeway traffic models and obtain easily checkable conditions which guarantee the GES of the uncongested equilibrium point.

Vector Lyapunov functions are useful to large-scale discrete-time systems. Sufficient stability conditions by means of vector Lyapunov functions have been proposed in [11] (pages 792-798). More recently, small-gain conditions have been proposed in [22], which can be expressed by means of a vector Lyapunov function formulation (as shown in [13], Chapter 5). In this work, we propose a set of conditions expressed by means of vector Lyapunov functions, which guarantee GAS and GES (Theorem 2.3) and can be applied easily to nonlinear, large-scale, uncertain discrete-time systems. The basis for the applicability is the expression of the stability condition by means of a condition on the spectral radius of a nonnegative matrix. Therefore, we can apply recent results on nonnegative matrices that provide upper bounds for the spectral radius (see [3]; Chapter 2). The stability notions used in this work are the standard stability notions for discrete-time systems used in [11] (Chapter 13), [12], [13] (Chapter 2) and [16] (Chapter 4), but we also allow the discrete-time, uncertain system to be defined on a subset of a finite-dimensional space. Discrete-time systems defined on a subset of a finite-dimensional space were studied in [28] (Chapter 1).

The conservatism of the obtained stability conditions can be reduced significantly if we have an accurate description of a trapping region of the system: this feature is exploited throughout the present work. A nonlinear system with a trapping region is a system for which all solutions enter a specific set after an initial transient period (for continuous-time systems without inputs the name "global uniform ultimate boundedness" is used in [14] (page 211) when the corresponding set is compact; the term "dissipative system" is used in the literature of continuous-time systems with compact corresponding sets; see [28], page 180).

The obtained stability results are applied to traffic networks (Section 3). More specifically, we develop a general model for traffic networks, which consists of an arbitrary number of elementary components. The components can be interconnected to form any two-dimensional structure for the overall traffic network. This general formulation allows for a plethora of diverse traffic network infrastructures to be addressed on the basis of a unifying modeling approach; specific instances of the proposed general model may result in systems which are similar to other models in the literature (see for example [6, 8, 25]). In particular, the traffic network structures and problems that can be considered as special cases of the proposed network model include: urban road networks consisting of interconnected links which are modelled as store-and-forward components [1] or cell-transmission links [4]; large urban networks consisting of smaller homogeneous sub-networks [2]; freeway stretches or networks consisting of series of links which are modelled via the discretized LWR (Lighthill-Whitham-Richards) model [17] or its simplified CTM (Cell Transmission

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Model) version [7]; large mixed (corridor) networks consisting of urban and freeway links [24]. As a matter of fact, the same generic approach may also be used for modeling water networks consisting of interconnected links which are modelled by discretized versions of the Lighthill-Whitham model [21], see [5, 23]. Our main related result (Theorem 3.1) provides explicit formulas for the elements of a specific nonnegative matrix whose spectral radius is critical for the GES of the uncongested equilibrium point of the traffic network. Therefore, our results can be used for the determination of the stability properties in a given traffic network within this framework (see Example 3.3).

The obtained results are specialized to the case of a freeway stretch (Section 4). The overall model in this specific constellation consists of a series of subsequent cells and is similar to the known first-order discrete Godunov approximations (see [9]) to the kinematic-wave partial differential equation of the LWR-model (see [21, 26]) with nonlinear ([17]) or piecewise linear (CTM, [7]) outflow functions. However, the presented framework can also accommodate recent modifications of the LWR-model as in [18] to reflect the so-called capacity drop phenomenon. Our main related result (Corollary 4.3) provides an easily implementable algorithm for the determination of the stability properties of the uncongested equilibrium point of the freeway stretch. The results are different from other results in the literature on the CTM (see [6, 10]), since our methodology is different from the methodology used in [6, 10]. More specifically, in [10] the dynamical analysis is based on monotone systems theory and in [6] the results concerning the uncongested equilibrium point are local. On the other hand, in this work we provide global stability results based on a vector Lyapunov function analysis.

<u>Notation.</u>

* $\Re_+ := [0,+\infty)$. For every set S, $S^n = \underbrace{S\times\ldots\times S}_{n\ times}$ for every positive integer n. $\Re_+^n := (\Re_+)^n$. For every $x\in\Re$, $[x]$ denotes the integer part of $x\in\Re$.
* We say that an increasing function $\rho\in C^0(\Re_+;\Re_+)$ is of class K_∞ if $\rho(0)=0$ and $\lim_{s\to+\infty}\rho(s)=+\infty$. By KL we denote the set of functions $\sigma\in C^0(\Re_+\times\Re_+\,;\Re_+)$ with the properties: (i) for each $t\geq 0$ the mapping $\sigma(\cdot,t)$ is increasing with $\sigma(0,t)=0$; (ii) for each $s\geq 0$, the mapping $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to+\infty}\sigma(s,t)=0$.
* Let $x,y\in\Re^n$. We say that $x\leq y$ iff $(y-x)\in\Re_+^n$. By $|x|$ we denote the Euclidean norm of $x\in\Re^n$. Let $A\in\Re^{n\times n}$ be a real matrix. By $|A|$ we denote the induced matrix norm. The spectral radius of $A\in\Re^{n\times n}$ is denoted by $\rho(A)$. When all elements of A are non-negative, then we say that A is non-negative and we write $A\in\Re_+^{n\times n}$.

II. Vector Lyapunov Stability Criteria For Discrete-Time Networks

Consider the discrete-time system:

$$x^+ = F(d,x)\,,\; x\in S\subseteq\Re^n\,, d\in D \tag{2.1}$$

where $S\subseteq\Re^n$ is a non-empty closed set with $x^*\in S$, $D\subseteq\Re^l$ is a non-empty, compact set, $F: D\times S\to S$ is a locally bounded mapping, being continuous on the set $D\times\{x^*\}$ with $F(d,x^*)=x^*$ for all $d\in D$. We suppose that $\{x\in S: 0<|x-x^*|\leq\delta\}\neq\emptyset$ for every $\delta>0$.

In order to develop the Vector Lyapunov Stability criteria we need the notion of a Trapping Region (TR). A nonlinear system with a Trapping Region is a system for which all solutions enter a specific set after an initial transient period.

Definition 2.1: *A Trapping Region (TR) for system (2.1) is a set $A\subseteq S$ for which there exists an integer $m\geq 0$ such that for every $x_0\in S$, $\{d_i\in D\}_{i=0}^{\infty}$, the solution $x(t)$ of (2.1) with initial condition $x(0)=x_0$ corresponding to input $\{d_i\in D\}_{i=0}^{\infty}$ satisfies $x(t)\in A$ for all $t\geq m$.*

A direct consequence of Definition 2.1 is that every TR for (2.1) must contain all equilibrium points. We next define the robust stability notions used for (2.1).

Definition 2.2: *We say that $x^*\in S$ is Robustly Globally Asymptotically Stable (RGAS) for system (2.1), if there exists a function $\sigma\in KL$ such that for every $x_0\in S$, $\{d_i\in D\}_{i=0}^{\infty}$, the solution $x(t)$ of (2.1) with $x(0)=x_0$ corresponding to $\{d_i\in D\}_{i=0}^{\infty}$ satisfies $|x(t)-x^*|\leq\sigma(|x_0-x^*|,t)$ for all $t\geq 0$. We say that $x^*\in S$ is Robustly Globally Exponentially Stable (RGES) for system (2.1) if there exist constants $M,\sigma>0$ such that for every $x_0\in S$, $\{d_i\in D\}_{i=0}^{\infty}$, the solution $x(t)$ of (2.1) with $x(0)=x_0$ corresponding to $\{d_i\in D\}_{i=0}^{\infty}$ satisfies $|x(t)-x^*|\leq M\exp(-\sigma t)|x_0-x^*|$ for all $t\geq 0$.*

We are now ready to state the main result of the section.

Theorem 2.3: *Consider (2.1) and suppose that $A\subseteq S$ is a TR for (2.1). Moreover, suppose that there exist functions $a_1,a_2\in K_\infty$ with $a_1(s)\leq a_2(s)$ for all $s\geq 0$, $V_i: A\to\Re_+$ $(i=1,...,l)$ and a matrix $\Gamma=\{\gamma_{i,j}\geq 0, i,j=1,...,l\}\in\Re_+^{l\times l}$ such that the following inequalities hold for all $x\in A, d\in D$ and $i=1,...,l$:*

$$a_1(|x-x^*|)\leq\max_{i=1,\ldots,l}(V_i(x))\leq a_2(|x-x^*|) \tag{2.2}$$

$$V_i(F(d,x))\leq\sum_{j=1}^{l}\gamma_{i,j}V_j(x) \tag{2.3}$$

Moreover, suppose that the spectral radius $\rho(\Gamma)$ of the matrix Γ is less than 1. Then $x^*\in S$ is RGAS for (2.1).

Moreover, if there exist constants $L \ge 0$, $0 < K_1 \le K_2$, $p > 0$ *such that* $\sup\left\{ \left| F(d,x) - x^* \right| : d \in D \right\} \le L \left| x - x^* \right|$ *for all* $x \in S \setminus A$ *and if* $a_i(s) = K_i s^p$ $(i = 1,2)$ *for all* $s \ge 0$ *then* $x^* \in S$ *is RGES for (2.1).*

Since the matrix Γ is non-negative, there are effective tools for the computation of its spectral radius ([3]; Chapter 2). For example, if there exists $\varepsilon > 0$ such that

$$\max_{i=1,\ldots,n}\left(\sum_{j=1}^{n} \gamma_{i,j} \right) < 1 \text{ or } \max_{i=1,\ldots,n}\left(\frac{\sum_{j=1}^{n}(\varepsilon + \gamma_{i,j}) \sum_{k=1}^{n}(\varepsilon + \gamma_{j,k})}{n\varepsilon + \sum_{j=1}^{n}\gamma_{i,j}} \right) < 1$$

then the spectral radius of Γ is less than 1. The above conditions can be used for large-scale systems easily.

It should be emphasized that the novelty of Theorem 2.3 with respect to existing results lies in the presence of deterministic uncertainty and the exploitation of the TR.

Proof: Let $x_0 \in S$, $\{d_i \in D\}_{i=0}^{\infty}$ be given and consider the solution $x(t)$ of (2.1) with $x(0) = x_0$ corresponding to $\{d_i \in D\}_{i=0}^{\infty}$. Let $m \ge 0$ be the integer in Definition 2.1. Let $j \in \{0,\ldots,m\}$ be the smallest integer for which it holds that $x(t) \in A$ for all $t \ge j$ (the fact that j exists and satisfies $j \in \{0,\ldots,m\}$ is a consequence of the fact that A is a TR for (2.1)). We next show that there exists $b \in K_\infty$ such that

$$\max_{k=0,\ldots,j} \left| x(k) - x^* \right| \le b\left(\left| x_0 - x^* \right| \right) \tag{2.4}$$

Indeed, if there exists a constant $L \ge 0$ such that $\max\left\{ \left| F(d,x) - x^* \right| : d \in D \right\} \le L \left| x - x^* \right|$ for all $x \in S \setminus A$, then we may define $b(s) := \max(1, L^m)s$ for all $s \ge 0$. The fact that (2.4) holds is a consequence of the fact that $j \le m$, equation (2.1) and the resulting inequality $\left| x(t+1) - x^* \right| \le \max(1, L)\left| x(t) - x^* \right|$ which holds for all $t = 0,\ldots,j-1$, for the case that $j \ge 1$.

For the general case, we define $F(d,x) = x^*$ for all $d \in D, x \in \Re^n \setminus S$ and

$$a(s) := \sup\left\{ \left| F(d,x) - x^* \right| : (d,x) \in D \times \Re^n, \left| x - x^* \right| \le s \right\}. \tag{2.5}$$

Clearly, $a(s)$ is well-defined by (2.5) for all $s \ge 0$, since F is a locally bounded mapping. Continuity of F on the set $D \times \{x^*\}$ in conjunction with the fact that $D \subseteq \Re^l$ is a compact set with $F(d,x^*) = x^*$ for all $d \in D$ implies that

$$\lim_{s \to 0^+} a(s) = a(0) = 0 . \tag{2.6}$$

By virtue of Lemma 2.4 on page 65 in [13] there exists $\bar{a} \in K_\infty$ such that $s + a(s) \le \bar{a}(s)$ for all $s \ge 0$. We define:

$$b := \underbrace{\bar{a} \circ \ldots \circ \bar{a}}_{m \text{ times}} \tag{2.7}$$

Definition (2.7) shows that $b \in K_\infty$. Using the fact that $j \le m$, inequality $\left| x(t+1) - x^* \right| \le \bar{a}(\left| x(t) - x^* \right|)$ which holds for all $t = 0,\ldots,j-1$ (a consequence of (2.5) and (2.1)) for the case that $j \ge 1$ and definition (2.7), we obtain (2.4). When $j = 0$, then (2.4) holds automatically.

Since $\rho(\Gamma) < 1$ there exist $M \ge 1$, $\sigma > 0$ such that

$$\left| \Gamma^t \right| \le M \exp(-\sigma t), \text{ for all integers } t \ge 0 \tag{2.8}$$

(see [27], page 212 and page 231). Next define:

$$\xi(t) = (V_1(x(t+j)),\ldots,V_l(x(t+j)))' \in \Re_+^l \text{ for all } t \ge 0 . \tag{2.9}$$

Equation (2.1) in conjunction with inequalities (2.3) imply that the following recursive relation holds for all $t \ge 0$:

$$\xi(t+1) \le \Gamma \xi(t) \tag{2.10}$$

Using the fact that Γ is a non-negative matrix (and consequently satisfies $\Gamma x \le \Gamma y$ for all vectors $x, y \in \Re^l$ with $x \le y$), we obtain from (2.10):

$$\xi(t) \le \Gamma^t \xi(0), \text{ for all } t \ge 0 \tag{2.11}$$

Using (2.8), (2.11), definition (2.9) and (2.2), we get:

$$a_1\left(\left| x(j+t) - x^* \right| \right) \le M \exp(-\sigma t)\sqrt{l}\, a_2\left(\left| x(j) - x^* \right| \right), \text{ for all } t \ge 0 \tag{2.12}$$

Using (2.4) and (2.12), we get:

$$a_1\left(\left| x(j+t) - x^* \right| \right) \le M \exp(-\sigma t)\sqrt{l}\, a_2\left(b\left(\left| x_0 - x^* \right| \right) \right), \text{ for all } t \ge 0 \tag{2.13}$$

Since $a_1(s) \le a_2(s)$ for all $s \ge 0$, and since $M \ge 1$, $j \le m$, it follows from (2.4), (2.13) that the following estimate holds for all $t \ge 0$:

$$a_1\left(\left| x(t) - x^* \right| \right) \le M \exp(-\sigma(t-m))\sqrt{l}\, a_2\left(b\left(\left| x_0 - x^* \right| \right) \right) \tag{2.14}$$

Inequality (2.14) shows that the estimate $\left| x(t) - x^* \right| \le \sigma\left(\left| x_0 - x^* \right|, t \right)$ holds for all $t \ge 0$ with $\sigma(s,t) := a_1^{-1}\left(M \exp(-\sigma(t-m))\sqrt{l}\, a_2(b(s)) \right)$ (notice that $\sigma \in KL$) and consequently $x^* \in S$ is RGAS for system (2.1). If there exist constants $L \ge 0$, $0 < K_1 \le K_2$, $p > 0$ such that $\max\left\{ \left| F(d,x) - x^* \right| : d \in D \right\} \le L \left| x - x^* \right|$ for all $x \in S \setminus A$ and if $a_i(s) = K_i s^p$ ($i = 1,2$) for all $s \ge 0$ then inequality (2.13) implies that

$$K_1 \left| x(j+t) - x^* \right|^p \le M\sqrt{l}\, K_2 \exp(-\sigma t) \max\left(1, L^{pm}\right)\left| x_0 - x^* \right|^p, \text{ for all } t \ge 0 \tag{2.15}$$

Here we have used the fact that (2.4) holds with $b(s) := \max(1, L^m)s$ for all $s \ge 0$. It follows from (2.15), (2.4) with $b(s) := \max(1, L^m)s$ and the facts that $j \le m$, $0 < K_1 \le K_2$ that the following estimate holds for all $t \ge 0$:

$$K_1 \left| x(t) - x^* \right|^p \le M\sqrt{l}\, K_2 \exp(-\sigma(t-m)) \max\left(1, L^{pm}\right)\left| x_0 - x^* \right|^p$$

which directly implies that $x^* \in S$ is RGES for system (2.1). The proof is complete. $\triangleleft$

III. Global Stability Results For Traffic Networks

This section is devoted to the derivation of sufficient conditions that guarantee RGES for the equilibrium point of a traffic network. We consider a generic traffic network which consists of n components (see Section I for several specific instances of the generic model). The number of vehicles at time $t \geq 0$ in component $i \in \{1,...,n\}$ is denoted by $x_i(t)$. The outflow and the inflow of vehicles of the component $i \in \{1,...,n\}$ at time $t \geq 0$ are denoted by $q_i(t) \geq 0$ and $F_i(t) \geq 0$, respectively. All flows during a time interval are measured in [veh]. Consequently, the balance of vehicles for each component $i \in \{1,...,n\}$ gives:

$$x_i(t+1) = x_i(t) - q_i(t) + F_i(t)\,,\ i = 1,...,n\,,\ t \geq 0\,. \quad (3.1)$$

Each component of the network has storage capacity $a_i > 0$ ($i = 1,...,n$). Our first assumption states that the inflow of vehicles at the cell $i \in \{1,...,n\}$ at time $t \geq 0$, denoted by $F_i(t) \geq 0$, cannot exceed the number of free positions for vehicles of cell $i \in \{1,...,n\}$ at time $t \geq 0$, i.e.,

$$F_i(t) = \min\left(c_i(a_i - x_i(t)), \widetilde{F}_i(t)\right),\ i = 1,...,n\,,\ t \geq 0 \quad (3.2)$$

where $\widetilde{F}_i(t) \geq 0$ is the attempted inflow of vehicles at the component $i \in \{1,...,n\}$ at time $t \geq 0$ and $c_i \in (0,1]$ ($i = 1,...,n$) are constants.

Our second assumption is dealing with the attempted outflows and inflows. We assume that there exist functions $f_i \in C^0(D \times [0,a_i]; \Re_+)$ with $f_i(d,x_i) \leq x_i$ for all $(d,x_i) \in D \times [0,a_i]$, where $D \subseteq \Re^l$ is a non-empty, compact set, constants $p_{i,j} \geq 0$, $i,j = 1,...,n$, with $p_{i,i} = 0$ for $i = 1,...,n$, and constants $Q_i \geq 0$, $i = 1,...,n$ so that:

$$\begin{pmatrix} \textit{attempted flow of vehicles} \\ \textit{from component } i \textit{ to component } j \end{pmatrix} = p_{i,j} f_i(d,x_i)\,,$$
$$\text{for } i,j = 1,...,n \quad (3.3)$$

$$\begin{pmatrix} \textit{attempted flow of vehicles from} \\ \textit{component } i \textit{ to regions out of the network} \end{pmatrix} = Q_i f_i(d,x_i)\,,$$
$$\text{for } i = 1,...,n \quad (3.4)$$

We also assume that:

$$\sum_{j=1}^{n} p_{i,j} + Q_i = 1\,. \quad (3.5)$$

Some explanations are needed at this point. The function $f_i : D \times [0,a_i] \to \Re_+$ is what in the specialized literature of Traffic Engineering is called the demand-part of the fundamental diagram of the i-th cell, i.e. the flow that will exit the cell if there is sufficient space in the downstream cells; while $p_{i,j}$ are turning rates and Q_i are exit rates. The uncertainty $d \in D$ has been introduced in order to accommodate the uncertain nature of the fundamental diagram. Finally, equation (3.5) implies that the total attempted outflow from the i-th cell is exactly equal to the demand-part of the fundamental diagram, $f_i(d,x_i)$.

Let $v_i > 0$ ($i = 1,...,n$) denote the attempted inflow to component $i \in \{1,...,n\}$ from the region out of the network. Our assumptions lead us to the following equations:

$$\widetilde{F}_i(t) = v_i + \sum_{j=1}^{n} p_{j,i} f_j(d(t), x_j(t))\,,\ i = 1,...,n\,,\ t \geq 0\,. \quad (3.6)$$

Equations (3.2) and (3.6) imply that the percentage of the attempted inflow of vehicles at cell i at time $t \geq 0$, which becomes actual inflow of vehicles at cell i at time $t \geq 0$, denoted by $s_i(t) \in [0,1]$ for $i = 1,...,n$, $t \geq 0$ is given by:

$$s_i(t) = \frac{\min\left(c_i(a_i - x_i(t)), v_i + \sum_{j=1}^{n} p_{j,i} f_j(d(t), x_j(t))\right)}{v_i + \sum_{j=1}^{n} p_{j,i} f_j(d(t), x_j(t))} \quad (3.7)$$

Our final assumption relates the actual inflows with the outflows. Many rules for the outflows of road links have been proposed in the literature; see for example [6, 15, 19, 20]. Here we employ a similar rule to the so-called proportional priority, first-in-first-out (PP/FIFO) rule for junctions (see [6, 15]). We assume that, if cell i cannot accommodate all inflows, then the actual inflows from other cells of the network (or from regions out of the network) to cell i are equal percentages of the attempted inflows, i.e.

$$\begin{pmatrix} \textit{actual flow of vehicles} \\ \textit{from component } j \textit{ to component } i \end{pmatrix} =$$
$$s_i(t) \begin{pmatrix} \textit{attempted flow of vehicles} \\ \textit{from component } j \textit{ to component } i \end{pmatrix}$$
$$i,j = 1,...,n\,. \quad (3.8)$$

Other assumptions could be accommodated in this modeling framework if required. Combining (3.3) with (3.8) we get:

$$\begin{pmatrix} \textit{actual flow of vehicles} \\ \textit{from component } j \textit{ to component } i \end{pmatrix} = s_i(t) p_{j,i} f_j(d,x_j)\,,$$
$$i,j = 1,...,n\,. \quad (3.9)$$

Moreover, we assume that the actual flow of vehicles from cell $i \in \{1,...,n\}$ to regions out of the network is equal to the corresponding attempted flow of vehicles. Thus, the outflow $q_i(t) \geq 0$ from cell $i \in \{1,...,n\}$ is:

$$q_i(t) = \left(Q_i + \sum_{j=1}^{n} s_j(t) p_{i,j}\right) f_i(d(t), x_i(t)) \quad (3.10)$$

Combining equations (3.1), (3.2), (3.6), (3.7) and (3.10) we obtain the following discrete-time dynamical system:

$$x_i^+ = x_i + s_i\left(v_i + \sum_{j=1}^{n} p_{j,i} f_j(d,x_j)\right) - \left(Q_i + \sum_{j=1}^{n} s_j p_{i,j}\right) f_i(d,x_i)\,,$$
$$\text{for } i = 1,...,n \quad (3.11)$$

Define $S=[0,a_1]\times\ldots\times[0,a_n]$. Since the functions f_i satisfy $f_i(d,x_i)\le x_i$ for all $(d,x_i)\in D\times[0,a_i]$, it follows that (3.11) is an (uncertain) dynamical system on S.

A component $i\in\{1,\ldots,n\}$ of the traffic network (3.11) is said to be "congested" at time t if $c_i(a_i-x_i(t))<v_i+\sum_{j=1}^{n}p_{j,i}f_j(d(t),x_j(t))$ (or, equivalently, if $s_i(t)<1$). In this case, the actual inflow to component $i\in\{1,\ldots,n\}$ is less than the attempted inflow. We will assume next that there exists an equilibrium point for which no congestion phenomena are present: the uncongested equilibrium point of the network.

(H) *The matrix* $P=\{p_{i,j}:i,j=1,\ldots,n\}$ *satisfies* $\det(I-P')\neq 0$. *There exists a point* $x^*=(x_1^*,\ldots,x_n^*)'\in S$ *that satisfies the following for all* $d\in D$ *and* $i=1,\ldots,n$*:*

$$v_i+c_ix_i^*+\sum_{j=1}^{n}p_{j,i}f_j(d,x_j^*)\le c_ia_i \tag{3.12}$$

$$f_i(d,x_i^*)=f_i^*=v_i+\sum_{j=1}^{n}p_{j,i}f_j(d,x_j^*) \tag{3.13}$$

We are now in a position to prove the following theorem.

Theorem 3.1: *Consider system (3.7), (3.11) under assumption (H). Assume that there exist constants* $L\ge 0$, $0\le \underline{b}_i<\bar{b}_i\le a_i$ $\lambda_i,\mu_i\ge 0$, $\omega_i\in[x_i^*,a_i)$ $(i=1,\ldots,n)$ *such that the set* $A=[\underline{b}_1,\bar{b}_1]\times\ldots\times[\underline{b}_n,\bar{b}_n]$ *is a TR for system (3.7), (3.11) and such that the following inequalities hold for all* $i=1,\ldots,n$*:*

$$\left|f_i(d,x_i)-f_i^*\right|\le L\left|x_i-x_i^*\right|,\ \text{for all } (d,x_i)\in D\times[0,a_i] \tag{3.14}$$

$$\left|x_i-x_i^*-G_i(\theta,f_i(d,x_i))f_i(d,x_i)+\min\left(c_i(a_i-x_i),v_i+\sum_{j=1}^{n}p_{j,i}f_j^*\right)\right|\le\lambda_i\left|x_i-x_i^*\right|$$

for all $(d,x_i)\in D\times[\underline{b}_i,\bar{b}_i]$, $\theta=(\theta_1,\ldots,\theta_n)\in[0,1]^n$, $i=1,\ldots,n$ (3.15)

$$\left|f_i^*-f_i(d,x_i)\right|\le\mu_i\left|x_i-x_i^*\right|,$$

for all $(d,x_i)\in D\times[\underline{b}_i,\bar{b}_i]$ and $i=1,\ldots,n$. (3.16)

where $G_i(\theta,y):=Q_i+\sum_{j=1}^{n}\dfrac{\min\left(c_j(a_j-\theta_j\omega_j),v_j+p_{i,j}y+\sum_{k\neq i}p_{k,j}f_k^*\right)}{v_j+p_{i,j}y+\sum_{k\neq i}p_{k,j}f_k^*}p_{i,j}$.

Define $F_i=\max\left\{f_i(d,s):s\in[\underline{b}_i,\bar{b}_i],d\in D\right\}$ $(i=1,\ldots,n)$ *and assume that*

$$f_j^*+p_{i,j}(F_i-f_i^*)\le c_ja_j \ \text{ for all } i,j=1,\ldots,n \tag{3.17}$$

Define the matrix $\Gamma=\{\gamma_{i,j}:i,j=1,\ldots,n\}$ *by:*

$$\gamma_{i,i}:=\lambda_i,\ \text{for } i=1,\ldots,n \tag{3.18}$$

$$\gamma_{i,j}:=\frac{F_ip_{i,j}c_j\max(0,\bar{b}_j-\omega_j)}{\left(f_j^*+p_{i,j}\left(F_i-f_i^*\right)\right)\left(\bar{b}_j-x_j^*\right)}+\left(p_{j,i}+\sum_{k=1}^{n}\frac{F_ip_{i,k}p_{j,k}}{f_k^*+p_{i,k}\left(F_i-f_i^*\right)}\right)\mu_j$$

$$\text{for } i,j=1,\ldots,n \text{ with } i\neq j. \tag{3.19}$$

If $\rho(\Gamma)$ *is less than 1, then* x^* *is RGES for (3.11).*

Remark 3.2:

(a) Assumption (3.17) is not restrictive: since we are studying the properties of the uncongested equilibrium point, the equilibrium flow values f_i^* for $i=1,\ldots,n$ are far smaller than the quantities c_ia_i, and condition (3.17) holds.

(b) It should be pointed out that Theorem 3.1 is based on the estimation of the constants $\lambda_i,\mu_i\ge 0$ ($i=1,\ldots,n$) which satisfy inequalities (3.15), (3.16). The numerical evaluation of the magnitude of the constants $\lambda_i,\mu_i\ge 0$ ($i=1,\ldots,n$) can be performed independently for each cell, no matter how many interconnections are present. This implies that the computational complexity for the evaluation of the constants $\lambda_i,\mu_i\ge 0$ ($i=1,\ldots,n$) is of order n and is independent of the number of interconnections. This feature is important for the analysis of large-scale networks.

Proof of Theorem 3.1: We use Theorem 2.3 for

$$V_i(x):=\left|x_i-x_i^*\right|\ (i=1,\ldots,n) \tag{3.20}$$

and the dynamical system (3.7), (3.11). Since the inequality

$$\frac{1}{\sqrt{n}}\left|x-x^*\right|\le\max_{i=1,\ldots,n}\left(V_i(x)\right)\le\left|x-x^*\right|$$

holds for all $x\in A$ and since (3.14) implies the condition $\max\left\{\left|F(d,x)-x^*\right|:d\in D\right\}\le\tilde{L}\left|x-x^*\right|$ for all $x\in S$, for certain constant $\tilde{L}\ge 0$, where $F(d,x)=(F_1(d,x),\ldots,F_n(d,x))'\in\Re^n$ and

$$F_i(d,x):=x_i+s_i\left(v_i+\sum_{j=1}^{n}p_{j,i}f_j(d,x_j)\right)-\left(Q_i+\sum_{j=1}^{n}s_jp_{i,j}\right)f_i(d,x_i)$$

it suffices to show that (2.3) holds for all $x\in A$, $i=1,\ldots,n$.

The rest part of proof is devoted to the proof of (2.3). Indeed, using (3.7), (3.20) we get for all $(d,x)\in D\times A$, $\theta\in[0,1]^n$ and $i=1,\ldots,n$:

$$\begin{aligned}V_i(F(d,x))=\left|x_i^+-x_i^*\right|&\le f_i(d,x_i)\sum_{j=1}^{n}p_{i,j}\left|w_{i,j}\right|\\&+\left|x_i-x_i^*-G_i\left(\theta,f_i(d,x_i)\right)f_i(d,x_i)+\min\left(c_i(a_i-x_i),f_i^*\right)\right|\\&+\left|s_i\left(v_i+\sum_{j=1}^{n}p_{j,i}f_j(d,x_j)\right)-\min\left(c_i(a_i-x_i),f_i^*\right)\right|\end{aligned} \tag{3.21}$$

where

$$w_{i,j} := \frac{\min\left(c_j(a_j - x_j), v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)\right)}{v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)} - \frac{\min\left(c_j(a_j - \theta_j\omega_j), v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*} \tag{3.22}$$

Using (3.15), (3.16) and the fact that $\left|\min(a,x) - \min(a,y)\right| \leq |x-y|$ for all $a,x,y \in \Re$, we obtain from (3.21) for all $(d,x) \in D \times A$, $\theta \in [0,1]^n$ and $i = 1,...,n$:

$$V_i(F(d,x)) \leq \lambda_i \left|x_i - x_i^*\right| + f_i(d,x_i) \sum_{j=1}^{n} p_{i,j} \left|w_{i,j}\right| + \sum_{j=1}^{n} p_{j,i} \mu_j \left|x_j - x_j^*\right| . \tag{3.23}$$

We next show that for every $(d,x) \in D \times A$ and $i = 1,...,n$ we can select $\theta_j \in [0,1]$ in a way so that we can minimize the values of $\left|w_{i,j}\right|$ ($j = 1,...,n$). Continuity of the mapping $[0,1] \ni \theta_j \to \min\left(\frac{c_j(a_j - \theta_j \omega_j)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*}, 1\right)$ implies the existence of $\theta_j \in [0,1]$ with $w_{i,j} = 0$, provided that:

$$\min\left(\frac{c_j(a_j - \omega_j)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*}, 1\right) \leq s_j \leq \min\left(\frac{c_j a_j}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*}, 1\right) \tag{3.24}$$

where s_j is defined in (3.7). If (3.24) does not hold, then $\min\left(\frac{c_j(a_j - \omega_j)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*}, 1\right) > s_j$. This follows from $\min\left(\frac{c_j a_j}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*}, 1\right) = 1$ (a consequence of (3.13), (3.17)) and $s_j \leq 1$. Consequently, (3.22) implies

$$\left|w_{i,j}\right| = \frac{\min\left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*} - \frac{\min\left(c_j(a_j - x_j), v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)\right)}{v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)}$$

when (3.24) does not hold. Moreover, since $s_j = \frac{\min\left(c_j(a_j - x_j), v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)\right)}{v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)} < 1$, we get $c_j(a_j - x_j) < v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)$ and consequently,

$$\left|w_{i,j}\right| = \frac{\min\left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*} - \frac{c_j(a_j - x_j)}{v_j + \sum_{k=1}^{n} p_{k,j} f_k(d,x_k)} \tag{3.25}$$

provided that (3.24) does not hold.

Hence, when (3.24) does not hold, we have from (3.25), (3.16) and (3.24):

$$\left|w_{i,j}\right| \left(v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^* + \sum_{k \neq i} p_{k,j} \mu_k \left|x_k - x_k^*\right| \right) \leq$$
$$\min\left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right) - c_j\left(a_j - x_j\right)$$
$$+ \frac{\min\left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right)}{v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*} \sum_{k \neq i} p_{k,j} \mu_k \left|x_k - x_k^*\right|$$

Using (3.13), the inequalities $x_j - \omega_j \leq \max(0, x_j - \omega_j)$ and $\min\left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d,x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right) \leq c_j(a_j - \omega_j)$, we get:

$$\left|w_{i,j}\right| \leq \frac{c_j \max(0, x_j - \omega_j) + \sum_{k \neq i} p_{k,j} \mu_k \left|x_k - x_k^*\right|}{f_j^* + p_{i,j}\left(f_i(d,x_i) - f_i^*\right) + \sum_{k \neq i} p_{k,j} \mu_k \left|x_k - x_k^*\right|}$$

The above inequality holds when (3.24) holds as well. Using the above inequality in conjunction with (3.23), we obtain for all $(d,x) \in D \times A$ and $i = 1,...,n$:

$$V_i(F(d,x)) \leq \lambda_i \left|x_i - x_i^*\right| + \sum_{j=1}^{n} p_{j,i} \mu_j \left|x_j - x_j^*\right| + f_i(d,x_i) \sum_{j=1}^{n} p_{i,j} \frac{c_j \max(0, x_j - \omega_j) + \sum_{k \neq i} p_{k,j} \mu_k \left|x_k - x_k^*\right|}{f_j^* + \sum_{k \neq i} p_{k,j} \mu_k \left|x_k - x_k^*\right| + p_{i,j}\left(f_i(d,x_i) - f_i^*\right)} . \tag{3.26}$$

Using the facts that $x_j \in [\underline{b}_j, \bar{b}_j]$ and $\omega_j \ge x_j^*$, we obtain $\max(0, x_j - \omega_j) \le \frac{\max(0, \bar{b}_j - \omega_j)}{\bar{b}_j - x_j^*}\left|x_j - x_j^*\right|$. Therefore, we obtain from (3.26) for all $(d,x) \in D \times A$ and $i = 1,...,n$:

$$
\begin{aligned}
V_i(F(d,x)) &\le \lambda_i\left|x_i - x_i^*\right| + \sum_{j=1}^{n} p_{j,i}\mu_j\left|x_j - x_j^*\right| \\
&+ \sum_{j=1}^{n} \frac{p_{i,j} f_i(d,x_i) c_j \max(0, \bar{b}_j - \omega_j)}{\left(f_j^* + p_{i,j}\left(f_i(d,x_i) - f_i^*\right)\right)\left(\bar{b}_j - x_j^*\right)}\left|x_j - x_j^*\right| \qquad (3.27) \\
&+ \sum_{j=1}^{n} \frac{p_{i,j} f_i(d,x_i)}{f_j^* + p_{i,j}\left(f_i(d,x_i) - f_i^*\right)} \sum_{k \ne i} p_{k,j}\mu_k\left|x_k - x_k^*\right|
\end{aligned}
$$

Finally, using definitions (3.20), the fact that $\frac{p_{i,j} f_i(d,x_i)}{f_j^* + p_{i,j}\left(f_i(d,x_i) - f_i^*\right)} \le \frac{p_{i,j} F_i}{f_j^* + p_{i,j}\left(F_i - f_i^*\right)}$ for all $(d,x) \in D \times A$, where $F_i = \max\left\{f_i(d,s) : s \in [\underline{b}_i, \bar{b}_i], d \in D\right\}$ and the fact that $p_{i,i} = 0$, we obtain (2.3). ◁

Example 3.3: Consider the traffic network shown in Figure 1, for which the matrix $P = \left\{p_{i,j} : i,j = 1,...,5\right\}$ is:

$$
P = \begin{bmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & \tilde{p} \\ p & 0 & 0 & \tilde{p} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.28)
$$

where $p, \tilde{p} > 0$ are constants with $p + \tilde{p} \le 1$. The external inflows and the capacities of the network are:

$$
\begin{gathered}
v_1 = v_2 = v_3 = v > 0\,,\ v_4 = v_5 = \tilde{v} > 0\,, \\
c_i = 1\,,\ a_i = a > 0\ \ (i = 1,...,5) \qquad (3.29)
\end{gathered}
$$

where $v, \tilde{v}, a > 0$ are constants. Finally, we assume that all functions f_i ($i = 1,...,5$) are given by:

$$
f_i(x) = f(x) := \begin{cases} rx \text{ for } x \in [0,\delta] \\ r\delta - q(x - \delta) \text{ for } x \in (\delta, a] \end{cases} \quad (i = 1,...,5) \qquad (3.30)
$$

where $\delta \in (0,a)$, $r \in (0,1]$, $q \in \left[0, \delta r/(a - \delta)\right]$ are constants. Note that the lower part of the right-hand side of (3.30) allows for the modeling of capacity drop at the outflow of congestion according to [18]. The network has the (uncongested) equilibrium point

$$
x^* = \left(c, c, c, \kappa, \kappa\right) \qquad (3.31)
$$

where $c := v/(r(1-p))$, $\kappa := (\tilde{v}(1-p) + \tilde{p}v)/(r(1-p))$, which satisfies (H) provided that $c \le \delta$, $c(r+1) \le a$, $\kappa \le \delta$ and $(r+1)\kappa \le a$. We next apply Theorem 3.1 under the assumption

$$
v + pr\delta \le a \ \text{ and } \ \tilde{v} + \tilde{p}r\delta \le a \qquad (3.32)
$$

with $A = S = [0,a]^5$. Assumption (3.32) is assumption (3.17) for the given network. The matrix Γ is equal to:

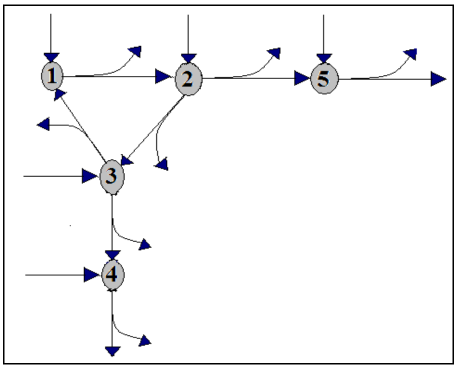

Fig. 1: The traffic network of Example 3.3

$$
\Gamma = \begin{bmatrix} \lambda_1 & (a - \omega_2)\varphi & p\mu & 0 & 0 \\ p\mu & \lambda_2 & (a - \omega_3)\varphi & 0 & (a - \omega_5)\zeta \\ (a - \omega_1)\varphi & p\mu & \lambda_3 & (a - \omega_4)\zeta & 0 \\ 0 & 0 & \tilde{p}\mu & \lambda_4 & 0 \\ 0 & \tilde{p}\mu & 0 & 0 & \lambda_5 \end{bmatrix} \qquad (3.33)
$$

where

$$
\varphi = \frac{pr^2\delta(1-p)}{(pr\delta + v)(ar(1-p) - v)}\,,\ \zeta = \frac{r^2\delta\tilde{p}(1-p)}{(\tilde{v} + \tilde{p}r\delta)(ar(1-p) - \tilde{v}(1-p) - \tilde{p}v)}
$$

$$
\mu := \sup\left\{ r\frac{\left|v - (1-p)f(s)\right|}{\left|rs(1-p) - v\right|} : s \in [0,a], s \ne \frac{v}{r(1-p)} \right\} \qquad (3.34)
$$

$$
\lambda_4 = \lambda_5 = \sup\left\{ \frac{\left|s - \kappa - f(s) + \min(a - s, r\kappa)\right|}{\left|s - \kappa\right|} : s \in [0,a], s \ne \kappa \right\} \qquad (3.35)
$$

$$
\lambda_i = \max(u_i, w_i)\,,\ i = 1,2,3 \qquad (3.36)
$$

where $h(\omega, s) := \left(1 - p + \frac{\min(a - \omega, v + pf(s))}{v + pf(s)}p\right)f(s)$,

$$
\tilde{h}(\omega, s) := \left(1 - \tilde{p} + \frac{\min(a - \omega, \tilde{v} + \tilde{p}f(s))}{\tilde{v} + \tilde{p}f(s)}\tilde{p}\right)f(s)\,,
$$

$g(s) := s - c + \min(a - s, cr)$, $B = [0,a] \setminus \{c\}$,

$$
u_1 := \sup\left\{ \frac{g(s) - h(\omega_2, s)}{\left|s - c\right|} : s \in B \right\}
$$

$$
w_1 = \sup\left\{ \frac{h(0,s) - g(s)}{\left|s - c\right|} : s \in B \right\}
$$

$$
u_2 = \sup\left\{ \frac{g(s) + f(s) - h(\omega_3, s) - \tilde{h}(\omega_5, s)}{\left|s - c\right|} : s \in B \right\}
$$

$$
w_2 = w_3 = \sup\left\{ \frac{h(0,s) - f(s) + \tilde{h}(0,s) - g(s)}{\left|s - c\right|} : s \in B \right\}
$$

$$
u_3 = \sup\left\{ \frac{g(s) + f(s) - h(\omega_1, s) - \tilde{h}(\omega_4, s)}{\left|s - c\right|} : s \in B \right\}
$$

and $\omega_i \in \left[c, a\right)$ $(i = 1,2,3)$, $\omega_i \in \left[\kappa, a\right)$ $(i = 4,5)$ are constants. For $a = 10$, $v = 0.4$, $\tilde{v} = 0.4$, $\delta = 5$, $p = 0.2$,

$\widetilde{p} = 0.1$, $r = 0.55$, $q = 0.1$, the selection $\omega_1 = 9.14$, $\omega_2 = 8.53$, $\omega_3 = 9.559$, $\omega_4 = 9.37$, $\omega_5 = 9.329$ gives:

$$\Gamma = \begin{bmatrix} 0.7905 & 0.0281 & 0.11 & 0 & 0 \\ 0.11 & 0.8166 & 0.0281 & 0 & 0.0298 \\ 0.0548 & 0.11 & 0.7905 & 0.028 & 0 \\ 0 & 0 & 0.055 & 0.7869 & 0 \\ 0 & 0.055 & 0 & 0 & 0.7869 \end{bmatrix}$$

Since $\max_{i=1,...,5}\left(\sum_{j=1}^{5}\gamma_{i,j}\right) = 0.9845 < 1$, we can conclude that $\rho(\Gamma) < 1$ and consequently, Theorem 3.1 implies that the (uncongested) equilibrium point is GES. $\triangleleft$

IV. Global Exponential Stability For Freeways

A freeway divided in $n \geq 3$ sections or cells is a traffic network of the form (3.7), (3.11) with $p_{i,j} = 0$ for all $i, j = 1,...,n$ with $j \neq i+1$. Defining $p_{i,i+1} = p_i$ for $i = 1,...,n-1$ and if we further suppose that $v_i = 0$ for $i = 2,...,n$, $p_i = 1$, $f_i(d, x_i) = f_i(x_i)$ for $i = 1,...,n$, $v_1 = v > 0$, we obtain from (3.5), (3.7) and (3.11):

$$x_1^+ = x_1 - \min\left(c_2(a_2 - x_2), f_1(x_1)\right) + \min\left(c_1(a_1 - x_1), v\right) \quad (4.1)$$

$$x_i^+ = x_i - \min\left(c_{i+1}(a_{i+1} - x_{i+1}), f_i(x_i)\right) + \min\left(c_i(a_i - x_i), f_{i-1}(x_{i-1})\right), \text{ for } i = 2,...,n-1 \quad (4.2)$$

$$x_n^+ = x_n - f_n(x_n) + \min\left(c_n(a_n - x_n), f_{n-1}(x_{n-1})\right). \quad (4.3)$$

Again $c_i \in (0,1]$, $f_i \in C^0([0,a_i]; \Re_+)$ ($i = 1,...,n$) are functions with $f_i(s) \leq s$ for all $s \in [0, a_i]$. We suppose that there exists a vector $x^* = (x_1^*,...,x_n^*) \in [0,a_1] \times \ldots [0,a_n]$ with $f_i(x_i^*) = v$ and $c_i x_i^* + v < c_i a_i$ ($i = 1,...,n$). It follows that assumption (H) holds for the equilibrium point $x^* \in \Re^n$. The following corollary is a direct consequence of Theorem 3.1 (although Theorem 3.1 was applied to the model (3.7), (3.11) which required $v_i > 0$ for $i = 1,...,n$, all arguments in the proof of Theorem 3.1 can be repeated).

Corollary 4.1: *Consider (4.1), (4.2), (4.3) with $n \geq 3$. Assume that there exist constants $0 \leq \underline{b}_i < \bar{b}_i \leq a_i$ ($i = 1,...,n$) such that the set $A = [\underline{b}_1, \bar{b}_1] \times \ldots \times [\underline{b}_n, \bar{b}_n]$ is a TR for (4.1), (4.2), (4.3). Moreover, assume that there exists $L \geq 0$ such that the following inequalities hold for $i = 1,...,n$:*

$$\left|f_i(x) - v\right| \leq L\left|x - x_i^*\right|, \text{ for all } x \in [0, a_i]. \quad (4.4)$$

Furthermore, assume that there exist constants $\lambda_i \geq 0$ ($i = 1,...,n$), $\mu_i \geq 0$ ($i = 1,...,n-1$), $\omega_i \in [x_i^*, a_i)$ ($i = 2,...,n$) such that

$$\left|s - x_i^* - \min\left(c_{i+1}(a_{i+1} - \omega_{i+1}), f_i(s)\right) + \min\left(c_i(a_i - s), v\right)\right| \leq \lambda_i\left|s - x_i^*\right|, \text{ for } s \in [\underline{b}_i, \bar{b}_i],\ i = 1,...,n-1 \quad (4.5)$$

$$\left|s - x_i^* - \min\left(c_{i+1}a_{i+1}, f_i(s)\right) + \min\left(c_i(a_i - s), v\right)\right| \leq \lambda_i\left|s - x_i^*\right|, \text{ for } s \in [\underline{b}_i, \bar{b}_i],\ i = 1,...,n-1 \quad (4.6)$$

$$\left|s - x_n^* - f_n(s) + \min\left(c_n(a_n - s), v\right)\right| \leq \lambda_n\left|s - x_n^*\right|, \text{ for all } s \in [\underline{b}_n, \bar{b}_n] \quad (4.7)$$

$$\left|v - f_i(s)\right| \leq \mu_i\left|s - x_i^*\right|, \text{ for all } s \in [\underline{b}_i, \bar{b}_i],\ i = 1,...,n-1. \quad (4.8)$$

Assume that $\max\left\{f_i(s) : s \in [\underline{b}_i, \bar{b}_i]\right\} \leq c_{i+1}a_{i+1}$ for all $i = 1,...,n-1$. Define the tridiagonal matrix $\Gamma = \left\{\gamma_{i,j} : i, j = 1,...,n\right\}$:

$$\gamma_{i,i} := \lambda_i, \text{ for } i = 1,...,n \quad (4.9)$$

$$\gamma_{i,i+1} := \frac{c_{i+1}\max(0, \bar{b}_{i+1} - \omega_{i+1})}{\bar{b}_{i+1} - x_i^*} \text{ for } i = 1,...,n-1 \quad (4.10)$$

$$\gamma_{i,i-1} := \mu_{i-1}, \text{ for } i = 2,...,n. \quad (4.11)$$

If $\rho(\Gamma) < 1$, then x^* is GES for (4.1), (4.2), (4.3).

Corollary 4.1 shows that the TR is crucial for the stability properties of the system (4.1), (4.2), (4.3). Indeed, if $\omega_i \geq \bar{b}_i$ for $i = 2,...,n$ then $\rho(\Gamma) = \max_{i=1,...,n}(\lambda_i)$ (because in this case Γ is lower triangular). The crudest TR that can be used is $A = [0, a_1] \times \ldots [0, a_n]$. However, we can generate "smaller" TRs by means of the following proposition.

Proposition 4.2: *Suppose that that there exist constants $0 \leq \underline{b}_i < \bar{b}_i \leq a_i$ ($i = 1,...,n$) such that the set $A = [\underline{b}_1, \bar{b}_1] \times \ldots \times [\underline{b}_n, \bar{b}_n]$ is a TR for (4.1), (4.2), (4.3) with $n \geq 3$. Let $i \in \{1,...,n\}$, $\delta \in [0, \bar{b}_i]$ be a constant such that one of the following hold:*

If $i = 1$ and $\delta \geq x_1^*$ then

$$\min_{\delta \leq s \leq \bar{b}_1}\left(\min\left(c_2(a_2 - \bar{b}_2), f_1(s)\right) - \min\left(c_1(a_1 - s), v\right)\right) > 0$$

$$\text{and } \max_{\underline{b}_1 \leq s \leq \delta}\left(s - \min\left(c_2(a_2 - \bar{b}_2), f_1(s)\right) + \min\left(c_1(a_1 - s), v\right)\right) \leq \delta. \quad (4.12)$$

If $i \in \{2,...,n-1\}$ and $\delta \geq x_i^*$ then

$$\min_{\delta \leq s \leq \bar{b}_i}\left(\min\left(c_{i+1}(a_{i+1} - \bar{b}_{i+1}), f_i(s)\right) - \min\left(c_i(a_i - s), F_{i-1}\right)\right) > 0 \text{ and}$$

$$\max_{\underline{b}_i \leq s \leq \delta}\left(s - \min\left(c_{i+1}(a_{i+1} - \bar{b}_{i+1}), f_i(s)\right) + \min\left(c_i(a_i - s), F_{i-1}\right)\right) \leq \delta,$$

$$\text{where } F_{i-1} = \max\left\{f_{i-1}(s) : s \in [\underline{b}_{i-1}, \bar{b}_{i-1}]\right\}. \quad (4.13)$$

If $i = n$ and $\delta \geq x_n^*$ then

$$\min_{\delta \leq s \leq \bar{b}_n}\left(f_n(s) - \min\left(c_n(a_n - s), F_{n-1}\right)\right) > 0$$

$$\text{and } \max_{\underline{b}_n \leq s \leq \delta}\left(s - f_n(s) + \min\left(c_n(a_n - s), F_{n-1}\right)\right) \leq \delta,$$

$$\text{where } F_{n-1} = \max\left\{f_{n-1}(s) : s \in [\underline{b}_{n-1}, \bar{b}_{n-1}]\right\}. \quad (4.14)$$

Then the set $B \subseteq A$ which results from the replacement of $\bar{b}_i$ by δ in the formula $[\underline{b}_1, \bar{b}_1] \times \ldots \times [\underline{b}_n, \bar{b}_n]$ is a TR for (4.1), (4.2), (4.3).

Proof: We consider the case $i=1$ (all other cases are similar). We consider the case $\delta < \bar{b}_1$. Notice that since $A=[\underline{b}_1,\bar{b}_1]\times\ldots\times[\underline{b}_n,\bar{b}_n]$ is a TR for (4.1), (4.2), (4.3), there exists $m \geq 0$ such that for every $x_0 \in S$ the solution $x(t)$ of (4.1), (4.2), (4.3) with $x(0)=x_0$ satisfies $x(t)\in A$ for all $t \geq m$. Consequently, (4.1) implies for all $t \geq m$:

$$x_1(t+1) \leq x_1(t) - \min\left(c_2(a_2-\bar{b}_2), f_1(x_1(t))\right) + \min\left(c_1(a_1-x_1(t)), v\right) \tag{4.15}$$

It follows from (4.12) and (4.15) that, if $x_1(t) \leq \delta$ for certain $t \geq m$ then $x_1(t+1)\leq \delta$. Thus, the following property holds:

(P): If there exists $T \geq m$ with $x_1(T)\leq\delta$ then it holds that $x_1(t)\leq\delta$ for all $t \geq T$.

Let $\varepsilon := \min\limits_{\delta\leq s\leq \bar{b}_1}\left(\min\left(c_2(a_2-\bar{b}_2), f_1(s)\right) - \min\left(c_1(a_1-s),v\right)\right) > 0$.

We claim that the solution $x(t)$ of (4.1), (4.2), (4.3) with arbitrary initial condition $x(0)=x_0\in S$ satisfies $x_1(t)\leq\delta$ for all $t \geq m + \left\lfloor \left(\bar{b}_1-\delta\right)/\varepsilon \right\rfloor + 1$. The proof is made by contradiction. Suppose that there exists $x_0 \in S$ and $t \geq m + \left\lfloor \left(\bar{b}_1-\delta\right)/\varepsilon \right\rfloor + 1$ such that $x_1(t) > \delta$. Notice that property (P) guarantees that $x_1(j)>\delta$ for all $j=m,...,t$. It follows from (4.15) and definition $\varepsilon := \min\limits_{\delta\leq s\leq \bar{b}_1}\left(\min\left(c_2(a_2-\bar{b}_2), f_1(s)\right) - \min\left(c_1(a_1-s),v\right)\right) > 0$ that the following inequality holds for all $j=m,...,t$:

$$x_1(j+1) \leq x_1(j) - \varepsilon . \tag{4.16}$$

Inequality (4.16) implies that $x_1(t)\leq x_1(m)-(t-m)\varepsilon$. The previous inequality in conjunction with $x_1(t)>\delta$ and the fact that $x_1(m)\leq\bar{b}_1$ implies $(t-m)\varepsilon < \bar{b}_1 - \delta$ which contradicts the fact that $t \geq m + \left\lfloor \left(\bar{b}_1-\delta\right)/\varepsilon \right\rfloor + 1$. ◁

Using Proposition 4.2 and Corollary 4.1, we can construct an algorithm that provides easily checkable sufficient conditions for the GES of x^*.

Corollary 4.3: *Consider system (4.1), (4.2), (4.3) with $n\geq 3$. Suppose that $0<f_n(s)$ for all $s\in(0,a_n]$, $0<f_i(s)<c_{i+1}a_{i+1}$ for all $s\in(0,a_i]$ and $i=1,...,n-1$. Perform the following algorithm:*

Step 1: Find $k_n \in [x_n^*, a_n)$ such that $\min\limits_{k_n\leq s\leq a_n}\left(f_n(s)-\min\left(c_n(a_n-s),F_{n-1}\right)\right)>0$ and $\max\limits_{0\leq s\leq k_n}\left(s-f_n(s)+\min\left(c_n(a_n-s),F_{n-1}\right)\right)\leq k_n$, where $F_{n-1} := \max\limits_{s\in[0,a_{n-1}]}\left(f_{n-1}(s)\right)$.

Step $n+1-i$, where $i\in\{2,...,n-1\}$: Find $k_i\in[x_i^*,a_i)$ such that $\min\limits_{k_i\leq s\leq a_i}\left(\min\left(c_{i+1}(a_{i+1}-k_{i+1}),f_i(s)\right)-\min\left(c_i(a_i-s),F_{i-1}\right)\right)>0$ and $\max\limits_{0\leq s\leq k_i}\left(s-\min\left(c_{i+1}(a_{i+1}-k_{i+1}),f_i(s)\right)+\min\left(c_i(a_i-s),F_{i-1}\right)\right)\leq k_i$, where $F_{i-1} := \max\limits_{s\in[0,a_{i-1}]}\left(f_{i-1}(s)\right)$.

Step n: Find $\bar{b}_1\in[x_1^*,a_1)$ such that $\min\limits_{\bar{b}_1\leq s\leq a_1}\left(\min\left(c_2(a_2-k_2),f_1(s)\right)-\min\left(c_1(a_1-s),v\right)\right)>0$ and $\max\limits_{0\leq s\leq \bar{b}_1}\left(s-\min\left(c_2(a_2-k_2),f_1(s)\right)+\min\left(c_1(a_1-s),v\right)\right)\leq\bar{b}_1$.

Step $n+i-1$, where $i\in\{2,...,n-1\}$: Find $\bar{b}_i\in[x_i^*,k_i]$ such that $\min\limits_{\bar{b}_i\leq s\leq k_i}\left(\min\left(c_{i+1}(a_{i+1}-k_{i+1}),f_i(s)\right)-\min\left(c_i(a_i-s),F_{i-1}\right)\right)>0$ and $\max\limits_{0\leq s\leq \bar{b}_i}\left(s-\min\left(c_{i+1}(a_{i+1}-k_{i+1}),f_i(s)\right)+\min\left(c_i(a_i-s),F_{i-1}\right)\right)\leq\bar{b}_i$, where $F_{i-1} := \max\limits_{s\in[0,\bar{b}_{i-1}]}\left(f_{i-1}(s)\right)$.

Step $2n-1$: Find $\bar{b}_n\in[x_n^*,k_n]$ such that $\min\limits_{\bar{b}_n\leq s\leq k_n}\left(f_n(s)-\min\left(c_n(a_n-s),F_{n-1}\right)\right)>0$ and $\max\limits_{0\leq s\leq \bar{b}_n}\left(s-f_n(s)+\min\left(c_n(a_n-s),F_{n-1}\right)\right)\leq\bar{b}_n$, where $F_{n-1} := \max\limits_{s\in[0,\bar{b}_{n-1}]}\left(f_{n-1}(s)\right)$.

Assume that there exist $\lambda_i\in[0,1)$ ($i=1,...,n$) such that

$$\left|s-x_i^*-\min\left(c_{i+1}(a_{i+1}-\bar{b}_{i+1}),f_i(s)\right)+\min\left(c_i(a_i-s),v\right)\right|\leq\lambda_i\left|s-x_i^*\right|,$$
$$\text{for } s\in[0,\bar{b}_i],\ i=1,...,n-1 \tag{4.17}$$

$$\left|s-x_i^*-\min\left(c_{i+1}a_{i+1},f_i(s)\right)+\min\left(c_i(a_i-s),v\right)\right|\leq\lambda_i\left|s-x_i^*\right|,$$
$$\text{for } s\in[0,\bar{b}_i],\ i=1,...,n-1 \tag{4.18}$$

$$\left|s-x_n^*-f_n(s)+\min\left(c_n(a_n-s),v\right)\right|\leq\lambda_n\left|s-x_n^*\right|,$$
$$\text{for all } s\in[0,\bar{b}_n] \tag{4.19}$$

Finally, assume that inequalities (4.4) hold for certain constant $L\geq 0$. Then x^* is GES for (4.1), (4.2), (4.3).

All steps of the algorithm can be performed since $0<f_n(s)$ for all $s\in(0,a_n]$, $0<f_i(s)<c_{i+1}a_{i+1}$ for all $s\in(0,a_i]$ and $i=1,...,n-1$.

Example 4.4: Consider the network (4.1), (4.2), (4.3) with $n=5$, $c_i=1$, $a_i=10$ ($i=1,...,5$),

$$f_i(s)=f(s):=\begin{cases} 0.5s & \text{for } s\in[0,5] \\ 3-0.1s & \text{for } s\in(5,10] \end{cases} \quad (i=1,...,4),$$

$$f_5(s):=\begin{cases} 0.4s & \text{for } s\in[0,5] \\ 2-p(s-5) & \text{for } s\in(5,10] \end{cases},\ v=1 \tag{4.20}$$

where $p\in[0,0.4)$. We have $x_i^*=2$ ($i=1,...,4$), $x_5^*=2.5$ and (H) holds. We consider the following question: "For what values of $p\in[0,0.4)$ x^* is GES?". The algorithm of Corollary 4.3 was performed for values of $p\in[0,0.4)$ in the following way: for a given integer $N>0$ a grid of points $s_i = ia/N$ ($i=0,1,...,N$) was generated. Then $\bar{b}_i$ ($i=1,...,5$) and k_i ($i=2,...,5$) were chosen to be the smallest grid points $s_j = ja/N$ so that $\min_{j \le l \le N}\left(q(la/N)\right) > 0$ and $\max_{0 \le l \le j}\left(la - Nq(la/N)\right) < ja$, where:

- $q(s) := f_5(s) - \min(10-s, 2.5)$ and $F := 2.5$ for k_5,
- $q(s) := \min(10-k_{i+1}, f(s)) - \min(10-s, 2.5)$, for k_i ($i = 4,3,2$),
- $q(s) := \min(10-k_2, f(s)) - \min(10-s, 1)$ for $\bar{b}_1$,
- $q(s) := \min(10-k_{i+1}, f(s)) - \min(10-s, F_{i-1})$, $F_{i-1} := \max_{s\in[0,b_{i-1}]}(f(s))$ for $\bar{b}_i$ ($i = 2,3,4$),
- $q(s) := f_5(s) - \min(10-s, F_4)$ and $F_4 := \max_{s\in[0,c_4]}(f(s))$, for $\bar{b}_5$.

For all $N > 0$, there exists $p_N > 0$ such that the assumptions of Corollary 4.3 hold with $\lambda_i = 0.5$ ($i = 1,...,4$) and $\lambda_5 = 0.6$ for all $p \in [0, p_N]$. We obtained $p_{100} = 0.189$, $p_{1000} = 0.244$, $p_{2000} = 0.247$, indicating a sequence that tends to 0.25 as $N \to +\infty$. For $p = 0.25$ there exist additional equilibria and therefore, x^* cannot be GES. The results show that the sufficient conditions of Corollary 4.3 are virtually exact in this case. $\triangleleft$

V. Concluding Remarks

Sufficient conditions for GAS and GES have been given, by means of vector Lyapunov functions. The results were applied to traffic networks for the derivation of sufficient conditions of GES of the uncongested equilibrium point. Specific results were provided for freeway models.

The results of the present paper can be used for different purposes for future research:

- for the derivation of feedback laws which stabilize the uncongested equilibrium point,
- for the study of the dynamic behavior of traffic networks under the effect of external disturbances (varying inflows),
- for the study of complicated freeway models divided in $n \ge 3$ cells, each with one on-ramp and one off-ramp.

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References

[1] Aboudolas, K., M. Papageorgiou and E. Kosmatopoulos, "Store-and-Forward Based Methods for the Signal Control Problem in Large-Scale Congested Urban Road Networks", *Transportation Research Part C*, 17(2), 2009, 163-174.

[2] Aboudolas, K. and N. Geroliminis, "Perimeter and Boundary Flow Control in Multi-Reservoir Heterogeneous Networks", *Transportation Research Part B*, 55, 2013, 265-281.

[3] Berman, A. and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM Classics in Applied Mathematics, 1994.

[4] Buisson, C., J.-P. Lebacque and J.B. Lesort, "STRADA, a Discretized Macroscopic Model of Vehicular Traffic Flow in Complex Networks Based on the Godunov Scheme", *CESA'96 IMACS Multiconference*, 1996.

[5] Cantoni, M., E. Weyer, Y. Li, S. K. Ooi, I. Mareels and M. Ryan, "Control of Large-Scale Irrigation Networks", *Proceedings of the IEEE*, 95, 2007, 75-91.

[6] Coogan, S. and M. Arcak, "Dynamical Properties of a Compartmental Model for Traffic Networks", *Proc. of the American Control Conference*, 2014.

[7] Daganzo, C., "The Cell Transmission Model: A Dynamic Representation of Highway Traffic Consistent With the Hydrodynamic Theory", *Transportation Research, Part B*, 28(4), 1994, 269–287.

[8] Fermo, L. and A. Tosin, "A Fully-Discrete-State Kinetic Theory Approach to Modeling Vehicular Traffic", *SIAM Journal on Applied Mathematics*, 73(4), 2013, 1533-1556.

[9] Godunov, S., "A Difference Method for Numerical Calculation of Discontinuous Solutions of Hydrodynamic Equations", *Matematicheskii Sbornik*, 47, 1959, 271–290.

[10] Gomes, G., R. Horowitz, A. A. Kurzhanskiy, P. Varaiya and J. Kwon, "Behavior of the Cell Transmission Model and Effectiveness of Ramp Metering", *Transportation Research Part C*, 16(4), 2008, 485-513.

[11] Haddad, W. M. and V. Chellaboina, *Nonlinear Dynamical Systems and Control A Lyapunov-Based Approach*, Princeton University Press, 2008, Princeton.

[12] Jiang, Z.-P., and Y. Wang, "Input-to-State Stability for Discrete-Time Nonlinear Systems", *Automatica*, 37, 2001, 857-869.

[13] Karafyllis, I., and Z.-P. Jiang, *Stability and Stabilization of Nonlinear Systems*, Springer-Verlag London, 2011.

[14] Khalil, H. K., *Nonlinear Systems*, 2nd Edition, Prentice-Hall, 1996.

[15] Kurzhanskiy, A. A. and P. Varaiya, "Active Traffic Management on Road Networks: A Macroscopic Approach", *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(1928), 2010, 4607–4626.

[16] Lakshmikantham, V. and D. Trigiante, *Theory of Difference Equations Numerical Methods and Applications*, 2nd Edition, Marcel Dekker, New York, 2002.

[17] Lebacque, J. P., "The Godunov Scheme and What it Means for First Order Traffic Flow Models", *Proc. of the 13th International Symposium on Transportation and Traffic Theory*, 1996, Pergamon, 647–677.

[18] Lebacque, J. P., "Two-Phase Bounded Acceleration Traffic Flow Model Analytical Solutions and Applications", *Transportation Research Record 1852*, Paper No. 03-4397, 2003, 220-230.

[19] Lebacque, J. P. and M. Khoshyaran, "First Order Macroscopic Traffic Flow Models for Networks in the Context of Dynamic Assignment", *Transportation Planning*, 2004, 119–140.

[20] Lebacque, J. P., "Intersection Modeling, Application to Macroscopic Network Traffic Flow Models and Traffic Management", *Traffic and Granular Flow'03*, 2005, 261–278.

[21] Lighthill, M.,and G. Whitham, "On Kinematic Waves I: Flow Movement in Long Rivers. II: A Theory of Traffic Flow on Long Crowded Roads", *Proceedings of the Royal Society of London Part A,* 229(1178), 1955, 281–345.

[22] Liu, T. H., D. J. Hill and Z.-P. Jiang, "Lyapunov Formulation of the Large-Scale and ISS Cyclic Small-Gain Theorem: The Discrete-Time Case", *Systems and Control Letters*, 61(1), 2012, 266-272.

[23] Marinaki, M., and M. Papageorgiou, *Optimal Real-time Control of Sewer Networks*, Springer-Verlag, London, 2005.

[24] Papageorgiou, M., "An Integrated Control Approach for Traffic Corridors", *Transportation Research Part C*, 3(1), 1995, 19-30.

[25] Pisarski, D. and C. Canudas de Wit, "Analysis and Design of Equilibrium Points for the Cell-Transmission Traffic Model", *Proceedings of the American Control Conference*, 2012, 5763–5768.

[26] Richards, P., "Shock Waves on the Highway", *Operations Research*, 4(1), 1956, 42–51.

[27] Sontag, E. D., *Mathematical Control Theory Deterministic Finite Dimensional Systems*, 2nd Edition, Springer-Verlag, New York, 1998.

[28] Stuart, A.M. and A.R. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, 1998.