Convex Approximation Techniques for Joint Multiuser Downlink Beamforming and Admission Control

Master Thesis

By

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CONTENTS

Abstract

Multiuser downlink beamforming under quality of service (QoS) constraints has attracted considerable interest in recent years, because it is particularly appealing from a network operator's perspective (e.g., UMTS, 802.16e). When there are many co-channel users and/or the service constraints are stringent, the problem becomes infeasible and some form of admission control is necessary. We advocate a cross-layer approach to joint multiuser transmit beamforming and admission control, aiming to maximize the number of users that can be served at their desired QoS. It is shown that the core problem is NP-hard, yet amenable to convex approximation tools. Two computationally efficient convex approximation algorithms are proposed: one is based on semidefinite relaxation of an equivalent problem reformulation; the other takes a penalized second-order cone approach. Their performance is assessed in a range of experiments, using both simulated and measured channel data. In all experiments considered, the proposed algorithms work remarkably well in terms of the attained performance-complexity trade-off, consistently exhibiting close to optimal performance at an affordable computational complexity.

I. Introduction

Transmit antenna arrays are nowadays commonly employed or provisioned in cellular wireless networks (e.g., UMTS and the emerging UMTS-LTE [Long-Term Evolution]), wireless local area networks, and fixed wireless back-haul solutions (e.g., 802.16e). In the context of the cellular downlink (or point-to-multipoint distribution for fixed wireless), a transmit antenna array can be beneficial in a number of ways, depending on the available grade of channel state information at the transmitter (CSI-T). When accurate CSI-T is available, it becomes possible to multiplex a number of user streams in space, by appropriately designing transmit beamformers that steer energy in the directions of the intended users [2]. Transmit beamforming is also beneficial when there is only statistical CSI-T, in the form of channel correlation matrices; see [2] and Section IX where we revisit this issue.

Consider a single transmitter with N antenna elements and K receivers, each with a single antenna. Let $\mathbf{h}_k$ denote the $N \times 1$ complex vector that models the propagation loss and phase shift of the frequency-flat quasi-static channel from each transmit antenna to receiver k, and $\mathbf{w}_k^H$ denote the $1 \times N$ weight vector used to beamform towards receiver (user) k, $k \in \{1, \cdots, K\}$. Here and in the sequel, ^{T} denotes transpose, and ^{H} denotes Hermitian (conjugate) transpose. When $N \geq K$, and assuming that the channel matrix $\mathbf{H} := [\mathbf{h}_1, \cdots, \mathbf{h}_K]^T$ is full row-rank, it is possible to right-invert it at the transmitter, thus creating K separate streams. This is indicative of the spatial multiplexing potential, but channel inversion has drawbacks. In practice $K >> N$ (e.g., $N = 4$, $K = 40$ are typical for the cellular downlink), giving rise to a difficult user selection problem [5]. As we will see, it is often possible to simultaneously serve $K > N$ users, but this cannot be accomplished using channel inversion. Transmit power limitations are also an issue when channel inversion is used.

A more flexible alternative to channel inversion is to ensure a certain Signal to Interference plus Noise Ratio (SINR) at each receiver. This is well-motivated for voice, streaming media, and other interactive applications, and it is the prevailing design approach in cellular wireless today. The following joint multiuser transmit beamforming problem under individual SINR constraints as Quality of Service (QoS) metric has been considered in [7], and [2]:

$$\min_{\{\mathbf{w}_k \in \mathbb{C}^N\}_{k=1}^K} \sum_{k=1}^{K} \|\mathbf{w}_k\|_2^2 \tag{1}$$

$$\text{subject to}: \text{SINR}_k := \frac{|\mathbf{w}_k^H \mathbf{h}_k|^2}{\sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \geq c_k, \ \forall k \in \{1, \cdots, K\}, \tag{2}$$

where $||\cdot||_2$ denotes the Euclidean norm, σ_k^2 is the additive noise power at receiver k, and c_k is the SINR requirement of receiver k. For a fixed modulation and coding scheme, a target error rate requirement can be mapped to an appropriate c_k. Alternatively, $\log(1 + \frac{1}{\Gamma}\text{SINR}_k)$ is a measure of the practically attainable link capacity, where the Γ reflects the SINR loss due to modulation and coding.

As shown in [2] (see also [7]), the problem in (1)-(2) is convex (in fact, a *second-order cone program* - SOCP); it can be efficiently solved using modern interior-point methods [16], or specialized iterative algorithms [7]. The main difficulty with the formulation in (1)-(2) is that the problem can easily become infeasible, e.g., when the channel vectors of two or more users are co-linear or highly correlated, and/or the SINR targets are too high, or simply when the number of users, K, is much larger than the number of antennas N - which is the typical scenario in practice. In such a situation, interior point solutions provide an *infeasibility certificate*, whereas the custom-made algorithm in [7] diverges. Either way, infeasibility implies that some user(s) should be dropped (admission control) or rescheduled in orthogonal dimensions (time, frequency, code slot); or the SINR targets should be relaxed.

If users must be dropped / rescheduled, it makes sense to maximize the number of users that can be served at their desired QoS. A brute-force way of doing this is enumeration, each time solving a SOCP problem for a subset of users. This has prohibitive complexity for all practical purposes. In fact, we will show that the problem is NP-hard, which motivates the pursuit of approximate solutions of manageable complexity.

From a complexity point of view, it is appealing to consider a greedy approach: given already admitted users, consider adding one more user, until the problem becomes infeasible. This is still complex, because *testing* each candidate requires solving a separate SOCP problem from scratch. A low-complexity algorithm for admitting a new user was recently proposed in [4]. In order to keep complexity low, [4] advocates fixing the beampatterns of previously admitted users, and jointly optimizing the beampattern of the new user along with power control. This reduces to a generalized eigenvalue problem which can be efficiently solved. The algorithm in [4] can be iterated to grow the pool of admitted users, until the problem becomes infeasible. The overall algorithm is appealing from a complexity point of view. Its performance (in terms of the number of users served and the power required to do so) will be assessed in section VIII.

Practical and regulatory considerations typically dictate a non-trivial upper bound on transmission power, which is not enforced in (1)-(2). An explicit sum power constraint can be added to account for this, and the problem becomes

$$\min_{\{\mathbf{w}_k \in \mathbb{C}^N\}_{k=1}^K} \sum_{k=1}^K \|\mathbf{w}_k\|_2^2 \tag{3}$$

$$\text{subject to}: \sum_{k=1}^K \|\mathbf{w}_k\|_2^2 \leq P, \tag{4}$$

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2}{\sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \geq c_k, \ \forall k \in \{1, \cdots, K\}\,. \tag{5}$$

The problem of interest can now be concisely stated as follows: find a largest subset of users and associated optimum beamforming configuration for which (3)-(5) *restricted to the said subset* *of users* admits a feasible solution. Mathematically, the problem can be described in two stages. In the first stage,

$$S_o = \text{argmax}_{S \subseteq \{1,\cdots,K\}, \{\mathbf{w}_k \in \mathbb{C}^N\}_{k=1}^K} |S| \tag{6}$$

$$\text{subject to}: \sum_{k \in S} \|\mathbf{w}_k\|_2^2 \leq P, \tag{7}$$

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2}{\sum_{\ell \neq k,\ \ell \in S} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \geq c_k,\ \forall k \in S, \tag{8}$$

where $|S|$ denotes the cardinality of S, while in the second stage one solves

$$\min_{\{\mathbf{w}_k \in \mathbb{C}^N\}_{k \in S_o}} \sum_{k \in S_o} \|\mathbf{w}_k\|_2^2 \tag{9}$$

$$\text{subject to}: \sum_{k \in S_o} \|\mathbf{w}_k\|_2^2 \leq P, \tag{10}$$

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2}{\sum_{\ell \neq k,\ \ell \in S_o} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \geq c_k,\ \forall k \in S_o. \tag{11}$$

In this work we propose to solve the problem of transmit beamforming and admission control *jointly* by simultaneously maximizing the total number of users that can be served in the same slot at their desired QoS and minimizing the power required to serve them. In particular, we aim to approximate the original non-convex and NP-hard problem by means of a convex problem whose solution can be used to generate a feasible, close to optimal solution of the original problem. Although this convex approximation approach can only generate suboptimal solutions due to NP-hardness of the original problem, it is well motivated theoretically[1] and in certain cases can provide provably high quality approximate solutions [11].

In closing this section, we remark that there is considerable literature on the closely related topic of joint power and admission control when the coupling between communicating pairs of nodes is fixed (e.g., see [1], [6] and references therein). This is the case, for example, when code-division

[1]For example, admits a Lagrange dual interpretation, as we will see in section IV.

multiple access (CDMA) is used, and the spreading codes are fixed or long and pseudo-random. The same situation arises in our context, when the beampatterns of all users are fixed beforehand, and only power and admission control can be optimized. By the same token, the algorithms developed here can be used for joint design of spreading codes, power allocation, and admission control in CDMA wireless networks.

II. Single-stage Reformulation

In the following, we say that a user is *served* if the user is scheduled and its QoS target is supported. Our objective in this section is to come up with a convenient *single-stage* reformulation (cf. Claim 1 below) of the two-stage optimization problem in (6)–(11). Towards this end, introduce auxiliary binary scheduling variables $s_k \in \{-1,+1\}$, and consider

$$\min_{\{\mathbf{w}_k \in \mathbb{C}^N, s_k \in \{-1,+1\}\}_{k=1}^K} \epsilon \sum_{k=1}^K \|\mathbf{w}_k\|_2^2 + (1-\epsilon) \sum_{k=1}^K \lambda_k (s_k+1)^2 \tag{12}$$

$$\text{subject to}: \sum_{k=1}^K \|\mathbf{w}_k\|_2^2 \le P, \tag{13}$$

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2 + \delta^{-1}(s_k+1)^2}{\sum_{\ell \ne k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \ge c_k, \ \forall k \in \{1, \cdots, K\} \tag{14}$$

Here, the $\lambda_k > 0$ denote normalized weights[2], and ϵ, δ are suitably small positive constants. In particular, we take

$$\delta \le \min_k \frac{4c_k^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma_k^2},$$

which ensures (cf. the Cauchy-Schwartz inequality) that the constraint in (14) is satisfied when $s_k = +1$ even for $\mathbf{w}_k = \mathbf{0}_{N\times 1}$ and irrespective of the other $\mathbf{w}_\ell$, $\ell \ne k$. Since $\min_k \frac{4c_k^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma_k^2} \le \min_k \frac{4c_k^{-1}}{\sigma_k^2}$, this choice of δ also implies that $\mathbf{w}_k = \mathbf{0}_{N\times 1}$, $s_k = 1$, $\forall k$, is always admissible, i.e., the problem in (12)-(14) is *always feasible*. We also select $\epsilon < \frac{\min_k \lambda_k}{P/4 + \min_k \lambda_k}$ - this ensures that a user is not dropped unless it is necessary, cf. Claim 1.

[2]e.g., using λ_k proportional to the queue length of user k is throughput-optimal [18].

The binary slack / scheduling variables s_k play a key role: with $\left\{\check{\mathbf{w}}_k \in \mathbb{C}^N, \check{s}_k \in \{-1,+1\}\right\}_{k=1}^K$ denoting a solution of (12)-(14), it is easy to see that $\check{s}_k = -1$ implies that user k is served, whereas $\check{s}_k = +1$ implies that user k is dropped: $\check{\mathbf{w}}_k = \mathbf{0}_{N\times 1}$. This comes from the choice of δ and the cost function, and it also means that there is no need to explicitly account for dropped users in the denominator of (14). Formally:

Claim 1: With $\lambda_k = 1$, $\forall k$, $\delta \le \min_k \frac{4c_k^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma_k^2}$, and $\epsilon < \frac{1}{P/4+1}$, solution of (12)-(14) maximizes the number of users served and simultaneously yields the associated minimum sum-power beamforming vectors.

Proof of the above and all subsequent claims is deferred to the Appendix. A ruler analogy is useful for intuition. We wish to minimize two cost functions simultaneously. The important observation is that one is discrete-valued and the other is bounded. By proper weighting of the two, the weighted sum takes values on a ruler whose decimal ticks correspond to the discrete part, whereas the intervals in-between are (partially) spanned by the continuous part. Proper choice of weight ensures that these intervals *do not overlap*, i.e., there is an unattainable guard band between any two successive decimal ticks. In our context, this can be interpreted as follows: dropping any user costs more than can possibly be saved in terms of power by means of beamvector optimization for the remaining users.

Claim 1 shows that (12)–(14) provides a single stage optimization reformulation of the two-stage joint admission control and optimum beamforming problem described in (6)–(11). One advantage of this single stage reformulation is that it allows a convenient convex relaxation which can generate high quality approximate solutions efficiently (see Section IV). Another is that it facilitates complexity analysis, as discussed next.

III. Complexity Analysis

Claim 2: With $\delta \leq \min_k \frac{4c_k^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma_k^2}$ and $\epsilon < \frac{\min_k \lambda_k}{P/4 + \min_k \lambda_k}$, the problem in (12)-(14) is NP-hard for $N > 1$. For $N = 1$, the problem is polynomial time solvable.

To prove Claim 2, we will need a few definitions and an intermediate claim, which is of interest in its own right.

Let $G = (V, E)$ be an undirected graph, with $|V| = K$ vertices, one for each user, and edges $e_{k,\ell} \in E$, where $e_{k,\ell}$ denotes an edge between vertices k and ℓ.

Definition 1: A subset of vertices $S \subseteq V$ of $G = (V, E)$ is *independent* when no two vertices in S are connected by an edge in E.

Finding a largest independent subset in a graph is the *maximum independent set* problem, known to be NP-hard [8].

Definition 2: A subset of vertices $S \subseteq V$ of $G = (V, E)$ forms *independent two-hop neighborhoods* when every pair of vertices in S are separated by at least three hops (edges in E) from each other, i.e., no two vertices in S have a common one-hop neighbor.

Finding a largest subset of vertices that forms independent two-hop neighborhoods will be called the *maximum independent two-hop neighborhoods* problem (and the *two hop* qualifier will be dropped henceforth for brevity).

Claim 3: The maximum independent neighborhoods problem is NP-hard.

The proof of Claim 2 shows that an arbitrary instance of the maximum independent neighborhoods problem can be transformed to an instance of problem (12)-(14) with $\delta \leq \min_k \frac{4c_k^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma_k^2}$ and $\epsilon < \frac{\min_k \lambda_k}{P/4 + \min_k \lambda_k}$.

It is interesting to contrast the polynomial time solvability result for the $N = 1$ case with the NP-hardness of the joint power and admission control problem for CDMA wireless networks [6]. Although in both problems the coupling coefficients between communicating pairs of users are fixed, there is a key difference between the two. In a wireless CDMA network, the coupling constants between users are determined by the correlation coefficients between the users' spreading codes and therefore can be arbitrary and unequal, while in our context, the (normalized) coupling constants between all users are equal to 1 (cf. (38)). Hence, allowing arbitrary and unequal coupling constants can turn an otherwise polynomial time solvable joint power and admission control problem into a NP-hard problem which is computationally intractable.

IV. A SEMIDEFINITE RELAXATION APPROACH

The interest in the reformulation in (12)-(14) stems in part from its suitability for the application of Lagrangian relaxation tools. In particular, note that

$$(s_k+1)^2 = \left([s_k \quad 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 = \mathrm{Tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s_k \\ 1 \end{bmatrix} [s_k \quad 1] \right) = \mathrm{Tr}\left(\mathbf{1}_{2\times 2}\mathbf{S}_k\right),$$

where $\mathrm{Tr}(\cdot)$ denotes matrix trace, $\mathbf{S}_k := \mathbf{s}_k\mathbf{s}_k^T$, and $\mathbf{s}_k := [s_k \quad 1]^T$. By construction, $\mathbf{S}_k$ is positive semidefinite (denoted $\mathbf{S}_k \geq 0$), $\mathrm{rank}(\mathbf{S}_k) = 1$, and $\mathbf{S}_k(2,2) = 1$; if we further insist that $\mathbf{S}_k(1,1) = 1$, then there are only two possibilities for $\mathbf{S}_k$:

$$\mathbf{S}_k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \mathrm{Tr}\left(\mathbf{1}_{2\times 2}\mathbf{S}_k\right) = 4; \quad \mathit{or} \quad \mathbf{S}_k = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \mathrm{Tr}\left(\mathbf{1}_{2\times 2}\mathbf{S}_k\right) = 0.$$

As a result, the scalar binary variables s_k can be replaced by the 2×2 real matrix variables $\mathbf{S}_k$, and the ± 1 constraints can be replaced by positive semidefinite, rank-one, and linear equality constraints (see also [12]). Of the latter, only the rank-one constraint is non-convex, and thus difficult to handle.

In the same spirit, we may define rank-one positive semidefinite matrix variables $\mathbf{W}_k := \mathbf{w}_k\mathbf{w}_k^H$, and $\mathbf{H}_k := \mathbf{h}_k\mathbf{h}_k^H$, and rewrite the optimization problem in (12)-(14) equivalently as

$$\min_{\{\mathbf{W}_k \in \mathbb{C}^{N\times N}, \mathbf{S}_k \in \mathbb{R}^{2\times 2}\}_{k=1}^K} \epsilon \sum_{k=1}^K \mathrm{Tr}(\mathbf{W}_k) + (1-\epsilon)\sum_{k=1}^K \lambda_k \mathrm{Tr}(\mathbf{1}_{2\times 2}\mathbf{S}_k) \tag{15}$$

$$\text{subject to}: \sum_{k=1}^K \mathrm{Tr}(\mathbf{W}_k) \leq P, \tag{16}$$

$$\frac{\mathrm{Tr}(\mathbf{H}_k\mathbf{W}_k) + \delta^{-1}\mathrm{Tr}(\mathbf{1}_{2\times 2}\mathbf{S}_k)}{\sum_{\ell\neq k}\mathrm{Tr}(\mathbf{H}_k\mathbf{W}_\ell) + \sigma_k^2} \geq c_k, \ \forall k, \tag{17}$$

$$\mathbf{W}_k \geq 0, \mathrm{rank}(\mathbf{W}_k) = 1, \ \forall k, \tag{18}$$

$$\mathbf{S}_k \geq 0, \mathrm{rank}(\mathbf{S}_k) = 1, \mathbf{S}_k(1,1) = \mathbf{S}_k(2,2) = 1, \ \forall k. \tag{19}$$

Dropping the rank-one constraints, we obtain the following convex semidefinite relaxation (SDR) of (15)-(19):

$$\min_{\{\mathbf{W}_k \in \mathbb{C}^{N\times N}, \mathbf{S}_k \in \mathbb{R}^{2\times 2}\}_{k=1}^{K}} \epsilon \sum_{k=1}^{K} \mathrm{Tr}(\mathbf{W}_k) + (1-\epsilon) \sum_{k=1}^{K} \lambda_k \mathrm{Tr}(\mathbf{1}_{2\times 2}\mathbf{S}_k) \tag{20}$$

$$\text{subject to :} \sum_{k=1}^{K} \mathrm{Tr}(\mathbf{W}_k) \leq P, \tag{21}$$

$$\mathrm{Tr}(\mathbf{H}_k \mathbf{W}_k) + \delta^{-1}\mathrm{Tr}(\mathbf{1}_{2\times 2}\mathbf{S}_k) \geq c_k \sum_{\ell \neq k} \mathrm{Tr}(\mathbf{H}_k \mathbf{W}_\ell) + \sigma_k^2,\ \forall k, \tag{22}$$

$$\mathbf{W}_k \geq 0,\ \forall k, \tag{23}$$

$$\mathbf{S}_k \geq 0, \mathbf{S}_k(1,1) = \mathbf{S}_k(2,2) = 1,\ \forall k, \tag{24}$$

where we have also used the fact that the denominator in (17) is positive.

We note that problem (15)-(19) is a quadratically constrained quadratic program, and rank relaxation can be interpreted as its bi-dual problem [20], which further motivates rank relaxation from a Lagrangian perspective.

The problem in (20)-(24) is a semidefinite program, which can be efficiently solved using modern interior point solvers such as SeDuMi [3], [16]. Being a relaxation of (12)-(14), the problem in (20)-(24) is always feasible, provided that the constants ϵ, δ are chosen as in Claim 1.

It is interesting to recall that rank relaxation of the matrices $\mathbf{W}_k$ for the original problem (without user selection) is not a relaxation after all, as shown in [2]. It is also interesting to note that the matrices $\mathbf{S}_k$ are of rank at most two, hence the associated rank relaxation step is far milder than usual. In particular, the following can be shown by direct examination of eigenvalues:

Property 1: Consider a real symmetric positive semidefinite matrix with diagonal elements equal to one, i.e.,

$$\mathbf{S} = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \geq \mathbf{0}.$$

Then $\text{rank}(\mathbf{S}) = 1 \iff x \in \{-1, +1\}$, whereas $\text{rank}(\mathbf{S}) \in \{1, 2\} \iff x \in [-1, +1]$. Thus rank relaxation of $\mathbf{S}_k$ amounts to relaxing the $\{-1, +1\}$ constraint on its off-diagonal element to a $[-1, +1]$ interval constraint. The associated penalty (sum of elements) is always non-negative, in $[0, 4]$. These observations suggest that (20)-(24) is a relatively tight relaxation of (15)-(19).

Gaussian randomization coupled with multiuser power control (MPC) can be used to convert the optimal solution of (20)-(24) into an approximate solution of (15)-(19); e.g., see related approaches in [17], [10]. As an alternative to randomization / MPC, we may proceed as follows. The difficult part of the problem is the determination of which users to drop. Once this part is solved, the rest is SOCP. One idea is to try to determine this from the solution of the relaxed problem, by examining the 2×2 matrix variables $\mathbf{S}_k$, and/or the optimum of the cost function itself. For example, the optimum value can yield an upper bound on the maximum number of admissible users. From the various approaches that we tried, the following appears to work best in practice:

Algorithm 1: Deflation based on SDR (D-SDR):

1) Set $\mathcal{U} := \{1, ..., K\}$;
2) Solve problem (20)-(24) for the users in $\mathcal{U}$. Let $\{\check{\mathbf{W}}_k\}_{k \in \mathcal{U}}$ denote the resulting optimal transmit covariance matrices;
3) For each $k \in \mathcal{U}$, extract the principal component of $\check{\mathbf{W}}_k$, and scale it to power $Tr(\check{\mathbf{W}}_k)$; i.e., set $\check{\mathbf{w}}_k := \sqrt{Tr(\check{\mathbf{W}}_k)}\check{\mathbf{u}}_k$, where $\check{\mathbf{u}}_k$ is the unit-norm principal component of $\check{\mathbf{W}}_k$.
4) For each $k \in \mathcal{U}$, check whether $\frac{|\check{\mathbf{w}}_k^H \mathbf{h}_k|^2}{\sum_{\ell \neq k} |\check{\mathbf{w}}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \geq c_k$ holds; if so, stop (a feasible solution has been found); else pick the user with largest gap to its target SINR (smallest attained SINR if all the SINR targets are equal), remove from $\mathcal{U}$, and go to step 2.

D-SDR returns a feasible solution for the subset of selected users, which however need not be optimal in terms of sum power. Interestingly, our experiments indicate that further beamvector optimization by means of SOCP for the selected users does not improve the result of D-SDR.

V. A PENALIZED SECOND-ORDER CONE PROGRAMMING APPROACH

Exploiting the freedom to choose the phase of each beamvector, the problem in (3)-(5) can be equivalently formulated as an equivalent Second Order Cone Programming (SOCP) problem, as shown in [2]

$$\min_{\{\mathbf{w}_k \in \mathbb{C}^N\}_{k=1}^K} \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \tag{25}$$

$$\text{subject to: } \mathbf{w}_k^H \mathbf{h}_k \geq \sqrt{c_k \sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + c_k \sigma_k^2}, \quad \forall k, \tag{26}$$

$$\sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \leq P, \quad \mathrm{Im}(\mathbf{w}_k^H \mathbf{h}_k) = 0, \quad \forall k, \tag{27}$$

where $\mathrm{Im}(\cdot)$ extracts the imaginary part of its argument. The above problem is convex and can be solved efficiently via interior point methods. Another way towards accounting for infeasibility / admission control issues is to consider the following relaxed problem:

$$\min_{\{\mathbf{w}_k, \in \mathbb{C}^N, s_k \in \mathbb{R}\}_{k=1}^K} \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 + M \sum_{k=1}^{K} s_k^2 \tag{28}$$

$$\text{subject to: } \mathbf{w}_k^H \mathbf{h}_k + s_k \geq \sqrt{c_k \sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + c_k \sigma_k^2}, \quad \forall k, \tag{29}$$

$$\sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \leq P, \quad \mathrm{Im}(\mathbf{w}_k^H \mathbf{h}_k) = 0, \quad \forall k, \tag{30}$$

where M is a large positive constant. Notice that the s_k's here are different from those in (12)-(14): the former are unconstrained real variables, whereas the latter are binary before relaxation, and in the interval $[-1, 1]$ after relaxation. Problem (28)-(30) is a SOCP, and it is always feasible due to the presence of the auxiliary variables $\{s_k\}_{k=1}^K$. To see the latter, we only need to choose large enough s_k's to satisfy all the constraints, regardless of whether (25)-(27) is feasible or not. In fact, since the $\mathbf{w}_k$'s are bounded, it follows that the set of achievable SINRs is also bounded:

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2}{\sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \leq \gamma, \quad \forall\, k \text{ and } \mathbf{w}_k \text{ with } \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \leq P$$

for some $\gamma > 0$. Thus, even if the problem in (25)-(27) is infeasible, we can always choose finite s_k (say, $s_k = \sqrt{c_k/\gamma}$) to satisfy all the constraints in (28)-(30). Also notice that, if (25)-(27) is feasible, then $s_k = 0$, $\forall k$ is feasible in (28)-(30).

For any $M > 0$, let $\{\mathbf{w}_k(M) \in \mathbb{C}^N, s_k(M) \in \mathbb{R}\}_{k=1}^K$ denote the global optimal solution of (28)-(30). Clearly, $\{\mathbf{w}_k(M) \in \mathbb{C}^N, s_k(M) \in \mathbb{R}\}_{k=1}^K$ can be found efficiently using interior point methods.

Claim 4: Let

$$\mathcal{K} = \left\{ k \mid \lim_{M \to \infty} s_k(M) > 0 \right\}.$$

Then, the formulation in (25)-(27) is feasible if and only if $|\mathcal{K}| = 0$. Moreover, if $|\mathcal{K}| \neq 0$, then dropping the constraints in $\mathcal{K}$ will lead to a feasible beamforming problem; that is, the reduced problem

$$\min_{\{\mathbf{w}_k, \in \mathbb{C}^N\}_{k=1}^K} \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \tag{31}$$

$$\text{subject to: } \mathbf{w}_k^H \mathbf{h}_k \geq \sqrt{c_k \sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + c_k \sigma_k^2}, \quad \forall\, k \notin \mathcal{K} \tag{32}$$

$$\sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \leq P, \quad \mathrm{Im}(\mathbf{w}_k^H \mathbf{h}_k) = 0, \quad \forall\, k \notin \mathcal{K} \tag{33}$$

is always feasible.

The above claim suggests that we may want to solve the relaxed problem (28)-(30) for some sufficiently large M, and admit only those users for which $s_k(M)$ is small. Using a threshold is one possibility, but choosing the right threshold is not straightforward. A reasonable alternative is to sort $\{s_k(M)\}$ and prune one user at a time, until the problem becomes feasible. When a user is dropped, however, the remaining beamvectors are no longer optimal. This suggests the following deflation algorithm:

Algorithm 2: Deflation based on SOC programming (D-SOC):

1) Set $\mathcal{U} := \{1, ..., K\}$;
2) Solve problem (28)-(30) for the users in $\mathcal{U}$. Let $\{\check{\mathbf{w}}_k\}_{k\in\mathcal{U}}$ denote the resulting beamforming vectors, which are optimal for problem (28)-(30).
3) For each $k \in \mathcal{U}$, check whether $\frac{|\check{\mathbf{w}}_k^H \mathbf{h}_k|^2}{\sum_{\ell\neq k} |\check{\mathbf{w}}_\ell^H \mathbf{h}_k|^2 + \sigma_k^2} \geq c_k$ holds; if so, stop (a feasible solution has been found); else pick the user with largest gap to its target SINR (smallest attained SINR if all the SINR targets are equal)[3], remove from $\mathcal{U}$, and go to step 2.

[3]In our experiments, we have observed that dropping the user with the largest $s_k(M)$ yields identical (in most cases) or slightly worse results compared to using the SINR gap.

VI. Implementation Complexity

Due to the Cartesian product structure of the 2×2 SDP cones, the worst case complexity of solving the SDP in (20)-(24) is $O(K^{3.5} \log(1/\epsilon))$, where ϵ is the required relative accuracy of the duality gap at termination [19]. Similarly, the SOCP problem in (28)-(30) also has Cartesian structure, so its worst case complexity is of order $O(K^{3.5} \log(1/\epsilon))$ as well. The worst-case complexity of D-SDR and D-SOC is scaled up by a factor of K (total number of users), since we're using deflation. The final worst-case count for both algorithms is therefore $O(K^{4.5} \log(1/\epsilon))$.

VII. A simpler alternative

It is possible to conceive of simpler suboptimal solutions. From a complexity point of view, it is appealing to consider a greedy *inflation* (as opposed to deflation) approach: given already admitted users, consider adding one more user, until the problem becomes infeasible. Optimal new user admission entails solving the beamforming problem in (3)-(5) from scratch for each candidate new user, thereby leading to unacceptable complexity. For this reason, [4] suggests a simpler approach, namely, fixing the beampatterns (normalized beamvectors) of already admitted users and jointly optimizing the beampattern of the candidate user along with the powers of all (admitted and candidate) users. In on-line mode, this also has the benefit of minimizing service disruption for already admitted users.

When there is more than one option regarding which new user to admit, it makes sense to add the user that minimizes the overall power (albeit this strategy is not necessarily optimal in terms of the total number of users admitted at the end of the process). Putting everything together yields the following algorithm.

Algorithm 3: Inflation based on the Butussi - Bengtsson approach (I-BB):

1) Set $\mathcal{U} := \{1, ..., K\}$ and $\mathcal{A} := \emptyset$ (empty set);
2) Find $k_o \in \mathcal{U}$ whose QoS constraint $\frac{|\mathbf{w}_{k_o}^H \mathbf{h}_{k_o}|^2}{\sigma_{k_o}^2} \geq c_{k_o}$ (in the absence of interference from other users) can be satisfied at minimum power. This is the user with the largest channel norm, and the associated optimum beamvector is a scaled spatially matched filter. If the required power is less than P, set $\mathcal{A} = \mathcal{A} \bigcup \{k_o\}$, $\mathcal{U} = \mathcal{U} - \{k_o\}$ (i.e., admit the said user), and store its beampattern (normalized beamvector) and power; else exit (the problem is infeasible).
3) If $\mathcal{U} = \emptyset$ exit; else, for each candidate user in $\mathcal{U}$, fix the beampatterns of already admitted users in $\mathcal{A}$ and jointly optimize the beampattern of the candidate user along with the powers of all (admitted and candidate) users. This can be accomplished by solving a generalized eigenvalue problem, as shown in [4]. If no feasible solution can be found for the latter problem for any of the candidate users in $\mathcal{U}$, exit (no more users can be added); else pick the candidate, say k_*, which results in the smallest total transmit power. If this power is less than P, set $\mathcal{A} = \mathcal{A} \bigcup \{k_*\}$, $\mathcal{U} = \mathcal{U} - \{k_*\}$, store the new user beampattern and all powers, and return to step 3; else exit.

Algorithm I-BB has the lowest complexity among all algorithms considered. It is included as a reasonable low-complexity baseline in our experiments.

VIII. Experiments

Setup: We conducted experiments using both simulated and measured channel data. In both cases, we used SOCP enumeration (i.e., solving the problem in (3)-(5) using SeDuMi [16] for all possible user combinations) as a benchmark. SOCP enumeration provides the optimum solution(s), but its complexity grows exponentially in K. The maximum problem size that we could solve this way was $K = 18$ users, requiring over 7 hours of computation. We compared SOCP enumeration, the two proposed convex approximation algorithms (D-SDR, D-SOC), and I-BB. In all experiments reported, the number of transmit antennas is set to $N = 4$. Monte-Carlo results for a pool of $K = 14$ users and 30 channel matrices are presented in Tables I, II. The parameters of the various algorithms and the problem setup are listed in the table captions, for ease of reference.

Choice of parameters: For D-SDR, the choice of parameters ϵ, δ is governed by the respective upper bounds in Claim 1. The parameter ϵ can be taken to be smaller than the upper bound in Claim 1; the choice does not seem to be critical, so long as ϵ is not too small. For D-SOC, Claim 4 does not directly provide guidance on the choice of M. In practice, M should be large but not too large, because in the latter case the problem becomes badly conditioned and this slows the Newton iteration (increases the number of Newton steps) in the interior point algorithm [16]. We have observed that a factor of 10 change in M does not have significant performance/complexity effects.

Rayleigh channels: For Table I, the channel gains were i.i.d. complex normal with zero-mean and unit-variance ($\mathcal{CN}(0,1)$), and independent from realization to realization. For ease of visualization of the results in Table I, Figure 1 is a plot of the average number of users served by each algorithm versus target SINR, and Figure 2 is a plot of the average power per user served by each algorithm versus target SINR. Note that power does not scale linearly with the number of users served, due to interference; I-BB uses less power per user when serving a smaller number of users than the other algorithms. Also note that Table I contains additional information that cannot be easily conveyed in graphical form (average execution times, percentages in the number of users served).

Measured channels: Measured channel data (downloaded from the iCORE HCDC Lab web site, University of Alberta in Edmonton [9], http://www.ece.ualberta.ca/∼mimo/) were used for Table II. The site contains detailed descriptions of numerous measurement campaigns in the 902–928 MHz (ISM) band. The most pertinent scenario for our purposes is the stationary outdoor one, called Quad and illustrated in Figure 3. Quad is a 150 by 60 meters lawn surrounded by buildings with heights ranging from 15 to 30 meters. The transmitter (Tx) location was fixed while the receiver (Rx) was placed in 6 different locations (no measurements are actually provided for location 4). Both Tx and Rx were equipped with antenna arrays, each comprising four vertically polarized dipoles spaced $\lambda/2$ (≈ 16 cm) apart. The channels are frequency-flat, slowly time-selective fading, due to pedestrian movement and other factors (the chip rate used for sounding was low enough to safely assume that the channels are not frequency selective). For every Rx location, 9 different measurements were taken by shifting the Rx antenna array on a 3×3 square grid with $\lambda/4$ spacing. Each measurement contains about 100 4×4 channel snapshots, recorded 3 per second. We took $K = 14$ users (all depicted in Fig. 3 except 7, 10, 12, and 17), and took every third temporal channel snapshot, starting from the first one. For ease of comparison with the simulated Rayleigh case, all channel gains were normalized *by the same constant* (average amplitude over all channels and all snapshots). Note that this normalization maintains differences in path loss. Figure 4 (Figure 5) plots the average number of users served (average power per user served) by each algorithm versus target SINR, for the data in Table II.

A. Discussion of experimental results

In the vast majority of cases considered (99% for i.i.d. Rayleigh, 95% for measured channel data) D-SDR and D-SOC serve the maximum possible number of users at a small power penalty relative to the optimal solution provided by SOCP enumeration. In the remaining cases, both D-SDR and D-SOC serve one user less than the maximum. This is remarkable, given the associated reduction in execution time relative to SOCP enumeration, which is roughly by a factor of 10^3 for $K = 14$. There is no clear winner in terms of performance between D-SDR and D-SOC, albeit D-SOC does appear to be somewhat more effective in terms of the number of users served, especially at high target QoS. On the other hand, the run-time of D-SOC is triple that of D-SDR. This may seem curious at first sight, because solving a second-order cone program is generally simpler than solving a semidefinite program, and D-SDR actually entails more optimization variables than D-SOC. However, the constants hidden in complexity analysis depend on problem conditioning, which also affects average complexity. We stress that the M parameter for D-SOC has been manually tuned for best performance in our experiments.

I-BB further reduces the run-time by a factor of $10 - 10^2$ relative to D-SDR and D-SOC, but its performance is considerably worse, especially at high target QoS where it typically serves two users versus four served by D-SDR and D-SOC. Even at low target QoS (e.g., 3 dB), I-BB incurs ~ 6 dB power penalty relative to D-SDR, D-SOC, and the optimum solution.

It is worth noting that the performance (number of users served, required power) of all algorithms - including optimum SOCP enumeration - is somewhat worse in the case of measured channels compared to the case of simulated i.i.d. Rayleigh channels. There are reasons for this. Despite the presence of multipath, the measured channels exhibit directional selectivity and temporal correlation (slow fading); certain nodes are (approximately) behind others in the line of sight to the transmit antenna array. In this sense, the measured channels represent a more difficult scenario, and it is natural to expect that the number of users served will be smaller and the required power higher relative to the isotropic i.i.d. Rayleigh scenario.

IX. Concluding remarks

We have proposed two computationally efficient joint multiuser transmit beamforming and admission control algorithms. The objective is to maximize the number of users that can be supported at their desired SINR (and then minimize the total transmitted power) which is appealing from a network operator's perspective. The core problem is NP-hard, yet we have shown that it is well-suited to convex approximation tools. For a moderate user population, our experiments with simulated and measured channel data indicate that the proposed algorithms yield high-quality feasible solutions at a low computational cost.

There are some related problem formulations that can be easily dealt with using the tools that we developed. For example, we may adopt a min-max power criterion in place of the min-sum in (1). In this case, we can minimize an auxiliary optimization variable, t, subject to $||\mathbf{w}_k||_2^2 \leq t$, $\forall k$, in addition to sum-power and individual SINR constraints. The resulting problem can again be reformulated as SOCP. The single-stage reformulation of the joint beamforming and admission control problem can be worked out in a similar fashion, and the NP-hardness proof can be modified to accommodate the min-max formulation, provided that we keep the sum-power constraint. After semidefinite relaxation, which is again the Lagrange bi-dual problem, the newly added quadratic constraints become linear inequality constraints, so the resulting problem can again be solved via SDP.

Another variation emerges when the actual channel vectors $\mathbf{h}_k$ are not accurately known at the transmitter, e.g., due to mobility or delayed / low-rate feedback. In such cases, the channel correlation matrices $\mathbf{R}_k := E[\mathbf{h}_k\mathbf{h}_k^H]$ may be available, in which case it is possible to guarantee average SINRs. D-SDR works verbatim in this case, simply replacing $\mathbf{H}_k$ (previously defined as $\mathbf{h}_k\mathbf{h}_k^H$) by $\mathbf{R}_k$.

X. Appendix

Proof of Claim 1: Let $\left\{\check{\mathbf{w}}_k \in \mathbb{C}^N, \check{s}_k \in \{-1,+1\}\right\}_{k=1}^K$ be a solution of (12)-(14), and let $\left\{\tilde{\mathbf{w}}_k \in \mathbb{C}^N,\right.$ $\left.\tilde{s}_k \in \{-1,+1\}\right\}_{k=1}^K$ denote a feasible[4] alternative with $\sum_{k=1}^K 1(\tilde{s}_k = -1) > \sum_{k=1}^K 1(\check{s}_k = -1)$, where $1(\cdot)$ stands for the indicator function. Then $\left\{\tilde{\mathbf{w}}_k \in \mathbb{C}^N, \tilde{s}_k \in \{-1,+1\}\right\}_{k=1}^K$ serves at least one more user than $\left\{\check{\mathbf{w}}_k \in \mathbb{C}^N,\ \ \check{s}_k \in \{-1,+1\}\right\}_{k=1}^K$, and so $\sum_{k=1}^K(\tilde{s}_k+1)^2 \le \sum_{k=1}^K(\check{s}_k+1)^2-4$. From the total power constraint in (13) it follows that $\sum_{k=1}^K \|\tilde{\mathbf{w}}_k\|_2^2 \le P$, and so $\epsilon\sum_{k=1}^K \|\tilde{\mathbf{w}}_k\|_2^2 + (1-\epsilon)\sum_{k=1}^K(\tilde{s}_k+1)^2 \le \epsilon P+(1-\epsilon)\sum_{k=1}^K(\check{s}_k+1)^2-(1-\epsilon)4$. Now, $\epsilon < \frac{1}{P/4+1} \Leftrightarrow \epsilon P-(1-\epsilon)4 < 0$, therefore $\epsilon\sum_{k=1}^K \|\tilde{\mathbf{w}}_k\|_2^2 + (1-\epsilon)\sum_{k=1}^K(\tilde{s}_k+1)^2 < (1-\epsilon)\sum_{k=1}^K(\check{s}_k+1)^2 \le \epsilon\sum_{k=1}^K \|\check{\mathbf{w}}_k\|_2^2 + (1-\epsilon)\sum_{k=1}^K(\check{s}_k+1)^2$, which contradicts optimality of $\left\{\check{\mathbf{w}}_k \in \mathbb{C}^N, \check{s}_k \in \{-1,+1\}\right\}_{k=1}^K$ for the problem in (12)-(14). Therefore no other solution exists that serves a higher number of users under (13)-(14).

Given $\{\check{s}_k \in \{-1,+1\}\}_{k=1}^K$, it follows from the choice of δ and the cost function in (12) that $\check{s}_k = +1 \Rightarrow \check{\mathbf{w}}_k = \mathbf{0}_{N\times 1}$, thus the cost function becomes $\epsilon\sum_{k\ |\ \check{s}_k=-1}\|\mathbf{w}_k\|_2^2 +$ constant, the constraint in (13) becomes $\sum_{k\ |\ \check{s}_k=-1}\|\mathbf{w}_k\|_2^2 \le P$, the constraints in (14) corresponding to $\{k \mid \check{s}_k = +1\}$ are automatically satisfied (recall $\delta \le \min_k \frac{4c_k^{-1}}{P\max_m \|\mathbf{h}_m\|_2^2+\sigma_k^2}$), and those corresponding to $\{k \mid \check{s}_k = -1\}$ become

$$\frac{|\mathbf{w}_k^H\mathbf{h}_k|^2}{\sum_{\ell\neq k\ |\ \check{s}_\ell=-1}|\mathbf{w}_\ell^H\mathbf{h}_k|^2+\sigma_k^2} \ge c_k,\ \forall\left\{k \mid \check{s}_k = -1\right\}. \tag{34}$$

It follows that solution of (12)-(14) also yields optimal beamforming vectors for the admitted users.

■

Proof (of Claim 3): Given a connected graph $G = (V, E)$, construct a graph $G^{'} = (V^{'}, E^{'})$ as follows:

- Begin with $V^{'} = V$, $E^{'} = \emptyset$.

[4]i.e., satisfying (13)-(14).

- Replace each edge $e_{k,\ell} \in E$ by a a new vertex (node) $s_{k,\ell}$ that is added in $V^{'}$ and two new edges that are added in $E^{'}$: $(k, s_{k,\ell})$ and $(s_{k,\ell}, \ell)$.
- Add edges in $E^{'}$ between *all* newly created nodes $s_{k,\ell}$, for $e_{k,\ell} \in E$.

It is simple to verify the following observations (see Fig. 6 for an illustration):

- All old nodes (in $V \subset V^{'}$) are either two or three hops away from each other in $G^{'}$ (paths via newly created nodes).
- Two old nodes are two hops away in $G^{'}$ if and only if they are adjacent (one hop away) in G.
- Any newly created node is at most two hops away from any old node in $G^{'}$ (again, due to paths via new nodes).

We will show the following: Any independent set of size $|S| > 1$ in G corresponds to an independent neighborhoods set of the same size in $G^{'}$, and vice-versa.

The forward direction is easy, since, by construction of $G^{'}$, every two disjoint nodes (separated by more than one hop) in G will be exactly three hops away in $G^{'}$, so S is also an independent neighborhoods set in $G^{'}$.

The converse can be seen as follows. Given an independent neighborhoods set S in $G^{'}$ with $|S| > 1$, write $S = S_1 \cup S_2$, where $S_1 \subseteq V$ contains only old nodes and $S_2 \subseteq V^{'} - V$ contains only new nodes. We will show that S_2 is empty. Since the newly added nodes are all connected, S_2 can contain at most one node. This also implies that $|S_1| \geq |S| - 1 \geq 1$, so S_1 is non-empty. Moreover, if S_2 contains exactly one node, then any node in S_1 will be at most two hops away in $G^{'}$ from the node in S_2, contradicting the assumption that S is an independent neighborhoods set in $G^{'}$. So S_2 must be empty, i.e., S consists solely of old nodes which are three hops away in $G^{'}$. It follows that S is an independent set in G, and the proof is complete. ■

Proof (of Claim 2): Consider the following simplified version of problem (12)-(14), with $\lambda_k =$

$\sigma_k^2 = c_k = 1$, $\forall k$, and $P \geq 1$

$$\mathcal{A}: \min_{\{\mathbf{w}_k \in \mathbb{C}^N, s_k \in \{-1,+1\}\}_{k=1}^K} \epsilon \sum_{k=1}^{K} \|\mathbf{w}_k\|_2^2 + (1-\epsilon) \sum_{k=1}^{K} (s_k + 1)^2$$

$$\text{subject to}: \sum_{k=1}^{K} \|\mathbf{w}_k\|_2^2 \leq P,$$

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2 + \delta^{-1}(s_k+1)^2}{\sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + 1} \geq 1, \ \forall k \ \in \{1, \cdots, K\}$$

with $\epsilon P < (1-\epsilon)4 \Leftrightarrow \epsilon < \frac{1}{P/4+1}$, and $\delta \leq \frac{4}{P \max_m \|\mathbf{h}_m\|_2^2 + 1}$. From $P \geq 1$ it follows that $\epsilon < 4/5$.

We will show that an arbitrary instance of the maximum independent neighborhoods problem can be transformed to an instance of problem $\mathcal{A}$.

Given a graph $G = (V, E)$ with $|V| = K$ vertices, construct an instance of problem $\mathcal{A}$, denoted $\mathcal{A}(G)$, by setting

$$\mathbf{h}_k(\ell) = \begin{cases} \frac{1}{\sqrt{d(k)}}, & e_{k,\ell} \in E \\ 0, & \text{otherwise} \end{cases} \qquad k, \ell = 1, 2, ..., K, \tag{35}$$

where $d(k)$ is the degree of node $k \in V$ (i.e., the number of edges adjacent to node k). It can be seen that $\|\mathbf{h}_k\|^2 = 1$ for all $k \in V$ and $\mathbf{h}_k^H \mathbf{h}_\ell = 0$ if and only if nodes k and ℓ do not have a common one-hop neighbor, i.e., they are separated by at least three hops.

Remark 1: Note the relationship between G and $\mathcal{A}(G)$: vertices correspond to users and edges correspond to user interaction/interference. Also note that for $\|\mathbf{h}_k\|^2 = 1$, $\forall k$, as above, $P \geq 1$ is needed to ensure that $\mathcal{A}(G)$ is not trivial (in the sense that $s_k = 1$, $\mathbf{w}_k = \mathbf{0}$, $\forall k$ is the only feasible solution). On the technical side, $P \geq 1$ ensures that $\epsilon < 4/5$ which we need to invoke later in the proof.

We will prove the following: G contains an independent neighborhoods set $S \subseteq V$ of size $|S|$ if and only if $\mathcal{A}(G)$ with $P = |S|$ admits a solution of cost less than or equal to $\epsilon|S| + (1-\epsilon)4(K - |S|)$ $= (K - |S|)(4 - 5\epsilon) + \epsilon K$ (note that $\epsilon < 4/5$ has been assumed, so the leading term is non-negative).

Suppose G contains an independent neighborhoods set S. Consider $\mathcal{A}(G)$ with $P = |S|$ and set

$$s_k = \begin{cases} -1, & k \in S \\ +1, & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{w}_k = \begin{cases} \mathbf{h}_k, & k \in S \\ \mathbf{0}, & \text{otherwise,} \end{cases} \tag{36}$$

We first verify that the assignment in (36) is feasible for $\mathcal{A}(G)$. For the nodes in $V - S$, we have $\mathbf{w}_k = \mathbf{0}$ and $s_k = 1$, which together with the choice of δ implies that the QoS constraint

$$\frac{|\mathbf{w}_k^H \mathbf{h}_k|^2 + \delta^{-1}(s_k + 1)^2}{\sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + 1} \geq 1 \tag{37}$$

is satisfied. Moreover, for nodes in S, we have $\mathbf{w}_k = \mathbf{h}_k$ and $s_k = -1$ so that $\|\mathbf{w}_k^H \mathbf{h}_k\|_2^2 = \|\mathbf{w}_k\|_2^2 = 1$. By definition of independent neighborhoods set, there is no intra-group interference among the nodes in S, and the nodes in $V - S$ have been shut off; thus the interference power $\sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2$ for nodes $k \in S$ is zero, again implying that the QoS constraint (37) is satisfied. Finally, since $\|\mathbf{w}_k\|^2 = 1$ for $k \in S$ and $\mathbf{w}_k = \mathbf{0}$ for $k \notin S$, it follows that

$$\sum_{k=1}^{K} \|\mathbf{w}_k\|^2 = |S| = P.$$

We have thus established the feasibility of (36). It can be further verified that the feasible solution (36) yields an objective value of $\epsilon|S| + (1 - \epsilon)4(K - |S|)$. Thus $\mathcal{A}(G)$ with $P = |S|$ admits a solution of cost less than or equal to $\epsilon|S| + (1 - \epsilon)4(K - |S|)$ in this case.

For the converse, suppose that $\mathcal{A}(G)$ with $P = |S|$ (for some positive integer $|S|$) admits a solution $\{\check{\mathbf{w}}_k,\ \check{s}_k \in \{-1, +1\}\}_{k=1}^K$ of cost less than or equal to $\epsilon|S| + (1-\epsilon)4(K-|S|)$. We show below that graph G must have an independent neighborhoods set of cardinality at least $|S|$ in this case. Let $\check{S} := \{k \mid \check{s}_k = -1\}$ denote the set of served users. Then, for each $k \in \check{S}$ we have

$$\|\check{\mathbf{w}}_k\|^2 \geq |\check{\mathbf{w}}_k^H \mathbf{h}_k|^2 \geq \sum_{\ell \neq k} |\check{\mathbf{w}}_\ell^H \mathbf{h}_k|^2 + 1 \geq 1,$$

where the first step follows from Cauchy-Schwartz inequality and the second step is due to the QoS constraint for node k. This shows that the objective value corresponding to the solution

$\{\check{\mathbf{w}}_k,\ \check{s}_k \in \{-1,+1\}\}_{k=1}^K$ must be at least

$$\epsilon \sum_{k=1}^{K} \|\check{\mathbf{w}}_k\|_2^2 + (1-\epsilon) \sum_{k=1}^{K} (\check{s}_k+1)^2 \geq \epsilon \sum_{k\in\check{S}} \|\check{\mathbf{w}}_k\|_2^2 + (1-\epsilon) \sum_{k\notin\check{S}} (\check{s}_k+1)^2 \geq \epsilon|\check{S}| + 4(1-\epsilon)(K-|\check{S}|).$$

Combining this with our assumption, we obtain

$$\epsilon|S| + (1-\epsilon)4(K-|S|) \geq \epsilon|\check{S}| + 4(1-\epsilon)(K-|\check{S}|),$$

yielding $|\check{S}| \geq |S|$ (since $\epsilon < 4/5$). In other words, at least $|S|$ users must be served, for otherwise the objective value would be higher than the postulated upper bound. On the other hand, the total transmit power constraint implies that

$$|S| = P \geq \sum_{k=1}^{K} \|\check{\mathbf{w}}_k\|^2 \geq \sum_{s\in\check{S}} \|\check{\mathbf{w}}_k\|^2 \geq |\check{S}|.$$

Therefore, we must have $|S| = |\check{S}|$, $\|\check{\mathbf{w}}_k\|^2 = 1$ for all $k \in \check{S}$ and $\check{\mathbf{w}}_k = \mathbf{0}$ for all $k \notin \check{S}$. It follows that the corresponding objective value must be exactly equal to $\epsilon|S| + (1-\epsilon)4(K-|S|)$. Since $\|\check{\mathbf{w}}_k\|^2 = \|\mathbf{h}_k\|^2 = 1$ for $k \in \check{S}$, it follows that $|\check{\mathbf{w}}_k^H \mathbf{h}_k|^2 \leq 1$ (Cauchy-Schwartz inequality). For $\check{\mathbf{w}}_k$ to be sufficient for satisfying the QoS constraint of user $k \in \check{S}$, i.e.,

$$\frac{1}{\sum_{\ell\neq k} |\check{\mathbf{w}}_\ell^H \mathbf{h}_k|^2 + 1} \geq \frac{|\check{\mathbf{w}}_k^H \mathbf{h}_k|^2}{\sum_{\ell\neq k} |\check{\mathbf{w}}_\ell^H \mathbf{h}_k|^2 + 1} \geq 1$$

the interference term $\sum_{\ell\neq k} |\check{\mathbf{w}}_\ell^H \mathbf{h}_k|^2$ must be zero and $|\check{\mathbf{w}}_k^H \mathbf{h}_k| = 1$. Since the latter condition implies that the Cauchy-Schwartz inequality holds with equality, we must have $\mathbf{w}_k = \pm\mathbf{h_k}$ for all $k \in \check{S}$. This further shows that

$$\mathbf{h}_\ell^H \mathbf{h}_k = \pm\check{\mathbf{w}}_\ell^H \mathbf{h}_k = 0, \qquad \text{for all } k \neq \ell \text{ and } k, \ell \in \check{S}.$$

By the definition of $\mathbf{h}_k$ in (35), we can conclude that the set of served users $\check{S}$ forms an independent neighborhoods set of size $|\check{S}| = |S|$. This completes the proof of converse. Thus, problem (12)-(14) is NP-hard for $N > 1$.

It remains to show that for the case of $N = 1$ (one transmit antenna), the corresponding single stage problem (12)-(14) can be solved in polynomial time. Indeed, in this case, the problem reduces to

$$\begin{aligned} \min_{\{p_k \geq 0, s_k \in \{-1,+1\}\}_{k=1}^K} \ & \epsilon \sum_{k=1}^K p_k + (1-\epsilon) \sum_{k=1}^K \lambda_k (s_k+1)^2 \\ \text{subject to:} \ & \sum_{k=1}^K p_k \leq P, \\ & \frac{p_k |h_k|^2 + \delta^{-1}(s_k+1)^2}{|h_k|^2 (\sum_{\ell \neq k} p_\ell) + \sigma_k^2} \geq c_k, \ \forall k \ \in \{1, \cdots, K\} \end{aligned} \tag{38}$$

Let $\{p_k \geq 0, s_k \in \{-1,+1\}\}_{k=1}^K$ denote an optimal solution and $S = \{k \mid s_k = -1\}$ denote the set of served users. Notice that a user $k \in S$ is served means exactly

$$\frac{p_k |h_k|^2}{|h_k|^2 (\sum_{\ell \neq k} p_\ell) + \sigma_k^2} \geq c_k, \quad \text{or equivalently,} \ \ p_k \geq \frac{c_k}{c_k+1} \sum_{\ell=1}^K p_\ell + \frac{c_k \sigma_k^2}{(c_k+1)|h_k|^2}.$$

Thus, by a simple monotonicity argument, there holds

$$\frac{c_k}{c_k+1} u + \frac{c_k \sigma_k^2}{(c_k+1)|h_k|^2} \leq \frac{c_\ell}{c_\ell+1} u + \frac{c_\ell \sigma_\ell^2}{(c_\ell+1)|h_\ell|^2}, \quad \forall \, k \in S, \ \ell \notin S,$$

where $u := \sum_{\ell=1}^K p_\ell$ denotes the total transmit power. This is true because otherwise user ℓ could be (and should be) served instead of user k with less total transmit power, contradicting the optimality of S. For any $u \in [0, P]$, let us define

$$p_k(u) = \frac{c_k}{c_k+1} u + \frac{c_k \sigma_k^2}{(c_k+1)|h_k|^2}.$$

Then, S must be of the form $\{k \mid p_k(u) \leq \tau\}$ for some $\tau > 0$. For any fixed $u \in [0, P]$ there are only K such type of subsets of S, and they can be easily determined by sorting $p_k(u)$ in increasing order. We can then search over these K subsets to determine which gives the smallest objective value. When u increases monotonically from 0 to P, the ordering of $\{p_k(u)\}$ may vary. But the ordering changes only when $p_k(u) = p_\ell(u)$ for some pair of ℓ and k. As a result, there can be at most $K(K-1)/2$ different orderings of $p_k(u)$ for $u \in [0, P]$. This implies that there are at most $K^2(K-1)/2$ different subsets of S that need to be searched over, as u varies over $[0, P]$. These subsets are all computable in polynomial time (involving simple sorting operations and thresholding). Picking the smallest overall objective value from these subsets solves the original problem. ■

Proof of Claim 4: First, notice that $s_k(M) \geq 0$ for all k and M. This is due to the fact that if $s_k(M) < 0$, then setting this variable to zero would yield a better solution for the relaxed problem (28)-(30). Consequently, we have

$$\lim_{M\to\infty} s_k(M) \geq 0, \quad \forall\, k.$$

Let $R(M)$ denote the optimal value of (28)-(30). If (25)-(27) is feasible, then $\{s_k = 0\}_{k=1}^K$ together with some $\{\bar{\mathbf{w}}_k \in \mathbb{C}^N\}_{k=1}^K$ will satisfy all the constraints of (28)-(30) for all M. This shows

$$R(M) \leq \sum_{k=1}^{K} \|\bar{\mathbf{w}}_k\|^2, \quad \forall\, M,$$

implying that $\lim_{M\to\infty} R(M) < \infty$. This further shows that $|\mathcal{K}| = 0$, for otherwise we would have $\lim_{M\to\infty} R(M) \geq \lim_{M\to\infty} M \sum_{k=1}^K s_k^2(M) \geq \lim_{M\to\infty} M \sum_{k\in\mathcal{K}} s_k^2(M) = \infty$.

Conversely, if $|\mathcal{K}| = 0$, then $\lim_{k\to\infty} s_k(M) = 0$ for all $k = 1, 2, ..., K$. In this case, we claim that any limit point of $\{\mathbf{w}_k(M) \in \mathbb{C}^N\}_{k=1}^K$ (whose existence follows from the bounded transmission power constraint $\sum_{k=1}^K \|\mathbf{w}_k\|^2 \leq P$) will be a feasible solution of (25)-(27). This can be seen by taking limit with $M \to \infty$ in the following conditions:

$$\begin{cases} \mathbf{w}_k^H(M)\mathbf{h}_k + s_k(M) \geq \sqrt{c_k \sum_{\ell\neq k} |\mathbf{w}_\ell^H(M)\mathbf{h}_k|^2 + c_k\sigma_k^2}, \forall\, k \\ \sum_{k=1}^{K} \|\mathbf{w}_k(M)\|^2 \leq P, \quad \mathrm{Im}(\mathbf{w}_k^H(M)\mathbf{h}_k) = 0, \forall k. \end{cases}$$

To prove the second part of the claim, notice that $\lim_{k\to\infty} s_k(M) = 0$ for all $k \notin \mathcal{K}$. The above limiting argument readily shows that the constraints

$$\begin{cases} \mathbf{w}_k^H \mathbf{h}_k \geq \sqrt{c_k \sum_{\ell \neq k} |\mathbf{w}_\ell^H \mathbf{h}_k|^2 + c_k \sigma_k^2}, \quad \forall\, k \notin \mathcal{K} \\ \sum_{k=1}^{K} \|\mathbf{w}_k\|^2 \leq P, \quad \mathrm{Im}(\mathbf{w}_k^H \mathbf{h}_k) = 0, \quad \forall\, k \notin \mathcal{K} \end{cases}$$

are satisfied by any limit point of the sequence $\{\mathbf{w}_k(M) \in \mathbb{C}^N\}_{k=1}^K$ when $M \to \infty$. This implies that the reduced problem (31)-(33) is always feasible, as claimed. $\blacksquare$

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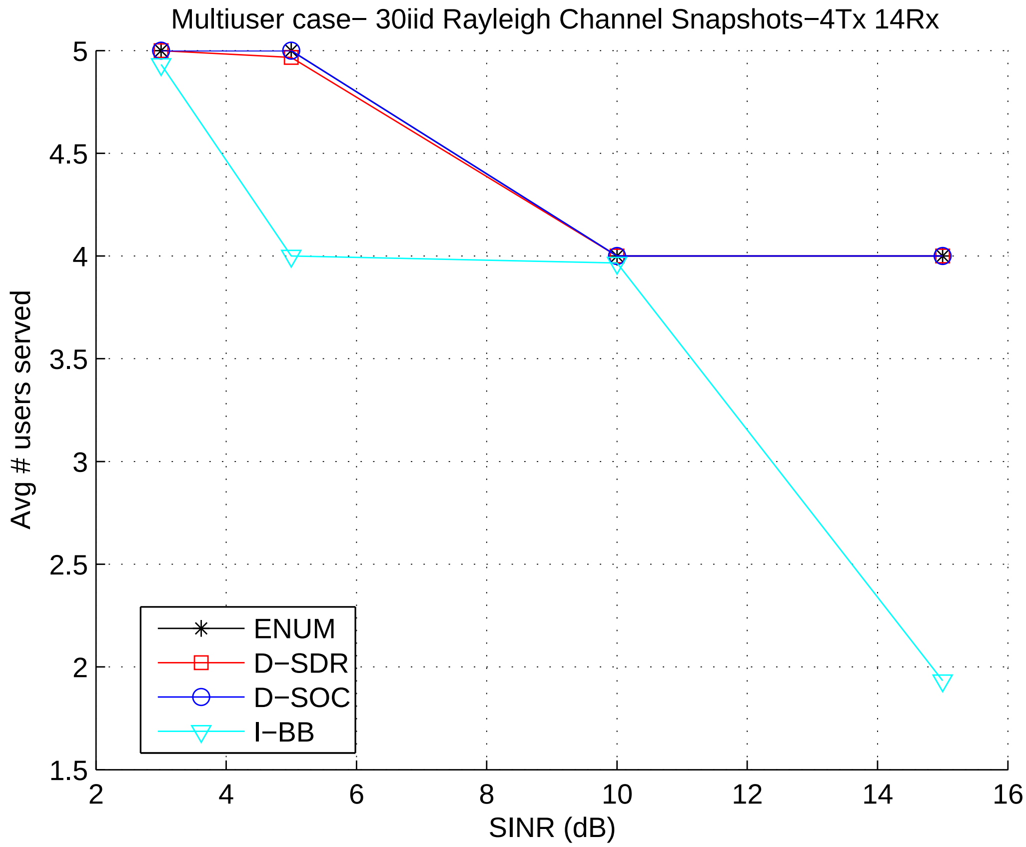

Fig. 1. Average number of users served versus target SINR: i.i.d. Rayleigh channels, 30 Monte-Carlo runs.

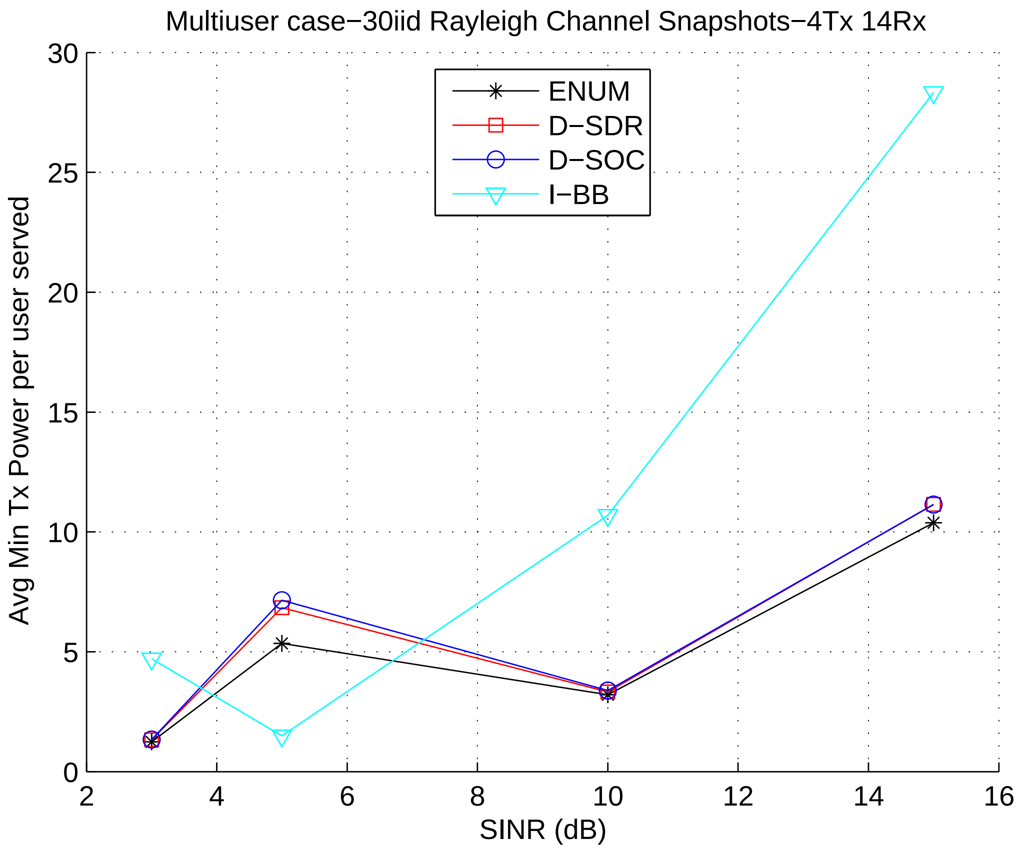

Fig. 2. Average power per user served versus target SINR: i.i.d. Rayleigh channels, 30 Monte-Carlo runs.

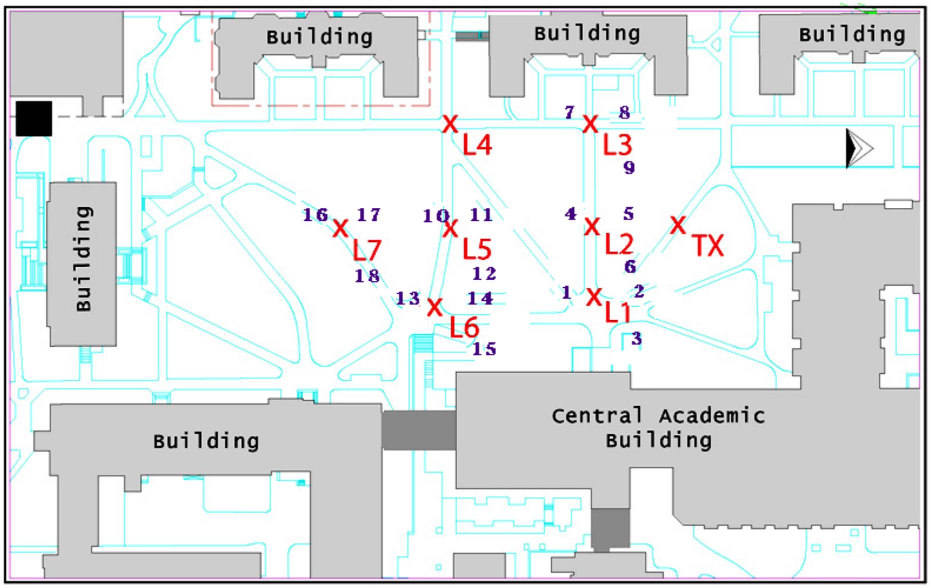

Fig. 3. Sample wireless channel measurement scenario from http://www.ece.ualberta.ca/∼mimo/

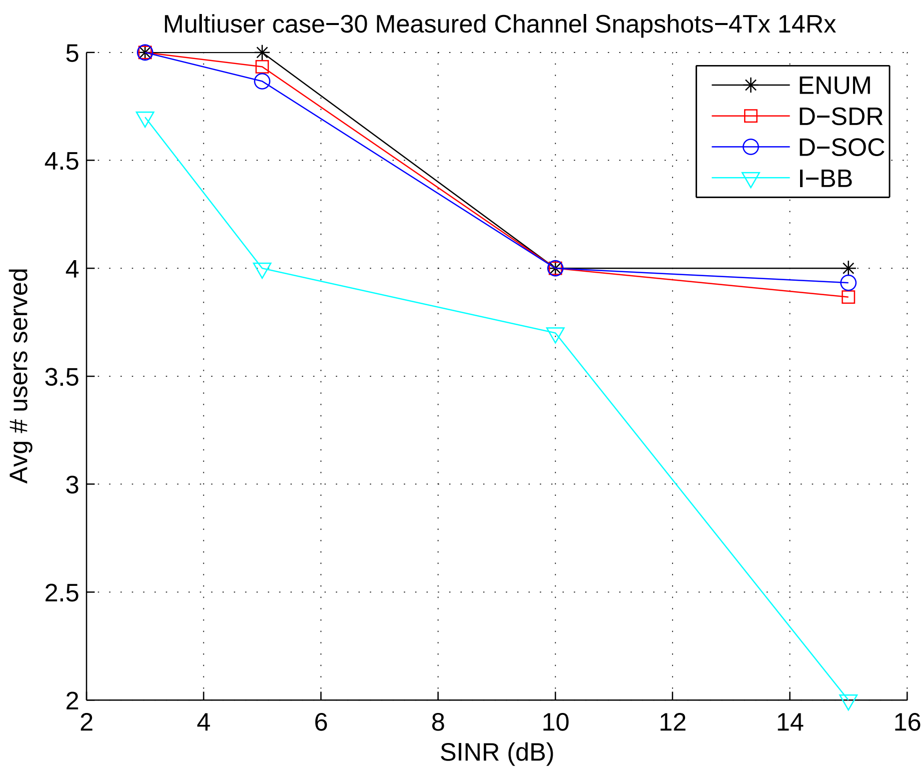

Fig. 4. Average number of users served versus target SINR: 30 measured channel snapshots.

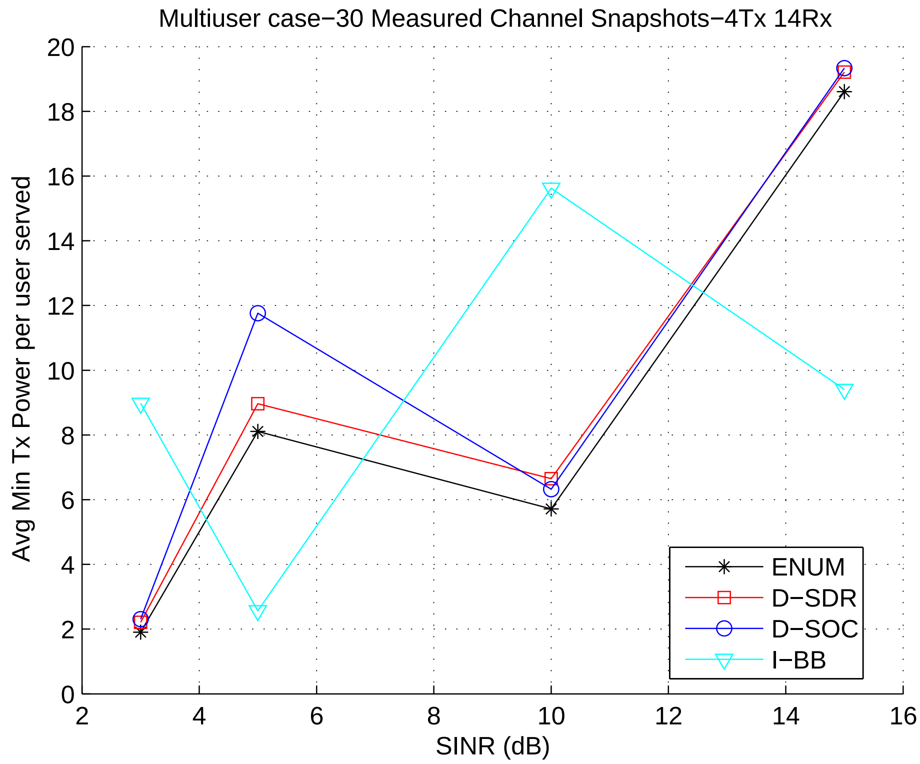

Fig. 5. Average power per user served versus target SINR: 30 measured channel snapshots.

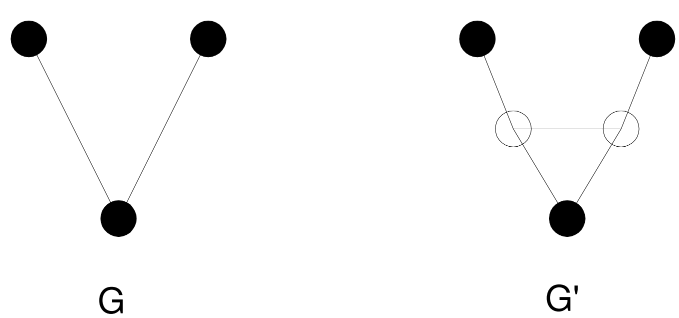

Fig. 6. Illustration of the construction of $G^{'}$ from G in the proof of Claim 3. Original nodes are in black, whereas newly introduced nodes are shown in white. The top two nodes form an independent set in G, and an independent two-hop neighborhoods set in $G^{'}$.

TABLE I

MONTE-CARLO RESULTS (30 i.i.d. Rayleigh channel snapshots): $N = 4$ TX ANT., $K = 14$ USERS, $P = 100$; $\sigma_k^2 = \sigma^2 = 1$, $c_k = c$, $\lambda_k = 1, \forall k$; $e = 0.0001 < \frac{1}{P/4+1}$, $\delta = \frac{4c^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma^2}$; FOR D-SOC, $M = 10^{10}$ FOR QOS TARGET $\in \{3, 5, 10\}$ dB; $M = 10^{11}$ FOR QOS TARGET 15 dB.

QoS target	Alg	♯ users served	Avg Min Tx Power	Max Min Tx.Power	Avg Time
3	SOCP enum	5	6.2043	9.5117	0.728 h
3	D-SDR	5	6.6267	11.094	0.89 s
3	D-SOC	5	6.6871	11.202	3.233 s
3	I-BB	5@ 93.33% 4 @ 6.66%	23.2382	73.3527	0.0534 s
5	SOCP enum	5	26.7524	41.8895	0.612 h
5	D-SDR	5@96.66% 4@ 3.33%	33.9469	78.7254	0.822 s
5	D-SOC	5	35.7631	80.3822	3.189 s
5	I-BB	4	5.9747	15.4648	0.0405 s
10	SOCP enum	4	12.812	18.8976	0.505 h
10	D-SDR	4	13.311	19.2041	0.8572 s
10	D-SOC	4	13.5383	19.2041	3.0324 s
10	I-BB	4 @ 96.66% 3 @ 3.33%	42.4219	85.039	0.0397 s
15	SOCP enum	4	41.5433	62.712	0.434 h
15	D-SDR	4	44.5851	67.2114	0.8372 s
15	D-SOC	4	44.5535	62.712	2.556 s
15	I-BB	4 @ 3.33% 2 @ 83.33% 1 @ 13.33%	54.7591	99.1921	0.0202 s

TABLE II

MONTE-CARLO RESULTS (30 measured channel snapshots): $N = 4$ TX ANT., $K = 14$ USERS (ALL DEPICTED IN FIG. 3 EXCEPT 7, 10, 12, AND 17), $P = 100$; $\sigma_k^2 = \sigma^2 = 1$, $c_k = c$, $\lambda_k = 1, \forall k$; $e = 0.0001 < \frac{1}{P/4+1}$, $\delta = \frac{4c^{-1}}{P \max_m \|\mathbf{h}_m\|_2^2 + \sigma^2}$; FOR D-SOC, $M = 10^{10}$ FOR QOS TARGET 5 DB AND 10 dB; $M = 10^{11}$ FOR QOS TARGET 3 dB AND 15 dB.

QoS target	Alg	♯ users served	Avg Min Tx Power	Max Min Tx.Power	Avg Time
3	SOCP enum	5	9.5342	13.1834	0.697 h
3	D-SDR	5	11.0522	15.7259	0.9978 s
3	D-SOC	5	11.5324	15.4238	3.24 s
3	I-BB	5@ 70% 4 @ 30%	42.1933	96.5993	0.0515 s
5	SOCP enum	5	40.5392	60.4637	0.554 h
5	D-SDR	5@93.33% 4@ 6.66%	44.2296	81.935	0.93 s
5	D-SOC	5@86.66% 4@13.33%	57.248	88.882	3.277 s
5	I-BB	4	10.2613	14.9659	0.0407 s
10	SOCP enum	4	22.8658	29.5651	0.488 h
10	D-SDR	4	26.5837	33.7893	0.8706 s
10	D-SOC	4	25.2875	44.7832	3.132 s
10	I-BB	4 @ 70% 3 @ 30%	57.8008	90.9868	0.0339 s
15	SOCP enum	4	74.4436	95.5118	0.429 h
15	D-SDR	4@86.66% 3@13.33%	74.2668	98.7509	0.8466 s
15	D-SOC	4@93.33% 3@6.66%	76.0445	99.089	2.577 s
15	I-BB	2	18.8126	31.1947	0.0137 s