Feedback Control of Nonlinear Hyperbolic PDE Systems Inspired by Traffic Flow Models

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***Abstract*—The paper investigates and provides results, including feedback control, for a nonlinear, hyperbolic, 1-D PDE system on a bounded domain. The considered model consists of two first-order PDEs with a dynamic boundary condition on the one end and actuation on the other. It is shown that, for all positive initial conditions, the system admits a globally defined, unique, classical solution that remains positive and bounded for all times; these properties are important, for example for traffic flow models. Moreover, it is shown that global stabilization can be achieved for arbitrary equilibria by means of an explicit boundary feedback law. The stabilizing feedback law depends only on collocated boundary measurements. The efficiency of the proposed boundary feedback law is demonstrated by means of a numerical example of traffic density regulation.**

***Index Terms*-hyperbolic PDEs, traffic flow, boundary feedback.**

I. Introduction

The study of vehicular traffic flow by means of hyperbolic Partial Differential Equations (PDEs) started in the 1950s with the LWR first-order model (see [27, 32]). In order to describe more accurately the mean speed dynamics, second-order models were later studied (see [1, 29, 39]). Many 1-D traffic flow models with no control were developed for unbounded domains (usually the whole real axis). Researchers working on second-order models as well as critics of second-order models (see [12]) have agreed that a valid traffic flow model must: (i) include the vehicle conservation equation, (ii) admit bounded solutions which predict positive values for both density and mean speed, (iii) obey the so-called anisotropy principle, i.e., the fact that a vehicle is influenced only by the traffic dynamics ahead of it, (iv) not allow waves traveling forward with a speed greater than the traffic speed. Recently, researchers have developed two phase models (see [8, 25]).

Recent advances in the boundary feedback control of hyperbolic systems of PDEs (see for instance [2, 3, 7, 9, 10, 11, 13, 14, 19, 22, 23, 30, 31, 35, 36]) as well as advances in the control of discrete-time, finite-dimensional traffic flow models (see [17, 18, 20, 28] and references therein) have motivated the study of well-posedness and control of traffic flow models on bounded domains. Both issues (well-posedness and control) for first-order models in bounded domains were studied in [4, 5, 33]. The stabilization of equilibrium profiles for linearized 2^{nd} order models in bounded domains by means of boundary feedback was also studied in [24, 37, 38, 40].

The present work considers a specific hyperbolic, nonlinear, second-order, 1-D PDE system on a bounded domain, which may be viewed as partial linearization of the ARZ model [1, 39] around an equilibrium point in a congested road. It consists of two quasilinear first-order PDEs with a dynamic nonlinear boundary condition that involves the time derivative of the speed, analogously to in-domain relaxation in typical second-order traffic flow models [1, 39]. The presence of this dynamic boundary condition renders the model non-standard (since standard systems of hyperbolic PDEs involve boundary conditions which do not contain the derivatives of the states), and thus, the existence and uniqueness of its solutions cannot be guaranteed by using standard results (see [2, 6, 21, 26]). The existence and uniqueness issues are first studied in the present work. Specifically, it is shown that for all physically meaningful initial conditions, the model admits a globally defined, unique, classical solution that remains positive and bounded for all times. As a result, we can guarantee that the proposed model has all of the four features mentioned in the first paragraph that are important from a traffic-theoretic point of view. The second contribution of the present work is the study of the control problem for the proposed model. Specifically, we design a simple, nonlinear, boundary feedback law, adjusting the inlet flow (via, e.g., ramp or mainline metering). The boundary feedback law employs only measurements of the inlet speed, and consequently, the measurement requirements for implementation of the proposed controller are minimal. Moreover, it is shown that the developed control design achieves global asymptotic stabilization of arbitrary equilibria, in the sup-norm of the

Accepted manuscript.

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logarithmic deviation of the state from its equilibrium point. The efficiency of the proposed feedback law is demonstrated by means of a numerical example.

Section II is devoted to the presentation of the model and the statement of the first main result (Theorem 2.1) which guarantees, for all physically meaningful initial conditions, the existence of a globally defined, unique, classical solution that remains positive and bounded for all times. The control design and the statement of the second main result, which guarantees global stabilization of arbitrary equilibria of the model (Theorem 3.1) are given in Section III. A simple illustrative example is presented in Section IV. The proofs of the main results as well as auxiliary results are provided in Section V. One of the auxiliary results has interest on its own (Proposition 5.2), because it covers a case not studied in [2, 6, 21, 26]: namely the case of a transport PDE with a non-negative (possibly zero at some points) transport speed. A unique, classical solution is shown to exist, which is differentiable and satisfies the PDE even on the boundary (something that cannot be guaranteed by the results in [21]). The concluding remarks are provided in Section VI. Finally, the Appendix contains the proofs of the two auxiliary results of Section V.

<u>Notation.</u>

* $\Re_+ := [0,+\infty)$. For a real number $x \in \Re$, $[x]$ denotes the integer part of x, i.e., the greatest integer which is less or equal to x.
* Let $U \subseteq \Re^n$ be a set with non-empty interior and let $\Omega \subseteq \Re$ be a set. By $C^0(U;\Omega)$, we denote the class of continuous mappings on U, which take values in Ω. By $C^k(U;\Omega)$, where $k \ge 1$, we denote the class of continuous functions on U, which have continuous derivatives of order k on U and take values in Ω. When Ω is omitted, i.e., when we write $C^k(U)$, it is implied that $\Omega = \Re$.
* Let $T \in (0,+\infty)$ and $u:[0,T]\times[0,1] \to \Re$ be given. We use the notation $u[t]$ to denote the profile at certain $t \in [0,T]$, i.e., $(u[t])(x) = u(t,x)$ for all $x \in [0,1]$. For a bounded $w:[0,1]\to\Re$ the sup-norm is given by $\|w\|_\infty := \sup_{0\le x\le 1}\left(|w(x)|\right)$.
* $W^{2,\infty}([0,1])$ is the Sobolev space of C^1 functions on $[0,1]$ with Lipschitz derivative.
* By K we denote the class of strictly increasing continuous functions $a:\Re_+ \to \Re_+$ with $a(0)=0$. By K_∞ we denote the class of functions $a \in K$ with $\lim_{s\to+\infty} a(s) = +\infty$. By KL we denote the set of all functions $\sigma \in C^0(\Re_+ \times \Re_+ ; \Re_+)$ with the properties: (i) for each $t \ge 0$, $\sigma(\cdot,t)$ is of class K; (ii) for each $s \ge 0$, $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to+\infty}\sigma(s,t)=0$.

II. The Model and Its Properties

In this section we present the nonlinear model of conservation laws, which is inspired by traffic flow PDE models. Moreover, we guarantee properties of the model, which are crucial from a traffic-theoretic point of view.

II.A. Description

Second-order traffic flow models involve a system of hyperbolic PDEs on the positive semi-axis. The state variables are the vehicle density $\rho(t,x)$ and the vehicle mean speed $v(t,x)$, where $t \ge 0$ is time and x is the spatial variable. All 2nd order traffic flow models involve the conservation equation

$$\frac{\partial \rho}{\partial t}(t,x) + v(t,x)\frac{\partial \rho}{\partial x}(t,x) + \rho(t,x)\frac{\partial v}{\partial x}(t,x) = 0 \quad (2.1)$$

and an additional PDE for the speed. In a congested road, the vehicle speed depends heavily on the speed of downstream vehicles. Therefore, the following equation may be appropriate for the evolution of the speed profile:

$$\frac{\partial v}{\partial t}(t,x) - c\frac{\partial v}{\partial x}(t,x) = 0\,, \quad (2.2)$$

where $c > 0$ is a constant related to the drivers' promptness in adjusting their speed. Equation (2.2) may also arise as a linearization of the speed PDE of the Aw-Rascle-Zhang model (ARZ; see [1, 39]) around a "congested equilibrium" without an in-domain relaxation term. By "congested equilibrium", we mean a spatially uniform equilibrium profile $\rho(x) \equiv \rho_{eq}$, $v(x) \equiv v_{eq}$ for which $v_{eq} + \kappa(\rho_{eq}) < 0$, where κ is the function involved in the speed PDE $\frac{\partial v}{\partial t}(t,x) + \left(v(t,x) + \kappa(\rho(t,x))\right)\frac{\partial v}{\partial x}(t,x) = 0$ of the ARZ model without an in-domain relaxation term. Therefore, model (2.1), (2.2) can be seen as a partial linearization of the Aw-Rascle-Zhang model. Here, we consider the model (2.1), (2.2) on a bounded domain, i.e., we assume that $x \in [0,1]$. The full model requires the specification of two boundary conditions. One boundary condition describes the inlet conditions, and more particularly the effect of the inlet demand $q(t) > 0$ and takes the form

$$\rho(t,0) = h\left(q(t)/v(t,0)\right), \text{ for } t \ge 0 \quad (2.3)$$

where $h \in C^2(\Re_+)$ is a non-decreasing function that satisfies

$$h(s) = s \text{ for } s \in [0, \rho_{\max} - \varepsilon]$$
$$\text{and } h(s) = \rho_{\max} \text{ for } s \ge \rho_{\max}\,, \quad (2.4)$$

where $\rho_{\max} > 0$ is a constant related to the physical upper bound of density in the particular road and $\varepsilon \in (0, \rho_{\max})$ is a sufficiently small constant. Notice that (2.3) implies that the inlet demand $q(t) > 0$ is equal to the vehicle inflow $\rho(t,0)v(t,0)$, provided that $q(t) \le (\rho_{\max} - \varepsilon)v(t,0)$. The boundary condition (2.3), as well as the rest of the model (2.1), (2.2), come together with the following requirement:

$$\rho(t,x) > 0 \text{ and } v(t,x) > 0\,, \text{ for all } (t,x) \in \Re_+ \times [0,1] \quad (2.5)$$

Condition (2.5) is an essential requirement for a model of a physical process, such as traffic flow. In what follows, we show that the proposed model meets this requirement.

In order to have a well-posed hyperbolic system, we also need a boundary condition at the outlet $x=1$. Assuming that the flow downstream of the outlet is uncongested (free), it is reasonable to assume that the relaxation term becomes dominant. So, we get

$$\frac{\partial v}{\partial t}(t,1) = -\mu\big(v(t,1) - f(\rho(t,1))\big), \text{ for } t \geq 0 \qquad (2.6)$$

where $\mu > 0$ is a constant and $f \in C^1(\Re_+)$ is a positive, bounded, non-increasing function that, in the case of traffic flow, expresses the fundamental diagram relation between density and speed. Condition (2.6) implies that there is no downstream influence at the downstream boundary. For traffic flow, this may be the case if the highway infrastructure downstream of the considered stretch has a higher capacity, e.g. due to an additional lane; or end of a tunnel or bridge; or end of a curvature or uphill stretch; or end of a speed-limited zone.

An important fact should be emphasized at this point: when $v(t,1)$ can be manipulated, then (2.6) can be seen as a dynamic feedback law at the outlet. Therefore, the boundary condition (2.6) can arise either for modeling purposes (to guarantee the relation $v = f(\rho)$ at equilibrium) or for control purposes (as a feedback law).

II.B. Traffic-Theoretic Features of the Model

Equations (2.1), (2.2), (2.3), (2.6) form a non-standard system of nonlinear hyperbolic PDEs. The reason that system (2.1), (2.2), (2.3), (2.6) cannot be studied by existing results in hyperbolic systems (see [2, 6, 21, 26]) is the non-standard boundary condition (2.6). However, in what follows, we show that system (2.1), (2.2), (2.3), (2.6) exhibits unique, positive, globally defined C^1 solutions for all positive initial conditions. Moreover, we show that density and speed are bounded from above by certain bounds that depend only on the initial conditions and the physical upper bounds of the density and speed, i.e., $\rho_{\max}$ and $v_{\max} = f(0)$, respectively. Before we show this, it is important to emphasize that (2.1), (2.2), (2.3), (2.6) may be viewed as a traffic flow model that

- can be applied to bounded domains, i.e., $x \in [0,1]$, without assuming knowledge of density/speed out of the domain,
- is completely anisotropic, i.e., the speed depends only on the speed of downstream vehicles,
- is a hyperbolic model of conservation laws of the form $\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0$, where $u = \begin{bmatrix} \rho \\ v \end{bmatrix}$, $A(u) = \begin{bmatrix} v & \rho \\ 0 & -c \end{bmatrix}$. The matrix $A(u)$ has two eigenvalues v and $-c$; consequently, information travels forward at exactly the same speed as traffic,
- allows only equilibria which satisfy the fundamental diagram law $v = f(\rho)$, i.e., when $q(t) \equiv q_{eq} > 0$ then the equilibrium profiles are given by $\rho(x) \equiv \rho_{eq}$, $v(x) \equiv f(\rho_{eq})$, where $\rho_{eq} > 0$ is a solution of $\rho_{eq} = h\big(q_{eq} / f(\rho_{eq})\big)$.

All the above features are important for a traffic flow model.

II.C. Characteristic Form

Let $\rho_{eq} \in (0, \rho_{\max})$ be a given constant. The nonlinear transformation of the density variable

$$\rho(t,x) = \rho_{eq}\big(c + f(\rho_{eq})\big)\big(c + v(t,x)\big)^{-1} \exp(w(t,x)) \qquad (2.7)$$

yields the equation

$$\frac{\partial w}{\partial t}(t,x) + v(t,x)\frac{\partial w}{\partial x}(t,x) = 0 \qquad (2.8)$$

with the boundary conditions

$$w(t,0) = \ln\left(\rho_{eq}^{-1} h\left(\frac{q(t)}{v(t,0)}\right)\frac{c + v(t,0)}{c + f(\rho_{eq})}\right),$$

$$\frac{\partial v}{\partial t}(t,1) = -\mu\left(v(t,1) - f\left(\rho_{eq}\exp(w(t,1))\frac{c + f(\rho_{eq})}{c + v(t,1)}\right)\right). \qquad (2.9)$$

The hyperbolic system (2.2), (2.8), (2.9) is nothing else but the hyperbolic system (2.1), (2.2), (2.3), (2.6) in Riemann coordinates. Provided that the initial conditions are positive, i.e., $\rho(0,x) > 0$, $v(0,x) > 0$, for $x \in [0,1]$, we are in a position to construct a unique solution to (2.1), (2.2), (2.3), (2.6) by constructing a unique solution to (2.2), (2.8), (2.9) and employing the nonlinear transformation (2.7).

II.D. Well-Posedness and Positivity of the System

The solution of (2.2), (2.8), (2.9) is constructed by using the following theorem. Its proof is provided in Section 5.

Theorem 2.1: *Let* $a \in C^2(\Re_+ \times \Re_+)$ *be any given function and let* $c > 0$, $\mu \geq 0$ *be given constants. Let* $g \in C^1(\Re_+ \times \Re)$ *be a given function for which there exists a constant* $v_{\max} > 0$ *such that the following inequality holds*

$$0 < g(0,w) \leq g(v,w) \leq v_{\max}, \text{ for all } v \in \Re_+, w \in \Re \qquad (2.10)$$

Let $\theta, \varphi \in W^{2,\infty}([0,1])$ *be given functions with* $\varphi(x) > 0$ *for all* $x \in [0,1]$, *for which the equalities*

$$\theta(0) = a(0, \varphi(0)), \quad \varphi'(1) = -\mu c^{-1}\big(\varphi(1) - g(\varphi(1), \theta(1))\big),$$

$$\frac{\partial a}{\partial t}(0, \varphi(0)) + c\frac{\partial a}{\partial v}(0, \varphi(0))\varphi'(0) = -\varphi(0)\theta'(0),$$

hold. Then the initial-boundary value problem

$$\frac{\partial w}{\partial t}(t,x) + v(t,x)\frac{\partial w}{\partial x}(t,x) = \frac{\partial v}{\partial t}(t,x) - c\frac{\partial v}{\partial x}(t,x) = 0,$$

$$\text{for all } (t,x) \in \Re_+ \times [0,1] \qquad (2.11)$$

$$w(t,0)-a(t,v(t,0))=\frac{\partial v}{\partial t}(t,1)+\mu\left(v(t,1)-g\left(v(t,1),w(t,1)\right)\right)=0\,,$$
$$\text{for all } t\geq 0 \tag{2.12}$$

$$w(0,x)-\theta(x)=v(0,x)-\varphi(x)=0\,,\ \text{for } x\in[0,1] \tag{2.13}$$

admits a unique solution $w,v\in C^{1}\left(\Re_{+}\times[0,1]\right)$. *Moreover, the solution* $w,v\in C^{1}\left(\Re_{+}\times[0,1]\right)$ *has Lipschitz derivatives on every compact* $S\subset\Re_{+}\times[0,1]$ *and satisfies the following inequalities for all* $(t,x)\in\Re_{+}\times[0,1]$:

$$\|w[t]\|_{\infty}\leq\max\left(B_{t}\,,\|\theta\|_{\infty}\right) \tag{2.14}$$

$$\begin{aligned}&\min\left(\min_{0\leq x\leq 1}\left(\varphi(x)\right),\min\left\{g(0,w):|w|\leq\max\left(B_{t}\,,\|\theta\|_{\infty}\right)\right\}\right)\\&\leq v(t,x)\leq\max\left(\max_{0\leq x\leq 1}\left(\varphi(x)\right),v_{\max}\right)\end{aligned} \tag{2.15}$$

where

$$B_{t}:=\max\left\{\,|a(s,v)|:s\in[0,t]\,,0\leq v\leq\max\left(\max_{0\leq x\leq 1}\left(\varphi(x)\right),v_{\max}\right)\right\}.$$

Notice that Theorem 2.1 holds for any (arbitrary) function $a\in C^{2}(\Re_{+}\times\Re_{+})$. It should be noted that the solution provided by Theorem 2.1 is a classical solution of the nonlinear system of conservation laws (2.11) with boundary conditions given by (2.12). The fact that systems of nonlinear conservation laws may admit classical solutions is well-known (see [2, 6, 21, 26]). However, an existence/uniqueness result for the case (2.11), (2.12) is not available in the literature and Theorem 2.1 is a novel result.

Remark 2.2: Theorem 2.1 shows that the appropriate (state) space for studying the hyperbolic system (2.1), (2.2), (2.3), (2.6) is the space X that contains all functions $(\rho,v)\in\left(W^{2,\infty}([0,1])\right)^{2}$ for which there exist numbers $a_{1}>0$, $a_{2}\in\Re$ such that

$$\begin{aligned}&\min(\rho(x),v(x))>0\ \textit{for all}\ x\in[0,1],\\&cv'(1)=-\mu\left(v(1)-f(\rho(1))\right),\rho(0)=h(a_{1}),\\&v(0)\rho'(0)+\rho(0)v'(0)=a_{2}h'(a_{1})\end{aligned} \tag{2.16}$$

In order to construct a solution $\left(\rho[t],v[t]\right)\in X$ of (2.1), (2.2), (2.3), (2.6) with initial conditions in $(\rho_{0},v_{0})\in X$, we apply Theorem 2.1 with

$$a(t,v):=\begin{cases}\ln\left(\rho_{eq}^{-1}h\left(\dfrac{q(t)}{v}\right)\dfrac{c+v}{c+f(\rho_{eq})}\right) & \textit{if } v>0\\[2ex] \ln\left(\rho_{eq}^{-1}\rho_{\max}\dfrac{c+v}{c+f(\rho_{eq})}\right) & \textit{if }\ v=0\end{cases},$$

$$g(v,w):=f\left(\rho_{eq}\exp(w)\frac{c+f(\rho_{eq})}{c+v}\right),\ v_{\max}:=f(0)\,,$$

$$\theta(x)=\ln\left(\frac{\rho_{0}(x)\left(c+v_{0}(x)\right)}{\left(c+f(\rho_{eq})\right)\rho_{eq}}\right),\ \varphi(x)=v_{0}(x)\ \text{for}\ x\in[0,1]$$

and we consider $q\in C^{2}\left(\Re_{+};(0,+\infty)\right)$ to be the input of the model. The set of admissible inputs consists of all functions $q\in C^{2}\left(\Re_{+};(0,+\infty)\right)$ that satisfy the conditions

$$\begin{aligned}&v_{0}(0)\rho_{0}'(0)+\rho_{0}(0)v_{0}'(0)+h'\left(\frac{q(0)}{v_{0}(0)}\right)\frac{\dot{q}(0)}{v_{0}(0)}\\&=ch'\left(\frac{q(0)}{v_{0}(0)}\right)\frac{q(0)}{v_{0}^{2}(0)}v_{0}'(0)\\&\text{and}\ \rho_{0}(0)=h\left(q(0)/v_{0}(0)\right).\end{aligned}$$

The solution $\left(\rho[t],v[t]\right)\in X$ of (2.1), (2.2), (2.3), (2.6) is found by using the solution $\left(w[t],v[t]\right)$ of (2.11), (2.12), (2.13) in conjunction with formula (2.7). Notice that if $v_{0}(x)\leq v_{\max}$ for all $x\in[0,1]$, then estimate (2.15) implies that $0<v(t,x)\leq v_{\max}$ for all $(t,x)\in\Re_{+}\times[0,1]$ and for all admissible $q\in C^{2}\left(\Re_{+};(0,+\infty)\right)$. Similarly, by performing more detailed calculations than those in the proof of Theorem 2.1, we are in a position to verify that if $\rho_{0}(x)\leq\rho_{\max}\left(c+v_{\max}\right)/c$ for $x\in[0,1]$, then the estimate $0<\rho(t,x)\leq\rho_{\max}\left(c+v_{\max}\right)/c$ holds for all $(t,x)\in\Re_{+}\times[0,1]$ and for all admissible $q\in C^{2}\left(\Re_{+};(0,+\infty)\right)$.

III. Collocated Boundary Control Design

The main result of the present section shows that stabilization of the equilibrium profile for a given desired equilibrium density $\rho_{eq}>0$ can be achieved by controlling the inlet flow. It is important to notice that the stabilizing feedback law depends *only* on the inlet speed.

We next describe the basic ideas behind the construction of the feedback law. In order to derive a globally stabilizing boundary feedback law for the traffic flow model (2.1), (2.2), (2.3), (2.6), we employ the characteristic form given by (2.2), (2.3), (2.7), (2.8) and (2.9). In order to drive the transformed state w to zero, we use the boundary condition $w(t,0)=0$, which can be expressed in terms of the density and speed by the collocated boundary feedback law:

$$q(t)=\rho_{eq}v(t,0)\left(c+v(t,0)\right)^{-1}\left(c+f(\rho_{eq})\right) \tag{3.1}$$

The feedback law (3.1) will not necessarily drive v to its equilibrium value $f\left(\rho_{eq}\right)$. To this purpose, we need to employ an assumption that deals with the outlet boundary condition, namely the assumption that the following inequality holds:

$$\left(v-f\left(\rho_{eq}\frac{c+f(\rho_{eq})}{c+v}\right)\right)\left(v-f\left(\rho_{eq}\right)\right)>0\,,$$
$$\text{for all } v\geq 0\,,\ v\neq f(\rho_{eq}) \tag{3.2}$$

In this way, existence and uniqueness of classical solutions for the closed-loop system may be guaranteed by means of Theorem 2.1 with

$a(t,v):=0$, $g(v,w):=f\left(\rho_{eq}\exp(w)\frac{c+f(\rho_{eq})}{c+v}\right)$, $v_{\max}:=f(0)$.

Our main result is stated next.

Theorem 3.1: *Consider the traffic flow model (2.1), (2.2), (2.3), (2.6) and let $\rho_{eq}>0$ be the desired equilibrium density. Suppose that $\rho_{eq}\le c\left(c+f(\rho_{eq})\right)^{-1}(\rho_{\max}-\varepsilon)$ and that (3.2) holds. Then there exists $Q\in KL$ such that for every $(\rho_0,v_0)\in X$ for which the equalities $\rho_0(0)=\rho_{eq}\frac{c+f(\rho_{eq})}{c+v_0(0)}$, $\rho_0'(0)=\rho_0(0)\left(c+v_0(0)\right)^{-1}v_0'(0)$ hold, the initial-boundary value problem (2.1), (2.2), (2.3), (2.6) with (3.1) and*

$$\rho(0,x)-\rho_0(x)=v(0,x)-v_0(x)=0\text{, for } x\in[0,1] \quad (3.3)$$

admits a unique solution $\rho,v\in C^1(\Re_+\times[0,1])$, with $(\rho[t],v[t])\in X$ for all $t\ge 0$ satisfying the following estimate for all $t\ge 0$:

$$\begin{aligned}&\max_{0\le x\le 1}\left(\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right|\right)+\max_{0\le x\le 1}\left(\left|\ln\left(\frac{v(t,x)}{f(\rho_{eq})}\right)\right|\right)\\ &\le Q\left(\max_{0\le x\le 1}\left(\left|\ln\left(\frac{\rho_0(x)}{\rho_{eq}}\right)\right|\right)+\max_{0\le x\le 1}\left(\left|\ln\left(\frac{v_0(x)}{f(\rho_{eq})}\right)\right|\right),t\right)\end{aligned} \quad (3.4)$$

Remark 3.2: A sufficient condition for (3.2) is the assumption that the function $F(\rho):=\rho\left(c+f(\rho)\right)$ is increasing on the interval $\left(\rho_{eq}\left(c+f(0)\right)^{-1}\left(c+f(\rho_{eq})\right),\rho_{eq}\left(1+c^{-1}f(\rho_{eq})\right)\right)$. Consequently, (3.2) holds automatically when $c+f(\rho)+\rho f'(\rho)>0$ for all $\rho\in\left(\rho_{eq}\left(c+f(0)\right)^{-1}\left(c+f(\rho_{eq})\right),\rho_{eq}\left(1+c^{-1}f(\rho_{eq})\right)\right)$. For example, when $f(\rho)=A\exp(-b\rho)$, where $A,b>0$ are constants (Underwood model), we guarantee that (3.2) holds when the inequality $c\exp(b\rho)+A>Ab\rho$ holds for $\rho\in\left(\rho_{eq}(c+A)^{-1}\left(c+A\exp(-b\rho_{eq})\right),\rho_{eq}\left(1+c^{-1}A\exp(-b\rho_{eq})\right)\right)$. It should be noticed that in this case (3.2) holds automatically when the speed ratio A/c is sufficiently small no matter what ρ_{eq} is: when $c\exp(2)\ge A$ the function $F(\rho):=\rho\left(c+A\exp(-b\rho)\right)$ is increasing on $\Re_+$.

Remark 3.3: Estimate (3.4) is a stability estimate in the sup-norm of the logarithmic deviation of the state from its equilibrium values. The use of logarithmic deviation variables is customary for systems with positive state values (e.g., biological systems, see [19]).

Remark 3.4: Another thing that should be noted at this point is that if the objective were local stabilization instead of global, then we would need to assume inequality (3.2) only in a neighborhood of $f\left(\rho_{eq}\right)$.

IV. Illustrative Example

We consider model (2.1), (2.2), (2.3), (2.6) with $f(\rho)=0.4\exp(1-\rho)$ (Underwood model; see for instance [34]), $c=5$, $\mu=10$, $\rho_{\max}=2.7$, $\varepsilon=10^{-6}$, $h(s)=s(1-g(s))+\rho_{\max}g(s)$ for $s\ge 0$, where

$$g(s)=0\text{, for } s\in[0,\rho_{\max}-\varepsilon],$$
$$g(s)=1\text{, for } s\ge\rho_{\max} \text{ and}$$
$$g(s)=\frac{\exp\left(-(s+\varepsilon-\rho_{\max})^{-1}\right)}{\exp\left(-(s+\varepsilon-\rho_{\max})^{-1}\right)+\exp\left(-(\rho_{\max}-s)^{-1}\right)},$$
$$\text{for } s\in(\rho_{\max}-\varepsilon,\rho_{\max}).$$

The objective is to stabilize the equilibrium point that maximizes the vehicle flow $\rho(x)\equiv\rho_{eq}=1$, $v(x)\equiv f(\rho_{eq})=2/5$. It should be noticed that the open-loop system (2.1), (2.2), (2.3), (2.6) with $q(t)\equiv q_{eq}=2/5$ has two equilibria: one is the desired equilibrium, and the other one is the fully congested equilibrium $\rho(x)\equiv\rho_{\max}=2.7$, $v(x)\equiv f(\rho_{\max})=0.4\exp(-1.7)$. Numerical experiments show that the fully congested equilibrium attracts the solution of the open-loop system (2.1), (2.2), (2.3), (2.6) with $q(t)\equiv q_{eq}=2/5$ for many initial conditions. We chose the initial conditions

$$\rho_0(x)=1 \text{ for } x\in[0,9/20],\ \rho_0(x)=2\text{, for } x\in[1/2,1],$$
$$\rho_0(x)=1+\frac{\exp\left(-(x-9/20)^{-1}\right)}{\exp\left(-(x-9/20)^{-1}\right)+\exp\left((x-1/2)^{-1}\right)},$$
$$\text{for } x\in(0.45,0.5)\text{, and}$$
$$v_0(x)=f(\rho_0(x))\text{, for } x\in[0,1].$$

For this particular initial condition (but also for many others) the solution of the open-loop system (2.1), (2.2), (2.3), (2.6) with $q(t)\equiv q_{eq}=2/5$ converges to the fully congested equilibrium $\rho(x)\equiv\rho_{\max}=2.7$, $v(x)\equiv f(\rho_{\max})=0.4\exp(-1.7)$. The deviation of the solution from the desired equilibrium is shown in Fig. 1, where the evolution of the sup-norm of the logarithmic deviation from the desired equilibrium $X(t):=\max_{0\le x\le 1}\left(\left|\ln\left(\rho(t,x)/\rho_{eq}\right)\right|\right)+\max_{0\le x\le 1}\left(\left|\ln\left(v(t,x)/f(\rho_{eq})\right)\right|\right)$ is shown for system (2.1), (2.2), (2.3), (2.6) with $q(t)\equiv 0.4$.

In this case we can apply Theorem 3.1, since the condition $\rho_{eq}\le c\left(c+f(\rho_{eq})\right)^{-1}(\rho_{\max}-\varepsilon)$ as well as condition (3.2) hold (recall Remark 3.2). Fig. 2 shows the evolution of the sup-norm of the logarithmic deviation from the desired equilibrium $X(t):=\max_{0\le x\le 1}\left(\left|\ln\left(\rho(t,x)/\rho_{eq}\right)\right|\right)+\max_{0\le x\le 1}\left(\left|\ln\left(v(t,x)/f(\rho_{eq})\right)\right|\right)$ for the closed-loop system (2.1), (2.2), (2.3), (2.6) with (3.1). It should be noted that at time $t=6.58$, the solution has become identical to the desired equilibrium. Fig. 3 shows the time evolution of the control input $q(t)$. The control input tries to keep the inlet density close to 1, while the heavy congestion belt is "washed out" slowly. Finally, the evolution of the density profile is shown in Fig. 4.

V. PROOFS

In this section we provide the proofs of all main results.

V.A. Technical Results

The proof of Theorem 2.1 requires two technical results. The first technical result is stated next and due to its simplicity, its proof is omitted.

Lemma 5.1: *Suppose that there exist constants* $a,b,p\geq 0$, $c>0$ *such that the sequence* $\{x(k)\geq 0\}_{k=0}^{\infty}$ *satisfies the following inequality for all* $k=0,1,...,m-1$*:*

$$x(k+1)\leq \max\left((1+a)x(k)+b,(1-c)x(k)+p\right). \quad (5.1)$$

Then the following estimate holds:

$$x(k)\leq \exp(ka)\left(x(0)+\frac{p}{a+c}+bk\right), \text{ for all } k=0,1,...,m \quad (5.2)$$

The following auxiliary result has interest on its own, because it covers a case not studied in [2, 6, 21, 26]: the case of a transport PDE with a non-negative (possibly zero at some points) transport speed. A unique, classical solution is shown to exist, which is differentiable and satisfies the PDE even on the boundary (something that cannot be guaranteed by the results in [21]): this is important for the proof of Theorem 2.1, because Lipschitz continuity of the derivatives of the solution is used in an instrumental way. The proof of Proposition 5.2 is given in the Appendix.

Proposition 5.2: *Consider the problem*

$$\frac{\partial w}{\partial t}(t,x)+v(t,x)\frac{\partial w}{\partial x}(t,x)=0, \text{ for } t\geq 0,\ x\in[0,1] \quad (5.3)$$

$$w(0,x)=\varphi(x), \text{ for } x\in[0,1] \quad (5.4)$$

$$w(t,0)=a(t), \text{ for } t\geq 0 \quad (5.5)$$

where $\varphi\in W^{2,\infty}([0,1])$, $a\in W^{2,\infty}([0,T])$ *for every* $T>0$ *with* $a(0)=\varphi(0)$, $\dot{a}(0)+v(0,0)\varphi'(0)=0$ *and* $v\in C^1(\Re_+\times[0,1])$ *is a non-negative function which has Lipschitz derivatives on* $[0,T]\times[0,1]$ *for every* $T>0$. *Assume that* $v(t,0)>0$ *for all* $t\geq 0$. *Then (5.3), (5.4), (5.5) has a unique solution* $w\in C^1(\Re_+\times[0,1])$, *which has Lipschitz derivatives on* $[0,T]\times[0,1]$ *for every* $T>0$ *and satisfies*

$$\|w[t]\|_{\infty}\leq \max\left(\max_{0\leq s\leq t}\left(|a(s)|\right),\|\varphi\|_{\infty}\right), \text{ for all } t\geq 0. \quad (5.6)$$

Moreover, if there exists a constant $v_{\min}>0$ *such that* $v(t,x)\geq v_{\min}$ *for all* $t\geq 0$, $x\in[0,1]$ *and if* $a\equiv 0$ *then* $w(t,x)=0$ *for all* $x\in[0,1]$ *and* $t\geq v_{\min}^{-1}$.

V.B. Proof of Main Results

Proof of Theorem 2.1: Let arbitrary $T>0$ be given. We will apply the method of finite differences (used in [15]).

Let $N>c^{-1}\mu$ be an integer and consider the parameterized (with parameter N) discrete-time system

$$\begin{aligned} &w_i((k+1)\delta)=\left(1-\lambda v_i(k\delta)\right)w_i(k\delta)+\lambda v_i(k\delta)w_{i-1}(k\delta) \quad i=1,...,N\\ &v_i((k+1)\delta)=\left(1-\lambda c\right)v_i(k\delta)+\lambda c v_{i+1}(k\delta) \quad i=0,...,N-1\\ &v_N((k+1)\delta)=\left(1-\mu\delta\right)v_N(k\delta)+\mu\delta g\left(v_N(k\delta),w_N(k\delta)\right) \end{aligned} \quad (5.7)$$

where $k=0,...,m-1$ is an integer (time of the discrete-time system), $\lambda:=\left(1+\left[T\max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max},c\right)\right]\right)^{-1}T$,

$$m:=N\left(1+\left[T\max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max},c\right)\right]\right),$$

$$w_0(k\delta)=a\left(k\delta,v_0(k\delta)\right), \text{ for } k=1,...,m \quad (5.8)$$

$$h:=1/N \quad , \quad \delta:=\lambda h \quad (5.9)$$

and initial condition

$$w_i(0)-\theta(ih)=v_i(0)-\varphi(ih)=0 \qquad i=0,...,N \quad (5.10)$$

Notice that the definition of λ guarantees that

$$\lambda\max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max},c\right)\leq 1. \quad (5.11)$$

Moreover, definitions (5.9) together with the definitions of λ and m guarantee that $T=m\delta$. We next prove that

$$0\leq v_i(k\delta)\leq \max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max}\right),$$

$$\text{for all } i=0,...,N \text{ and } k=0,...,m. \quad (5.12)$$

Indeed, by virtue of (5.10) it follows that (5.12) holds for $k=0$. Using (5.7) and (5.11) we guarantee that $0\leq v_i((k+1)\delta)\leq \max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max}\right)$, for $i=0,...,N-1$, provided that (5.12) holds for certain $k=0,...,m-1$. The fact that $N>c^{-1}\mu$, together with (5.11) and (5.9) implies that

$$\mu\delta\leq 1. \quad (5.13)$$

Moreover, using (2.10), (5.7) and (5.13), we can guarantee that $0\leq v_N((k+1)\delta)\leq \max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max}\right)$, provided that (5.12) holds for certain $k=0,...,m-1$. Consequently, we get $0\leq v_i((k+1)\delta)\leq \max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max}\right)$, for $i=0,...,N$, provided that (5.12) holds for certain $k=0,...,m-1$.

Define

$$B_T:=\max\left\{\left|a(t,v)\right|:t\in[0,T],0\leq v\leq \max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max}\right)\right\}. \quad (5.14)$$

We next prove that for all $i=0,...,N$ and $k=0,...,m$

$$\left|w_i(k\delta)\right|\leq \max\left(\|\theta\|_{\infty},B_T\right). \quad (5.15)$$

Indeed, by virtue of (5.10) it follows that (5.15) holds for $k=0$. Suppose that (5.15) holds for all $i=0,...,N$ and for certain $k=0,...,m-1$. Using (5.7), (5.8), (5.11), (5.12) and

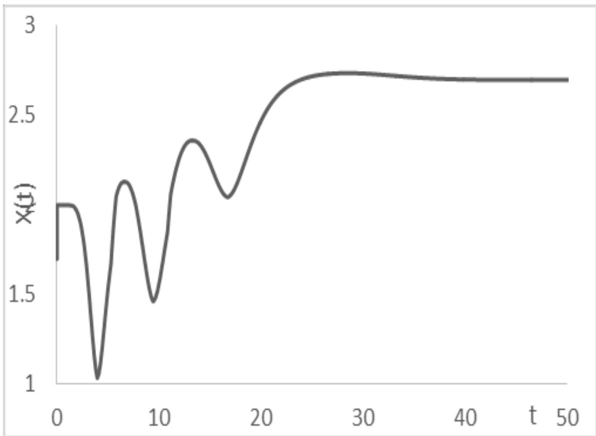

Fig. 1: The sup-norm of the logarithmic deviation from the desired equilibrium for the open-loop system (2.1), (2.2), (2.3), (2.6) with $q(t) \equiv q_{eq} = 2/5$.

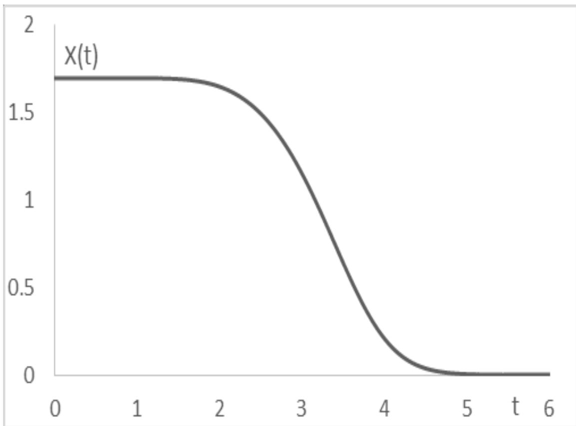

Fig. 2: The sup-norm of the logarithmic deviation from the desired equilibrium for the closed-loop system (2.1), (2.2), (2.3), (2.6) with (3.1).

the triangle inequality we guarantee that $|w_i((k+1)\delta)| \le \max\left(\|\theta\|_\infty, B_T\right)$, for $i = 1,\ldots,N$. Using (5.7), (5.12), (5.14) and the triangle inequality, we guarantee that $|w_0((k+1)\delta)| \le \max\left(\|\theta\|_\infty, B_T\right)$. Thus, (5.15) holds.

We next prove that

$$v_i(k\delta) \ge \min\left(\min_{0 \le x \le 1}(\varphi(x)), \min\left\{ g(0,w) : |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right),$$

$$\text{for all } i = 0,\ldots,N \text{ and } k = 0,\ldots,m. \qquad (5.16)$$

Indeed, by virtue of (5.10) it follows that (5.16) holds for $k = 0$. Suppose that (5.16) holds for all $i = 0,\ldots,N$ and for certain $k = 0,\ldots,m-1$. Using (5.7) and (5.11) we are in a position to guarantee that

$$v_i((k+1)\delta) \ge \min\left(\min_{0 \le x \le 1}(\varphi(x)), \min\left\{ g(0,w) : |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right),$$

for all $i = 0,\ldots,N-1$. Moreover, using (2.10), (5.7), (5.15) and (5.13), we can guarantee that $v_N((k+1)\delta) \ge \min\left(\min_{0 \le x \le 1}(\varphi(x)), \min\left\{ g(0,w) : |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right)$.

Thus, (5.16) holds.

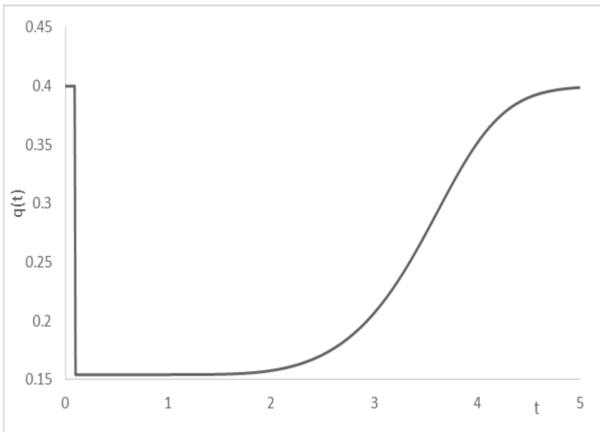

Fig. 3: The control input $q(t)$ for the closed-loop system (2.1), (2.2), (2.3), (2.6) with (3.1).

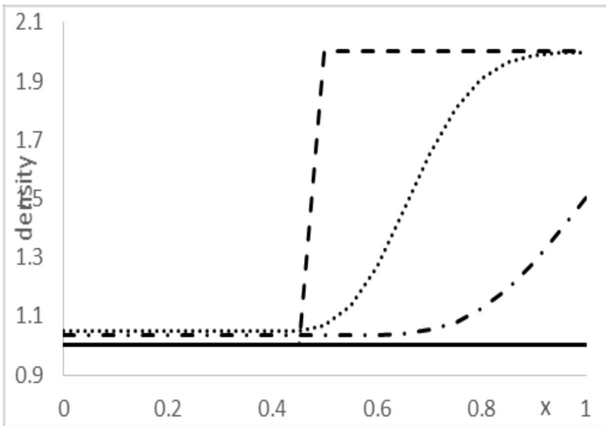

Fig. 4: The density profile for the closed-loop system (2.1), (2.2), (2.3), (2.6) with (3.1). The dashed line is the initial condition, the dotted line is for $t = 1.23$, the dotdash line is for $t = 3.11$ and the solid line is for $t = 7.04$.

We define for $(t,x) \in [0,T] \times [0,1]$ and for every integer $N > c^{-1}\mu$ (recall that $h = N^{-1}$, $\delta = \lambda h$, $m\delta = T$):

$$w(k\delta, x; N) = (i+1-xN)w_i(k\delta) + (xN-i)w_{i+1}(k\delta)$$
$$v(k\delta, x; N) = (i+1-xN)v_i(k\delta) + (xN-i)v_{i+1}(k\delta)$$
$$\text{with } i = [xN], \text{ for } x \in [0,1),\ k = 0,\ldots,m, \qquad (5.17)$$

$$w(k\delta, 1; N) = w_N(k\delta) \text{ and } v(k\delta, 1; N) = v_N(k\delta),$$
$$\text{for } k = 0,\ldots,m, \qquad (5.18)$$

$$w(t,x;N) = \left(k+1-\lambda^{-1}tN\right)w(k\delta,x;N) + \left(\lambda^{-1}tN-k\right)w((k+1)\delta,x;N)$$
$$v(t,x;N) = \left(k+1-\lambda^{-1}tN\right)v(k\delta,x;N) + \left(\lambda^{-1}tN-k\right)v((k+1)\delta,x;N)$$
$$\text{with } k = [\lambda^{-1}tN] \text{ for } x \in [0,1],\ t \in [0,T). \qquad (5.19)$$

It follows from (5.12), (5.15), (5.16) and definitions (5.17), (5.18), (5.19) that the following inequalities hold for all $(t,x) \in [0,T] \times [0,1]$ and for every integer $N > c^{-1}\mu$:

$$\min\left(\min_{0 \le x \le 1}(\varphi(x)), \min\left\{ g(0,w) : |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right)$$
$$\le v(t,x;N) \le \max\left(\max_{0 \le x \le 1}(\varphi(x)), v_{\max}\right) \qquad (5.20)$$

$$|w(t,x;N)| \le \max\left(\|\theta\|_\infty , B_T\right) \tag{5.21}$$

We next describe the major steps in the proof. We also use the notation $\Omega := [0,T]\times[0,1]$.

Step 1: We show that there exists a constant $L := L(T,\theta,\varphi,a) > 0$ such that for every $N > c^{-1}\mu$ the functions $w(\cdot;N)$, $v(\cdot;N)$ are Lipschitz on Ω with Lipschitz constant L. This step is very important because it allows the application of Arzela-Ascoli theorem. More specifically, it follows from (5.20), (5.21) that the sequences of functions $\{w(\cdot;N)\}_{N=N^*}^{\infty}$, $\{v(\cdot;N)\}_{N=N^*}^{\infty}$ with $N^* = [c^{-1}\mu]+1$, are uniformly bounded and equicontinuous. Therefore, compactness of Ω and the Arzela-Ascoli theorem implies that there exist Lipschitz functions $w,v:\Omega\to\Re$ and subsequences $\{w(\cdot;N_q)\}_{q=1}^{\infty}$, $\{v(\cdot;N_q)\}_{q=1}^{\infty}$ for an increasing index sequence $\{N_q\}_{q=1}^{\infty}$, which converge uniformly on Ω to w and v, respectively. Moreover, w and v are Lipschitz on Ω with Lipschitz constant L and satisfy the same bounds with $w(\cdot;N)$ and $v(\cdot;N)$, i.e., for $(t,x)\in\Omega$ it holds that

$$\begin{aligned}&\min\left(\min_{0\le x\le 1}(\varphi(x)), \min\left\{g(0,w): |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right)\\ &\le v(t,x) \le \max\left(\max_{0\le x\le 1}(\varphi(x)), v_{\max}\right)\end{aligned} \tag{5.22}$$

$$|w(t,x)| \le \max\left(\|\theta\|_\infty , B_T\right) \tag{5.23}$$

Using the fact that w, v are Lipschitz on Ω, (5.8), (5.10) and the fact that $\{w(\cdot;N_q)\}_{q=1}^{\infty}$, $\{v(\cdot;N_q)\}_{q=1}^{\infty}$ converge uniformly on Ω to w and v, we conclude that (2.13) holds and $w(t,0) = a(t, v(t,0))$ for $t\in[0,T]$.

Step 2: We define $\xi:[0,T]\to\Re$ by means of the equations

$$\dot{\xi}(t) = -\mu\,\left(\xi(t) - g(\xi(t), w(t,1))\right) = 0\text{, for } t\in[0,T] \tag{5.24}$$

$$\xi(0) = \varphi(1) \tag{5.25}$$

and we show that $\xi(t) = v(t,1)$ for $t\in[0,T]$. Notice that $\xi \in W^{2,\infty}([0,T])$.

Step 3: We define the function $\tilde{v}:[0,T]\times[0,1]\to\Re$

$$\tilde{v}(t,x) = \begin{cases}\varphi(x+ct) & \text{if} \quad x+ct\le 1\\ \xi\left(t-c^{-1}(1-x)\right) & \text{if} \quad x+ct>1\end{cases} \tag{5.26}$$

Due to the facts that $\xi\in W^{2,\infty}([0,T])$, $\varphi\in W^{2,\infty}([0,1])$ and since the compatibility conditions (5.25), $\varphi'(1) = -c^{-1}\mu\left(\varphi(1) - g(\varphi(1),\theta(1))\right)$ hold, it follows that $\tilde{v}\in C^1(\Omega)$ has Lipschitz derivatives satisfying $\frac{\partial\tilde{v}}{\partial t}(t,x) = c\frac{\partial\tilde{v}}{\partial x}(t,x)$ for $(t,x)\in\Omega$, $\tilde{v}(0,x) = \varphi(x)$ for $x\in[0,1]$ and $\tilde{v}(t,1) = \xi(t) = v(t,1)$ for $t\in[0,T]$. We show that $\tilde{v}(t,x) = v(t,x)$ for $(t,x)\in\Omega$. Thus, it follows from (5.24), (5.26) that the function v is of class $C^1(\Omega)$ with Lipschitz derivatives and satisfies $\frac{\partial v}{\partial t}(t,x) = c\frac{\partial v}{\partial x}(t,x)$ for $(t,x)\in\Omega$, $\frac{\partial v}{\partial t}(t,1) = -\mu\left(v(t,1) - g(v(t,1), w(t,1))\right)$ for $t\in[0,T]$ and $v(0,x) = \varphi(x)$ for $x\in[0,1]$.

Step 4: Proposition 5.2 implies that there exists a unique C^1 solution $\tilde{w}:\Omega\to\Re$ of the problem

$$\frac{\partial\tilde{w}}{\partial t}(t,x) + v(t,x)\frac{\partial\tilde{w}}{\partial x}(t,x) = 0\text{, for all } (t,x)\in\Omega \tag{5.27}$$

$$\tilde{w}(t,0) = a(t, v(t,0))\text{, for all } t\in[0,T] \tag{5.28}$$

$$\tilde{w}(0,x) = \theta(x)\text{, for all } x\in[0,1] \tag{5.29}$$

Moreover, $\tilde{w}$ has Lipschitz derivatives. We show that $\tilde{w}(t,x) = w(t,x)$ for $(t,x)\in\Omega$. It follows that the functions w,v are of class $C^1(\Omega)$ with Lipschitz derivatives and satisfy (2.11), (2.12), (2.13) on Ω.

Step 5: Finally, we prove that the solution is unique.

Step 1: Lipschitz Regularity

Define for every $i = 0,\ldots,N-1$ and $k = 0,\ldots,m$:

$$\begin{aligned}y_i(k\delta) &= h^{-1}\left(w_{i+1}(k\delta) - w_i(k\delta)\right)\\ p_i(k\delta) &= h^{-1}\left(v_{i+1}(k\delta) - v_i(k\delta)\right)\end{aligned} \tag{5.30}$$

Using (5.7), (5.8), (5.9), (5.10), the fact that $f(0) = a(0,\varphi(0))$, we are in a position to verify that the following equations hold for all $k = 0,\ldots,m-1$:

$$\begin{aligned}y_i((k+1)\delta) &= \left(1-\lambda\, v_{i+1}(k\delta)\right)y_i(k\delta)\\ &+ \lambda\, v_{i+1}(k\delta)y_{i-1}(k\delta) - \delta\, p_i(k\delta)y_{i-1}(k\delta),\\ &\text{for } i = 1,\ldots,N-1\end{aligned} \tag{5.31}$$

$$\begin{aligned}&y_0((k+1)\delta) = \left(1-\lambda v_1(k\delta)\right)y_0(k\delta)\\ &-\lambda\delta^{-1}\left(a((k+1)\delta, v_0(k\delta)) - a(k\delta, v_0(k\delta))\right)\\ &-h^{-1}\left(a\left((k+1)\delta, (1-\lambda c)v_0(k\delta) + \lambda c\, v_1(k\delta)\right) - a\left((k+1)\delta, v_0(k\delta)\right)\right)\end{aligned} \tag{5.32}$$

$$p_i((k+1)\delta) = (1-\lambda c)p_i(k\delta) + \lambda c\, p_{i+1}(k\delta)\text{, for } i = 0,1,\ldots,N-2 \tag{5.33}$$

$$\begin{aligned}p_{N-1}((k+1)\delta) &= (1-\lambda c)p_{N-1}(k\delta)\\ &+ \mu\lambda\left(g(v_N(k\delta), w_N(k\delta)) - v_N(k\delta)\right)\end{aligned} \tag{5.34}$$

Using (2.10), (5.11), (5.12), (5.16) and (5.33), (5.34), we get for all $k = 0,\ldots,m-1$:

$$\begin{aligned}&\max_{i=0,\ldots,N-1}\left(|p_i((k+1)\delta)|\right) \le\\ &\max\left(\max_{i=0,\ldots,N-1}\left(|p_i(k\delta)|\right), (1-\lambda c)\max_{i=0,\ldots,N-1}\left(|p_i(k\delta)|\right) + \mu\lambda\bar{v}_{\max}\right)\end{aligned} \tag{5.35}$$

where $\bar{v}_{\max} := \max\left(\max_{0\le x\le 1}(\varphi(x)), v_{\max}\right)$. Therefore, we obtain from (5.35) and Lemma 5.1 the estimate for $k = 0,\ldots,m$:

$$\max_{i=0,\ldots,N-1}\left(|p_i(k\delta)|\right) \le \max_{i=0,\ldots,N-1}\left(|p_i(0)|\right) + c^{-1}\mu\bar{v}_{\max}. \tag{5.36}$$

Definition (5.30) and (5.10) imply $|p_i(0)| \le \|\varphi'\|_\infty$ for $i=0,\dots,N-1$. Consequently, we get from (5.36) that

$$\max_{i=0,\dots,N-1}\left(|p_i(k\delta)|\right) \le P := \|\varphi'\|_\infty + c^{-1}\mu\,\bar{v}_{\max} \quad (5.37)$$

for $k=0,\dots,m$. Using (5.11), (5.12), (5.16), (5.31), (5.32), (5.37), we get for all $k=0,\dots,m-1$:

$$\begin{aligned}&H(k+1)\le\\ &\max\left((1+\delta P)H(k),\,(1-\lambda v_{\min})H(k)+\lambda R(1+cP)\right)\end{aligned} \quad (5.38)$$

where $v_{\min} := \min\left(\min_{0\le x\le 1}(\varphi(x)), \min\left\{g(0,w) : |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right)$,

$R := \max\left\{\left|\frac{\partial a}{\partial t}(t,v)\right| + \left|\frac{\partial a}{\partial v}(t,v)\right| : 0 \le v \le \bar{v}_{\max},\, t\in[0,T]\right\}$ and

$H(k) := \max_{i=0,\dots,N-1}\left(|y_i(k\delta)|\right)$. Using (5.38), Lemma 5.1 and the fact that $m\delta = T$, we get for all $k=0,\dots,m$:

$$\max_{i=0,\dots,N-1}\left(|y_i(k\delta)|\right) \le \exp(PT)\left(\max_{i=0,\dots,N-1}\left(|y_i(0)|\right) + R\frac{1+cP}{v_{\min}}\right) \quad (5.39)$$

Definition (5.30) in conjunction with (5.10) implies that $|y_i(0)| \le \|\theta'\|_\infty$ for all $i=0,\dots,N-1$. Consequently, we get from (5.36) for all $k=0,\dots,m$

$$\max_{i=0,\dots,N-1}\left(|y_i(k\delta)|\right) \le Y := \exp(PT)\left(\|\theta'\|_\infty + Rv_{\min}^{-1}(1+cP)\right). \quad (5.40)$$

It follows from (5.30), (5.37) and (5.40) that the following inequalities hold for all $i,j\in\{0,\dots,N\}$ and $k=0,\dots,m$:

$$|w_i(k\delta) - w_j(k\delta)| \le h|i-j|Y,\quad |v_i(k\delta) - v_j(k\delta)| \le h|i-j|P \quad (5.41)$$

Notice that P, Y, defined in (5.37) and (5.40), respectively, depend only on T, θ, φ and a.

Next define for every $i=0,\dots,N$ and $k=0,\dots,m-1$:

$$\begin{aligned}\zeta_i(k\delta) &= \delta^{-1}\left(w_i((k+1)\delta) - w_i(k\delta)\right)\\ \eta_i(k\delta) &= \delta^{-1}\left(v_i((k+1)\delta) - v_i(k\delta)\right)\end{aligned} \quad (5.42)$$

It follows from (5.7) and definitions (5.9), (5.30), (5.42) that the following equalities hold for $k=0,\dots,m-1$:

$$\begin{aligned}&\zeta_i(k\delta) = -v_i(k\delta)y_{i-1}(k\delta) && i=1,\dots,N\\ &\eta_i(k\delta) = c\,p_i(k\delta) && i=0,\dots,N-1\\ &\eta_N(k\delta) = \mu\left(g(v_N(k\delta), w_N(k\delta)) - v_N(k\delta)\right)\end{aligned} \quad (5.43)$$

Using (5.43), (2.10), (5.12), (5.37), we get for $k=0,\dots,m-1$:

$$\max_{i=0,\dots,N}\left(|\eta_i(k\delta)|\right) \le cP + \mu\,\bar{v}_{\max}. \quad (5.44)$$

It follows from (5.42), (5.7), (5.8), (5.10) and the fact that $f(0) = a(0,\varphi(0))$ that the following equalities hold

$$\begin{aligned}&\delta\,\zeta_0(k\delta) = a((k+1)\delta, v_0(k\delta)) - a(k\delta, v_0(k\delta)) +\\ &a\left((k+1)\delta, (1-\lambda c)v_0(k\delta) + \lambda c v_1(k\delta)\right) - a\left((k+1)\delta, v_0(k\delta)\right)\end{aligned} \quad (5.45)$$

for $k=0,\dots,m-1$. Equalities (5.43),(5.45) in conjunction with (5.12),(5.37),(5.40),(5.9),(5.30) imply for $k=0,\dots,m-1$

$$\max_{i=0,\dots,N}\left(|\zeta_i(k\delta)|\right) \le R(1+cP) + \bar{v}_{\max}Y. \quad (5.46)$$

It follows from (5.42), (5.44) and (5.46) that the following inequalities hold for $i=0,\dots,N$ and $k,l\in\{0,\dots,m\}$:

$$\begin{aligned}|w_i(k\delta) - w_i(l\delta)| &\le \delta|k-l|\left(R(1+cP) + \bar{v}_{\max}Y\right)\\ |v_i(k\delta) - v_i(l\delta)| &\le \delta|k-l|\left(cP + \mu\,\bar{v}_{\max}\right)\end{aligned} \quad (5.47)$$

It follows from (5.17), (5.18), (5.19),(5.41), (5.47) that there exists $L := L(T,\theta,\varphi,a)$ such that for every $N > c^{-1}\mu$ the following inequalities hold for all $x,z\in[0,1]$, $t,\tau\in[0,T]$:

$$|w(t,x;N) - w(\tau,z;N)| + |v(t,x;N) - v(\tau,z;N)| \le L\left(|x-z| + |t-\tau|\right). \quad (5.48)$$

Step 2: Solution of the ODE (5.24)

We define ξ by means of (5.24), (5.25). The fact that ξ can be defined on $[0,T]$ is a consequence of the formula

$$\xi(t) = \exp(-\mu t)\varphi(1) + \mu\int_0^t \exp(-\mu(t-s))g(\xi(s), w(s,1))ds,$$

which together with (2.10) and (5.23) imply the inequality $\min\left(\varphi(1), \min\left\{g(0,w) : |w| \le \max\left(B_T, \|\theta\|_\infty\right)\right\}\right) \le \xi(t) \le \max\left(\varphi(1), v_{\max}\right)$ for $t\in[0,T]$. Pick any $N > c^{-1}\mu$. It follows from (2.14), (5.24) that the following inequality holds for $k=0,\dots,m-1$:

$$\begin{aligned}&\left|\xi((k+1)\delta) - (1-\mu\delta)\xi(k\delta) - \mu\delta\, g\left(\xi(k\delta), w(k\delta,1)\right)\right|\\ &\le \mu\delta^2\left(\|\dot{\xi}\|(1+G) + GL\right)/2\end{aligned} \quad (5.49)$$

where $m := N\left(1 + \left[T\max\left(\max_{0\le x\le 1}(\varphi(x)), v_{\max}, c\right)\right]\right)$, $\delta = \lambda/N$,

$\lambda := \left(1 + \left[T\max\left(\max_{0\le x\le 1}(\varphi(x)), v_{\max}, c\right)\right]\right)^{-1}T$, L is the Lipschitz constant of w, $S := \left\{(v,w)\in\Re_+\times\Re : v \le \max(\varphi(1), v_{\max}), |w| \le \max\left(B_T, \|f\|_\infty\right)\right\}$,

$G := \max\left\{\left|\frac{\partial g}{\partial v}(v,w)\right| + \left|\frac{\partial g}{\partial w}(v,w)\right| : (v,w)\in S\right\}$ and $\|\dot{\xi}\| = \max_{0\le t\le T}\left(|\dot{\xi}(t)|\right)$.

Using (5.7), (5.49) and defining $e_N(k\delta) := \xi(k\delta) - v_N(k\delta)$ we get for $k=0,\dots,m-1$:

$$\begin{aligned}&|e_N((k+1)\delta)| \le (1+\mu\delta G)|e_N(k\delta)|\\ &+ \mu\delta G|w(k\delta,1) - w(k\delta,1;N)| + \mu\delta^2\left(\|\dot{\xi}\|(1+G) + LG\right)/2\end{aligned} \quad (5.50)$$

Inequality (5.50) in conjunction with Lemma 5.1 and the facts that $T = m\delta$, $e_N(0) = 0$ (a consequence of (5.25) and (2.13)) implies the following estimate for $k=0,\dots,m$:

$$\begin{aligned}&|e_N(k\delta)| \le\\ &\mu c T\exp(\mu c T G)\left(\max_{0\le t\le T}\left(|w(t,1) - w(t,1;N)|\right) + \frac{\delta}{2}\left(\|\dot{\xi}\|(1+G) + LG\right)\right)\end{aligned} \quad (5.51)$$

Pick any $t\in[0,T]$ and set $k = [t\delta^{-1}]$. Then we get

$$\begin{aligned}&|\xi(t) - v(t,1)| \le |\xi(t) - v(t,1) - \xi(k\delta) + v(k\delta,1)| + |\xi(k\delta) - v(k\delta,1)|\\ &\le \left(L + \|\dot{\xi}\|\right)\delta + |e_N(k\delta)| + \max_{0\le s\le T}|v(s,1) - v(s,1;N)|\end{aligned}$$

where L is the Lipschitz constant of v. Since $\{w(\cdot\,;N_q)\}_{q=1}^{\infty}$, $\{v(\cdot\,;N_q)\}_{q=1}^{\infty}$ converge uniformly to w and v as $q\to+\infty$, the above inequality in conjunction with (5.51) shows that $\xi(t)=v(t,1)$ for all $t\in[0,T]$.

Step 3: Solving the PDE for v

Pick any integer $N>c^{-1}\mu$. Define $h=1/N$, $m:=N\left(1+\left[\,T\max\left(\max_{0\le x\le 1}(\varphi(x)),v_{\max},c\right)\right]\right)$, $\delta=\lambda/N$, $\lambda:=\left(1+\left[\,T\max\left(\max_{0\le x\le 1}(\varphi(x)),v_{\max},c\right)\right]\right)^{-1}T$ and notice that since $\tilde{v}$ satisfies $\frac{\partial\tilde{v}}{\partial t}(t,x)=c\frac{\partial\tilde{v}}{\partial x}(t,x)$ for $(t,x)\in\Omega$, $\tilde{v}(0,x)=\varphi(x)$ for $x\in[0,1]$ and $\tilde{v}(t,1)=\xi(t)=v(t,1)$ for $t\in[0,T]$ then we get for $k=0,...,m-1$, $i=0,...,N-1$:

$$\tilde{v}((k+1)\delta,ih)=(1-\lambda c)\tilde{v}(k\delta,ih)+c\lambda\,\tilde{v}(k\delta,(i+1)h)+c\,err(k,i) \tag{5.52}$$

where

$$\begin{aligned}err(k,i)=&\int_{k\delta}^{(k+1)\delta}\left(\frac{\partial\tilde{v}}{\partial x}(t,ih)-\frac{\partial\tilde{v}}{\partial x}(k\delta,ih)\right)dt\\&-\lambda\int_{ih}^{(i+1)h}\left(\frac{\partial\tilde{v}}{\partial x}(k\delta,x)-\frac{\partial\tilde{v}}{\partial x}(k\delta,ih)\right)dx\end{aligned} \tag{5.53}$$

Defining $e_i^v(k\delta):=\tilde{v}(k\delta,ih)-v(k\delta,ih;N)$ for $k=0,...,m$ and $i=0,...,N$, we get from (5.7), (5.10), (5.17), (5.18), (5.52) and the facts that $\tilde{v}(0,x)=\varphi(x)$ for $x\in[0,1]$ and $\tilde{v}(t,1)=\xi(t)=v(t,1)$ for $t\in[0,T]$, $k=0,...,m-1$, $i=0,...,N-1$:

$$\begin{aligned}e_i^v((k+1)\delta)=&(1-\lambda c)e_i^v(k\delta,ih)\\&+c\lambda\,e_{i+1}^v(k\delta,(i+1)h)+c\,err(k,i)\end{aligned} \tag{5.54}$$

$$e_i^v(0)=0, \tag{5.55}$$

$$e_N^v(k\delta)=0. \tag{5.56}$$

Using (5.9),(5.11),(5.53),(5.54),(5.56), we get

$$\max_{i=0,...,N}\left(\left|e_i^v((k+1)\delta)\right|\right)\le\max_{i=0,...,N}\left(\left|e_i^v(k\delta)\right|\right)+2c\delta\,G(N), \tag{5.57}$$

for $k=0,...,m-1$, where

$$G(N):=\max\left\{\left|\frac{\partial\tilde{v}}{\partial x}(t,x)-\frac{\partial\tilde{v}}{\partial x}(\tau,z)\right|:(t,x),(\tau,z)\in\Omega,\ |t-\tau|+|x-z|\le(1+\lambda)N^{-1}\right\}. \tag{5.58}$$

Using Lemma 5.1, (5.55), (5.57) and $T=m\delta$, we get $\max_{i=0,...,N}\left(\left|e_i^v(k\delta)\right|\right)\le 2cT\,G(N)$. Thus, we obtain

$$\begin{aligned}|\tilde{v}(t,x)-v(t,x)|\le&\,2L(1+\lambda)N^{-1}+2cTG(N)\\&+\max_{(\tau,z)\in\Omega}\left(|v(\tau,z;N)-v(\tau,z)|\right)\end{aligned} \tag{5.59}$$

for $(t,x)\in\Omega$, where L is the Lipschitz constant of v and $\tilde{v}$. Definition (5.58), the fact that $\tilde{v}$ has Lipschitz derivatives on Ω implies that $\lim_{N\to+\infty}(G(N))=0$. Moreover, since $\{v(\cdot\,;N_q)\}_{q=1}^{\infty}$ converges uniformly to v as $q\to+\infty$, we get from (5.59) that $\tilde{v}(t,x)=v(t,x)$ for all $(t,x)\in\Omega$.

Step 4: Solving the PDE for w

Pick any integer $N>c^{-1}\mu$. Define $h=1/N$, $m:=N\left(1+\left[\,T\max\left(\max_{0\le x\le 1}(\varphi(x)),v_{\max},c\right)\right]\right)$, $\delta=\lambda/N$, $\lambda:=\left(1+\left[\,T\max\left(\max_{0\le x\le 1}(\varphi(x)),v_{\max},c\right)\right]\right)^{-1}T$ and notice that since $\tilde{w}$ satisfies (5.27) for $(t,x)\in\Omega$, we get

$$\begin{aligned}\tilde{w}((k+1)\delta,ih)=&(1-\lambda\,v(k\delta,ih))\tilde{w}(k\delta,ih)\\&+\lambda\,v(k\delta,ih)\tilde{w}(k\delta,(i-1)h)+Err(k,i)\end{aligned} \tag{5.60}$$

for $k=0,...,m-1$, $i=1,...,N$, where

$$\begin{aligned}Err(k,i)=&\,\lambda\,v(k\delta,ih)\int_{(i-1)h}^{ih}\left(\frac{\partial\tilde{w}}{\partial x}(k\delta,x)-\frac{\partial\tilde{w}}{\partial x}(k\delta,ih)\right)dx\\&-\int_{k\delta}^{(k+1)\delta}\left(v(t,ih)\frac{\partial\tilde{w}}{\partial x}(t,ih)-v(k\delta,ih)\frac{\partial\tilde{w}}{\partial x}(k\delta,ih)\right)dt\end{aligned} \tag{5.61}$$

Defining $e_i^w(k\delta):=\tilde{w}(k\delta,ih)-w(k\delta,ih;N)$ for $k=0,...,m$ and $i=0,...,N$, we get from (5.7), (5.9), (5.10), (5.17), (5.18), (5.28), (5.29), (5.30), (5.60):

$$\begin{aligned}&e_i^w((k+1)\delta)=(1-\lambda\,v(k\delta,ih))e_i^w(k\delta)\\&-\delta\left(v(k\delta,ih)-v(k\delta,ih;N)\right)y_{i-1}(k\delta)+\lambda\,v(k\delta,ih)e_{i-1}^w(k\delta)+Err(k,i),\\&\text{for } k=0,...,m-1,\ i=1,...,N\end{aligned} \tag{5.62}$$

$$e_i^w(0)=0,\ \text{for } i=0,...,N \tag{5.63}$$

$$e_0^w(k\delta)=0,\ \text{for } k=0,...,m \tag{5.64}$$

Using (2.15),(5.9),(5.11),(5.40),(5.61),(5.62), (5.64), we get

$$\max_{i=0,...,N}\left(\left|e_i^w((k+1)\delta)\right|\right)\le\max_{i=0,...,N}\left(\left|e_i^w(k\delta)\right|\right)+\delta\,\tilde{G}(N), \tag{5.65}$$

for $k=0,...,m-1$, where

$$\begin{aligned}\tilde{G}(N):=&\,\bar{v}_{\max}\max\left\{\left|\frac{\partial\tilde{w}}{\partial x}(t,x)-\frac{\partial\tilde{w}}{\partial x}(\tau,z)\right|:(t,x),(\tau,z)\in\Omega,\ |t-\tau|+|x-z|\le(1+\lambda)N^{-1}\right\}\\&+\lambda N^{-1}L\max\left\{\left|\frac{\partial\tilde{w}}{\partial x}(t,x)\right|:(t,x)\in\Omega\right\}+Y\max_{(t,x)\in\Omega}\left(|v(t,x)-v(t,x;N)|\right)\end{aligned} \tag{5.66}$$

and $\bar{v}_{\max}=\max\left(\max_{0\le x\le 1}(\varphi(x)),v_{\max}\right)$. Using Lemma 5.1, (5.63), (5.65) and the fact that $T=m\delta$, we get $\max_{i=0,...,N}\left(\left|e_i^w(k\delta)\right|\right)\le T\,\tilde{G}(N)$. Thus, the following inequality holds

$$\begin{aligned}|\tilde{w}(t,x)-w(t,x)|\le&\,2L(1+\lambda)N^{-1}+T\tilde{G}(N)\\&+\max_{(\tau,z)\in\Omega}\left(|w(\tau,z;N)-w(\tau,z)|\right)\end{aligned} \tag{5.67}$$

for $(t,x)\in\Omega$, where L is the Lipschitz constant of w and $\tilde{w}$. Definition (5.66), the fact that $\tilde{w}$ has Lipschitz derivatives on Ω and the fact that $\{v(\cdot\,;N_q)\}_{q=1}^{\infty}$ converges uniformly to v as $q\to+\infty$ imply that $\lim_{q\to+\infty}\left(\tilde{G}(N_q)\right)=0$. Moreover, since $\{w(\cdot\,;N_q)\}_{q=1}^{\infty}$ converges uniformly to w as $q\to+\infty$, we get from (5.67) that $\tilde{w}(t,x)\equiv w(t,x)$.

Step 5: Uniqueness of solutions

Consider two solutions $(w,v),(\bar{w},\bar{v})\in\left(C^1(\Re_+\times[0,1])\right)^2$ of (2.10), (2.11), (2.12). It then follows that the functions $e_w=\bar{w}-w$, $e_v=\bar{v}-v$ satisfy for $(t,x)\in\Re_+\times[0,1]$:

$$\begin{aligned}&\frac{\partial e_w}{\partial t}(t,x)+v(t,x)\frac{\partial e_w}{\partial x}(t,x)+e_v(t,x)\frac{\partial \bar{w}}{\partial x}(t,x)\\&=\frac{\partial e_v}{\partial t}(t,x)-c\frac{\partial e_v}{\partial x}(t,x)=0\end{aligned}\tag{5.68}$$

$$e_w(t,0)=a(t,v(t,0)+e_v(t,0))-a(t,v(t,0)),\tag{5.69}$$

$$\begin{aligned}\mu^{-1}\frac{\partial e_v}{\partial t}(t,1)&=g\left(v(t,1)+e_v(t,1),w(t,1)+e_w(t,1)\right)\\&-e_v(t,1)-g\left(v(t,1),w(t,1)\right)\end{aligned}\tag{5.70}$$

$$e_w(0,x)=e_v(0,x)=0,\tag{5.71}$$

Let $T>0$ and let $S\subseteq\Re_+\times\Re$ be a compact set that contains both solutions on $[0,T]$, i.e., $(v(t,x),w(t,x))\in S$ and $(\bar{v}(t,x),\bar{w}(t,x))\in S$ for all $(t,x)\in\Omega:=[0,T]\times[0,1]$. Let $M\geq 1$ be a constant that satisfies $Mc\geq Q^2\bar{v}_{\max}$, where

$$\begin{aligned}Q:=&\max\left\{\left|\frac{\partial g}{\partial v}(v,w)\right|+\left|\frac{\partial g}{\partial w}(v,w)\right|:(v,w)\in S\right\}\\&+\max\left\{\left|\frac{\partial a}{\partial v}(t,v)\right|:t\in[0,T],(v,w)\in S\right\}\end{aligned}$$

and $\bar{v}_{\max}=\max\left(\max_{0\leq x\leq 1}(\varphi(x)),v_{\max}\right)$. Define the functional:

$$V(t)=\frac{1}{2}\int_0^1 e_w^2(t,x)dx+\frac{M}{2}\int_0^1 e_v^2(t,x)dx+\frac{1}{2}e_v^2(t,1)\tag{5.72}$$

Using (5.68), (5.69), (5.70), (5.71) and the fact $Mc\geq Q^2\bar{v}_{\max}$, it follows that

$$\begin{aligned}\dot{V}(t)\leq&\frac{1}{2}\left(\|\bar{w}_x\|+\|v_x\|\right)\left(\int_0^1 e_w^2(t,x)dx+M\int_0^1 e_v^2(t,x)dx\right)\\&+\left(\frac{Mc}{2}+Q+\frac{Q^2\mu^2}{2v_{\min}}\right)e_v^2(t,1)\end{aligned}$$

for $t\in[0,T]$, where $\|\bar{w}_x\|:=\max\left\{\left|\frac{\partial\bar{w}}{\partial x}(t,x)\right|:(t,x)\in\Omega\right\}$, $\|v_x\|:=\max\left\{\left|\frac{\partial v}{\partial x}(t,x)\right|:(t,x)\in\Omega\right\}$. The above inequality in conjunction with (5.72) shows that there exists a constant $K>0$ such that $\dot{V}(t)\leq KV(t)$ for $t\in[0,T]$. Gronwall's lemma in conjunction with (5.71), (5.72) implies $V(t)\equiv 0$ on $[0,T]$. Since $T>0$ is arbitrary, we conclude that $\bar{w}(t,x)-w(t,x)=\bar{v}(t,x)-v(t,x)\equiv 0$. $\lhd$

Proof of Theorem 3.1: Let arbitrary $(\rho_0,v_0)\in X$ be given, for which the equalities $\rho_0(0)=\rho_{eq}\dfrac{c+f(\rho_{eq})}{c+v_0(0)}$, $\rho_0'(0)=\rho_0(0)(c+v_0(0))^{-1}v_0'(0)$ hold. The solution of (2.1), (2.2), (2.3), (2.6) with (3.1), (3.3) is constructed by applying the transformation (2.7), using (2.4) and the condition $\rho_{eq}\leq\left(1+c^{-1}f(\rho_{eq})\right)^{-1}(\rho_{\max}-\varepsilon)$. More specifically, we get the initial-boundary value problem (2.11) with

$$w(t,0)=\frac{\partial v}{\partial t}(t,1)+\mu\left(v(t,1)-f\left(\frac{\left(c+f(\rho_{eq})\right)\rho_{eq}\exp(w(t,1))}{c+v(t,1)}\right)\right)=0,$$
$$\text{for all } t\geq 0\tag{5.73}$$

$$w(0,x)-w_0(x)=v(0,x)-v_0(x)=0,\ \text{for } x\in[0,1]\tag{5.74}$$

$$w_0(x)=\ln\left(\left(c+f(\rho_{eq})\right)^{-1}\rho_{eq}^{-1}\rho_0(x)(c+v_0(x))\right),\ \text{for } x\in[0,1]\tag{5.75}$$

and $\rho(t,x)$ may be obtained by (2.7). Exploiting the fact that f is non-increasing and Theorem 2.1, we conclude that (2.11), (5.73), (5.74) admits a unique solution $w,v\in C^1(\Re_+\times[0,1])$, which has Lipschitz derivatives on every compact $S\subset\Re_+\times[0,1]$ and satisfies

$$\|w[t]\|_\infty\leq\|w_0\|_\infty\tag{5.76}$$

$$\begin{aligned}&\min\left(\min_{0\leq x\leq 1}(v_0(x)),f\left(\rho_{eq}\exp\left(\|w_0\|_\infty\right)\left(1+c^{-1}f(\rho_{eq})\right)\right)\right)\\&\leq v(t,x)\leq\max\left(\max_{0\leq x\leq 1}(v_0(x)),f(0)\right)\end{aligned}\tag{5.77}$$

for all $(t,x)\in\Re_+\times[0,1]$. Define for all $(t,x)\in\Re_+\times[0,1]$:

$$b(t,x):=\ln\left(v(t,x)/f(\rho_{eq})\right),\ b_0(x):=\ln\left(v_0(x)/f(\rho_{eq})\right)\tag{5.78}$$

Notice that (5.77) in conjunction with (5.78) gives:

$$\begin{aligned}v_{\min}:=&\min\left(f(\rho_{eq})\exp\left(-\|b_0\|_\infty\right),f\left(\rho_{eq}\left(1+c^{-1}f(\rho_{eq})\right)\exp\left(\|w_0\|_\infty\right)\right)\right)\\&\leq v(t,x)\leq\max\left(f(\rho_{eq})\exp\left(\|b_0\|_\infty\right),f(0)\right)\end{aligned}\tag{5.79}$$

Moreover, Proposition 5.2 implies that

$$w(t,x)=0\ \text{for all } x\in[0,1] \text{ and } t\geq v_{\min}^{-1}.\tag{5.80}$$

Equations (2.11) and (5.73) imply that the following equation holds for all $(t,x)\in\Re_+\times[0,1]$:

$$v(t,x)=\begin{cases}v_0(x+ct) & \text{if}\quad x+ct\leq 1\\ \xi\left(t-c^{-1}(1-x)\right) & \text{if}\quad x+ct>1\end{cases}\tag{5.81}$$

where $\xi:\Re_+\to\Re$ is the solution of the problem

$$\dot{\xi}(t)=-\mu\left(\xi(t)-f\left(\rho_{eq}\exp(w(t,1))\frac{c+f(\rho_{eq})}{c+\xi(t)}\right)\right)\tag{5.82}$$

$$\xi(0)=v_0(1).\tag{5.83}$$

Formula (5.81) implies for every $\sigma > 0$, $t \ge 0$:

$$\|b[t]\|_\infty \le \exp(-\sigma(ct-1))\|b_0\|_\infty + \max_{\max(0,t-c^{-1})\le s\le t}\left(\left|\ln\left(\frac{\xi(s)}{f(\rho_{eq})}\right)\right|\right) \tag{5.84}$$

Using the transformation

$$\zeta(t) = \ln\left(\xi(t)/f(\rho_{eq})\right), \tag{5.85}$$

we get from (5.82):

$$\dot\zeta(t) = -\mu\left(1 - \frac{\exp(-\zeta(t))}{f(\rho_{eq})} f\left(\frac{\rho_{eq}\exp(w(t,1))(c+f(\rho_{eq}))}{c+f(\rho_{eq})\exp(\zeta(t))}\right)\right) \tag{5.86}$$

Inequality (3.2) implies that $0 \in \Re$ is a globally asymptotically stable equilibrium point for system (5.82) with $w(t,1) \equiv 0$. Consequently, it follows from (5.80) and Theorem 2.2 in [16] that there exists a function $P \in KL$ such that the following estimate holds for all $t \ge v_{\min}^{-1}$:

$$\left|\ln\left(\xi(t)/f(\rho_{eq})\right)\right| \le P\left(\left|\ln\left(\xi(v_{\min}^{-1})/f(\rho_{eq})\right)\right|, t - v_{\min}^{-1}\right) \tag{5.87}$$

Since $(\zeta(t), w(t,1)) \in \Re^2$ takes values in a compact set $S(w_0, v_0) \subset \Re^2$ for all $t \ge 0$ (recall definition (5.85), (5.76), (5.79) and (5.81)) and since $F(\zeta, w) := -\mu\left(1 - \frac{\exp(-\zeta)}{f(\rho_{eq})} f\left(\frac{\rho_{eq}\exp(w)(c+f(\rho_{eq}))}{c+f(\rho_{eq})\exp(\zeta)}\right)\right)$ is a C^1 mapping, there exists a non-decreasing function $L: \Re_+ \to \Re_+$ such that $\left|\dot\zeta(t)\right| \le \left(|\zeta(t)| + |w(t,1)|\right) L\left(\|w_0\|_\infty + \|b_0\|_\infty\right)$. Using Gronwall's Lemma in conjunction with (5.76) and the previous inequality we get $|\zeta(t)| \le \exp(Lv_{\min}^{-1})\left(|\zeta(0)| + Lv_{\min}^{-1}\|w_0\|_\infty\right)$, for $t \in [0, v_{\min}^{-1}]$, where $L := L\left(\|w_0\|_\infty + \|b_0\|_\infty\right)$. Combining (5.87) with the previous estimate, (5.78), (5.85), (5.83), we get for $t \ge 0$:

$$\begin{aligned}
&\left|\ln\left(\xi(t)/f(\rho_{eq})\right)\right| \\
&\le P\left(\exp\left(Lv_{\min}^{-1}\right)\left(\|b_0\|_\infty + Lv_{\min}^{-1}\|w_0\|_\infty\right), \max(0, t - v_{\min}^{-1})\right) \\
&+ \exp\left((L+1)(2v_{\min}^{-1} - t)\right)\left(\|b_0\|_\infty + Lv_{\min}^{-1}\|w_0\|_\infty\right)
\end{aligned} \tag{5.88}$$

Using (5.76), (5.84), (5.88), (5.80), we get for $\sigma > 0$, $t \ge 0$:

$$\begin{aligned}
&\|b[t]\|_\infty + \|w[t]\|_\infty \\
&\le P\left(\exp\left(Lv_{\min}^{-1}\right)\left(\|b_0\|_\infty + Lv_{\min}^{-1}\|w_0\|_\infty\right), \max(0, t - c^{-1} - v_{\min}^{-1})\right) \\
&+ \exp\left((L+1)(2v_{\min}^{-1} + c^{-1} - t)\right)\left(\|b_0\|_\infty + Lv_{\min}^{-1}\|w_0\|_\infty\right) \\
&+ \exp\left(-\sigma(ct-1)\right)\|b_0\|_\infty + \exp\left(-\sigma(t - v_{\min}^{-1})\right)\|w_0\|_\infty
\end{aligned} \tag{5.89}$$

Estimate (5.89) in conjunction with the fact that $L := L\left(\|w_0\|_\infty + \|b_0\|_\infty\right)$, where L is a non-decreasing function and definition (5.79) of $v_{\min}$ implies that there exists $G \in KL$ such that the following estimate holds for $t \ge 0$:

$$\|b[t]\|_\infty + \|w[t]\|_\infty \le G\left(\|b_0\|_\infty + \|w_0\|_\infty, t\right). \tag{5.90}$$

Estimate (3.4) for certain $Q \in KL$ follows from (5.90), (2.7), which imply the following inequalities for $(t,x) \in \Re_+ \times [0,1]$

$$\begin{aligned}
\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right| &\le |w(t,x)| + f(\rho_{eq})\left|\frac{1-\exp(b(t,x))}{c+f(\rho_{eq})\exp(b(t,x))}\right| \\
&\le |w(t,x)| + c^{-1} f(\rho_{eq})\exp\left(|b(t,x)|\right)|b(t,x)|
\end{aligned}$$

$$\begin{aligned}
|w(t,x)| &\le \left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right| + f(\rho_{eq})\left|\frac{1-\exp(b(t,x))}{c+f(\rho_{eq})\exp(b(t,x))}\right| \\
&\le \left|\ln\left(\rho(t,x)/\rho_{eq}\right)\right| + c^{-1} f(\rho_{eq})\exp\left(|b(t,x)|\right)|b(t,x)|
\end{aligned}$$

Indeed, the two above inequalities imply the existence of a function $\varphi \in K_\infty$ such that the estimates

$$\max_{0\le x\le 1}\left(\left|\ln\left(\rho(t,x)/\rho_{eq}\right)\right|\right) \le \varphi\left(\|b[t]\|_\infty + \|w[t]\|_\infty\right),$$

$$\|w[t]\|_\infty \le \varphi\left(\|b[t]\|_\infty + \max_{0\le x\le 1}\left(\left|\ln\left(\rho(t,x)/\rho_{eq}\right)\right|\right)\right)$$

hold for $t \ge 0$. The proof is complete. ⊲

VI. Concluding Remarks

The paper provides results for a hyperbolic traffic flow model on a bounded domain. It has been shown that for all physically meaningful initial conditions, the model admits a unique, classical solution that remains positive and bounded for all times. Moreover, it has been shown that global stabilization in the sup-norm of the logarithmic deviation of the state from its equilibrium point can be achieved by means of a boundary feedback law. It is important to notice that the feedback law depends *only* on the inlet speed. Future work may involve the development of more complicated models, retaining the characteristics of the proposed model, to capture secondary features of traffic dynamics. More complicated models can also expand the validity of the model to uncongested roads with free flow.

Acknowledgments: I. Karafyllis and M. Papageorgiou were supported by the funding from the ERC under the European Union's 7th Framework Programme (FP/2007-2013)/ ERC Grant Agreement n. [321132], project TRAMAN21. N. Bekiaris-Liberis was supported by the funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 747898, project PADECOT.

Appendix

Proof of Proposition 5.2: Let $T > 0$ be given. We follow the methodology of finite-differences presented in [15].

Since $v(t,0) > 0$ for all $t \ge 0$, by continuity there exists $\varepsilon(T) > 0$ such that $v(t,x) > 0$ for $t \in [0,T]$, $x \in [0, \varepsilon(T)]$. Let $N^* > 1$ be an integer for which the inequalities

$$2\max\left\{\left|\frac{\partial v}{\partial t}(t,x)\right| : (t,x) \in [0,T]\times[0,1]\right\} \le N^* v_{\min}^2$$

$$\text{and } 2 \le N^*\varepsilon(T) \tag{A.1}$$

hold, where $v_{\min} := \min\{ v(t,x) : (t,x) \in [0,T]\times[0,\varepsilon(T)] \}$. Let $N \ge N^*$ and consider the discrete-time system

$$w_i((k+1)\delta) = (1-\lambda v_i(k\delta))w_i(k\delta) + \lambda v_i(k\delta) w_{i-1}(k\delta) ,$$
$$\text{for } i=1,\ldots,N ,\ k=0,1,\ldots,m-1 \quad (A.2)$$
$$w_0(k\delta) = a(k\delta) \text{ , for } k=0,1,\ldots,m \quad (A.3)$$
$$w_i(0) = \varphi(ih) \text{ , for } i=1,\ldots,N \quad (A.4)$$

where

$$h := 1/N ,\ \delta := \lambda h \quad (A.5)$$
$$v_i(k\delta) := v(k\delta, ih) \text{, for } i=0,1,\ldots,N ,\ k=0,1,\ldots,m \quad (A.6)$$
$$\lambda := T/([Tv_{\max}]+1) \quad (A.7)$$
$$m := N([Tv_{\max}]+1) \quad (A.8)$$
$$v_{\max} := \max\{ v(t,x) : (t,x) \in [0,T]\times[0,1] \}. \quad (A.9)$$

Notice that the above definitions guarantee that

$$T = m\delta \ , \quad (A.10)$$
$$\lambda v_{\max} \le 1 . \quad (A.11)$$

Using (A.2), (A.3), (A.6), (A.9) in conjunction with (A.11), we obtain the estimate for $k=0,1,\ldots,m-1$

$$\max_{i=0,\ldots,N}\left(|w_i((k+1)\delta)|\right) \le \max\left(\|a\| , \max_{i=0,\ldots,N}\left(|w_i(k\delta)|\right)\right) , \quad (A.12)$$

where $\|a\| := \max_{0\le s\le T}\left(|a(s)|\right)$. It follows from (A.3), (A.4) and (A.12) that the following estimate holds

$$\max_{i=0,\ldots,N}\left(|w_i(k\delta)|\right) \le \max\left(\|a\| , \|\varphi\|_\infty\right) \text{, for } k=0,1,\ldots,m . \quad (A.13)$$

We define $w(t,x;N)$ for $(t,x)\in[0,T]\times[0,1]$ and $N \ge N^*$:

$$w(k\delta,x;N) = (i+1-xN)w_i(k\delta) + (xN-i)w_{i+1}(k\delta)$$
$$\text{with } i=[xN] \text{, for } x\in[0,1) ,\ k=0,\ldots,m , \quad (A.14)$$
$$w(k\delta,1;N) = w_N(k\delta) \text{, for } k=0,\ldots,m , \quad (A.15)$$
$$w(t,x;N) =$$
$$(k+1-\lambda^{-1}tN)w(k\delta,x;N) + (\lambda^{-1}tN-k)w((k+1)\delta,x;N)$$
$$\text{with } k=[\lambda^{-1}tN] \text{ for } x\in[0,1] ,\ t\in[0,T) . \quad (A.16)$$

It follows from (A.13) and definitions (A.14), (A.15), (A.16) that the following estimate holds for every $N \ge N^*$:

$$\max_{0\le x\le 1}\left(|w(t,x;N)|\right) \le \max\left(\|a\| , \|\varphi\|_\infty\right) \text{, for } t\in[0,T] . \quad (A.17)$$

Next define for $i=0,1,\ldots,N-1$, $k=0,1,\ldots,m$

$$y_i(k\delta) := h^{-1}\left(w_{i+1}(k\delta) - w_i(k\delta)\right) \quad (A.18)$$
$$y_N(k\delta) := y_{N-1}(k\delta) . \quad (A.19)$$

Equations (A.2), (A.3) in conjunction with definitions (A.18), (A.19) imply that the following equalities hold:

$$y_i((k+1)\delta) = (1-\lambda v_{i+1}(k\delta))y_i(k\delta) + \lambda v_{i+1}(k\delta) y_{i-1}(k\delta)$$
$$-\lambda\left(v_{i+1}(k\delta) - v_i(k\delta)\right)y_{i-1}(k\delta) ,$$
$$\text{for } i=1,\ldots,N-1,\ k=0,1,\ldots,m-1, \quad (A.20)$$
$$y_0((k+1)\delta) = (1-\lambda v_1(k\delta))y_0(k\delta) - \lambda\delta^{-1}\left(a((k+1)\delta) - a(k\delta)\right),$$
$$\text{for } k=0,1,\ldots,m-1 . \quad (A.21)$$

Using the fact that $|v_{i+1}(k\delta) - v_i(k\delta)| \le h\|v_x\|$ for all $i=0,\ldots,N-1$, $k=0,1,\ldots,m$, where $\|v_x\| := \max\left\{ \left|\frac{\partial v}{\partial x}(t,x)\right| : (t,x)\in[0,T]\times[0,1] \right\}$, the fact $|a((k+1)\delta) - a(k\delta)| \le \delta\|\dot a\|$ for all $k=0,1,\ldots,m-1$, where $\|\dot a\| := \max\left\{ |\dot a(t)| : t\in[0,T] \right\}$ and the fact that $v_1(k\delta) \ge v_{\min} > 0$ for all $k=0,1,\ldots,m$, where $v_{\min} := \min\{ v(t,x) : (t,x)\in[0,T]\times[0,\varepsilon(T)] \}$ (a consequence of (A.1), (A.5), (A.6) which imply that $2h \le \varepsilon(T)$), in conjunction with (A.19), (A.20), (A.21), (A.5), (A.6), (A.9), (A.11), we get for $k=0,1,\ldots,m-1$:

$$\max_{i=0,\ldots,N}\left(|y_i((k+1)\delta)|\right) \le$$
$$\max\left((1+\delta\|v_x\|) \max_{i=0,\ldots,N}\left(|y_i(k\delta)|\right), (1-\lambda v_{\min}) \max_{i=0,\ldots,N}\left(|y_i(k\delta)|\right) + \lambda\|\dot a\| \right) \quad (A.22)$$

Using (A.22) in conjunction with (A.10), the fact that $|y_i(0)| \le \|\varphi'\|_\infty$ for $i=0,\ldots,N$ (a consequence of definitions (A.3), (A.4), (A.18), (A.19) and the fact that $a(0)=\varphi(0)$) and Lemma 5.1, we obtain the estimate for $k=0,1,\ldots,m$:

$$\max_{i=0,\ldots,N}\left(|y_i(k\delta)|\right) \le Y := \exp\left(T\|v_x\|\right)\left(\|\varphi'\|_\infty + v_{\min}^{-1}\|\dot a\|\right). \quad (A.23)$$

Next define for $i=0,1,\ldots,N$, $k=0,1,\ldots,m-1$

$$p_i(k\delta) := \delta^{-1}\left(w_i((k+1)\delta) - w_i(k\delta)\right) \quad (A.24)$$
$$p_i(m\delta) := p_i((m-1)\delta) . \quad (A.25)$$

Using (A.2), (A.3), (A.5), (A.9), (A.18), (A.24), (A.25) (which imply that $p_i(k\delta) = -v_i(k\delta)y_{i-1}(k\delta)$, for $i=1,\ldots,N$, $k=0,1,\ldots,m-1$ and $p_0(k\delta) = \delta^{-1}\left(a((k+1)\delta) - a(k\delta)\right)$, for $k=0,1,\ldots,m-1$), the fact $|a((k+1)\delta) - a(k\delta)| \le \delta\|\dot a\|$ for $k=0,1,\ldots,m-1$, and estimate (A.23), we obtain for $k=0,1,\ldots,m$

$$\max_{i=0,\ldots,N}\left(|p_i(k\delta)|\right) \le \max\left(v_{\max}Y , \|\dot a\| \right). \quad (A.26)$$

Definitions (A.18), (A.24) in conjunction with estimates (A.23), (A.26) imply for $i,j=0,1,\ldots,N$, $k,l=0,1,\ldots,m$:

$$|w_i(k\delta) - w_j(l\delta)| \le h|i-j|Y + \delta|k-l|\max\left(v_{\max}Y, \|\dot a\|\right). \quad (A.27)$$

Estimate (A.27) in conjunction with (A.14), (A.15), (A.16) imply that there exists $L_1 = L_1(T,a,v,\varphi)$ such that the following inequality holds for $N \ge N^*$, $t,\tau\in[0,T]$, $x,z\in[0,1]$:

$$|w(t,x;N) - w(\tau,z;N)| \le L_1\left(|x-z| + |t-\tau|\right). \quad (A.28)$$

Next define for $k=0,1,\ldots,m-1$:

$$\psi(k\delta) := h^{-1}\left(\delta^{-1}\left(a((k+1)\delta) - a(k\delta)\right) + v_1(k\delta)y_0(k\delta)\right). \quad (A.29)$$

It follows from (A.29) and (A.21) that

$$\psi((k+1)\delta) = \frac{v_1((k+1)\delta)}{v_1(k\delta)}(1-\lambda v_1(k\delta))\psi(k\delta)$$
$$+\lambda\delta^{-2}\left(a((k+2)\delta) - 2a((k+1)\delta) + a(k\delta)\right) \quad (A.30)$$
$$-\lambda v_1^{-1}(k\delta)\delta^{-2}\left(a((k+1)\delta) - a(k\delta)\right)\left(v_1((k+1)\delta) - v_1(k\delta)\right)$$

for $k=0,1,...,m-2$. Inequalities (A.1) in conjunction with (A.5), (A.6) and definition $v_{\min} := \min\{v(t,x):(t,x)\in[0,T]\times[0,\varepsilon(T)]\}$ imply that

$$v_1((k+1)\delta)(1-\lambda v_1(k\delta))/v_1(k\delta) \le 1-\lambda v_{\min}/2. \quad (A.31)$$

for $k=0,1,...,m-1$. It follows from (A.30), (A.31) in conjunction with the fact that $|a((k+1)\delta)-a(k\delta)| \le \delta\|\dot{a}\|$ for $k=0,1,...,m-1$, where $\|v_t\| := \max\left\{\left|\frac{\partial v}{\partial t}(t,x)\right| : (t,x)\in[0,T]\times[0,1]\right\}$ (recall definition (A.6)), the fact that $v_1(k\delta)\ge v_{\min}>0$ for all $k=0,1,...,m$, the fact that $|a((k+2)\delta)-2a((k+1)\delta)+a(k\delta)| \le \delta^2\|\ddot{a}\|$ for $k=0,1,...,m-2$, where $\|\ddot{a}\| := ess\sup\{|\ddot{a}(t)| : t\in[0,T]\}$, that the following inequality holds for $k=0,1,...,m-2$:

$$|\psi((k+1)\delta)| \le (1-\lambda v_{\min}/2)|\psi(k\delta)| + \lambda\|\ddot{a}\| + \lambda v_{\min}^{-1}\|\dot{a}\|\|v_t\|. \quad (A.32)$$

Consequently, we obtain (by induction) the following estimate for $k=0,1,...,m-1$:

$$|\psi(k\delta)| \le |\psi(0)| + 2v_{\min}^{-2}(\|\dot{a}\|\|v_t\| + v_{\min}\|\ddot{a}\|). \quad (A.33)$$

Definitions (A.3), (A.4), (A.6), (A.9), (A.18), (A.29) in conjunction with $\dot{a}(0)+v(0,0)\varphi'(0)=0$, $a(0)=\varphi(0)$, definition $\|\varphi''\|_\infty := ess\sup\{|\varphi''(x)| : x\in[0,1]\}$ and (A.11) imply that:

$$|\psi(0)| \le \|\ddot{a}\|/(2v_{\max}) + |\varphi'(0)|\|v_x\| + v_{\max}\|\varphi''\|_\infty/2. \quad (A.34)$$

Thus, we get from (A.33), (A.34) for $k=0,1,...,m-1$:

$$|\psi(k\delta)| \le M := \|\ddot{a}\|/(2v_{\max}) + |\varphi'(0)|\|v_x\| + v_{\max}\|\varphi''\|_\infty/2 + 2v_{\min}^{-2}(\|\dot{a}\|\|v_t\| + v_{\min}\|\ddot{a}\|) \quad (A.35)$$

We define the function $y(t,x;N)$ for $(t,x)\in[0,T]\times[0,1]$ and for every integer $N\ge N^*$:

$$y(k\delta,x;N) = (i+1-xN)y_i(k\delta) + (xN-i)y_{i+1}(k\delta)$$
$$\text{with } i=[xN], \text{ for } x\in[0,1), \ k=0,...,m, \quad (A.36)$$

$$y(k\delta,1;N) = y_N(k\delta), \text{ for } k=0,...,m, \quad (A.37)$$

$$y(t,x;N) = (k+1-\lambda^{-1}tN)y(k\delta,x;N) + (\lambda^{-1}tN-k)y((k+1)\delta,x;N)$$
$$\text{with } k=[\lambda^{-1}tN] \text{ for } x\in[0,1], \ t\in[0,T). \quad (A.38)$$

It follows from (A.23) and (A.36), (A.37), (A.38) that the following estimate holds for every $N\ge N^*$ and $t\in[0,T]$:

$$\max_{0\le x\le 1}(|y(t,x;N)|) \le Y := \exp(T\|v_x\|)(\|\varphi'\|_\infty + v_{\min}^{-1}\|\dot{a}\|). \quad (A.39)$$

We also define the function $p(t,x;N)$ for $(t,x)\in[0,T]\times[0,1]$ and for every integer $N\ge N^*$:

$$p(k\delta,x;N) = (i+1-xN)p_i(k\delta) + (xN-i)p_{i+1}(k\delta)$$
$$\text{with } i=[xN], \text{ for } x\in[0,1), \ k=0,...,m, \quad (A.40)$$

$$p(k\delta,1;N) = p_N(k\delta), \text{ for } k=0,...,m, \quad (A.41)$$

$$p(t,x;N) = (k+1-\lambda^{-1}tN)p(k\delta,x;N) + (\lambda^{-1}tN-k)p((k+1)\delta,x;N)$$
$$\text{with } k=[\lambda^{-1}tN] \text{ for } x\in[0,1], \ t\in[0,T). \quad (A.42)$$

It follows from (A.26) and definitions (A.40), (A.41), (A.42) that the following estimate holds for every $N\ge N^*$:

$$\max_{0\le x\le 1}(|p(t,x;N)|) \le \max(v_{\max}Y, \|\dot{a}\|), \text{ for } t\in[0,T]. \quad (A.43)$$

Next define for $i=0,1,...,N-1$, $k=0,1,...,m$

$$\omega_i(k\delta) := h^{-1}(y_{i+1}(k\delta) - y_i(k\delta)). \quad (A.44)$$

Definitions (A.19), (A.29), (A.44) in conjunction with (A.20), (A.21) imply that the following equalities hold:

$$\omega_i((k+1)\delta) = (1-\lambda v_{i+2}(k\delta))\omega_i(k\delta) + \lambda v_{i+2}(k\delta)\omega_{i-1}(k\delta)$$
$$-2\lambda(v_{i+2}(k\delta)-v_{i+1}(k\delta))\omega_{i-1}(k\delta)$$
$$-\lambda h^{-1}(v_{i+2}(k\delta)-2v_{i+1}(k\delta)+v_i(k\delta))y_{i-1}(k\delta)$$
$$\text{for } i=1,...,N-2, \ k=0,1,...,m-1 \quad (A.45)$$

$$\omega_{N-1}(k\delta)=0, \text{ for } k=0,1,...,m. \quad (A.46)$$

$$\omega_0((k+1)\delta) = (1-\lambda v_2(k\delta))\omega_0(k\delta)$$
$$-\lambda h^{-1}(v_2(k\delta)-v_1(k\delta))y_0(k\delta) + \lambda\psi(k\delta),$$
$$\text{for } k=0,1,...,m-1. \quad (A.47)$$

Using the facts that $|v_{i+2}(k\delta)-2v_{i+1}(k\delta)+v_i(k\delta)| \le h^2\|v_{xx}\|$, $|v_{i+2}(k\delta)-v_{i+1}(k\delta)| \le h\|v_x\|$ for $i=0,...,N-2$, $k=0,1,...,m$, where $\|v_{xx}\| := \sup\left\{\frac{\left|\frac{\partial v}{\partial x}(t,x)-\frac{\partial v}{\partial x}(t,z)\right|}{|x-z|} : (t,x,z)\in[0,T]\times[0,1]^2, x\ne z\right\}$ and the fact that $v_2(k\delta)\ge v_{\min}>0$ for all $k=0,1,...,m$, where $v_{\min} := \min\{v(t,x):(t,x)\in[0,T]\times[0,\varepsilon(T)]\}$ (a consequence of (A.1), (A.5), (A.6) which imply that $2h\le\varepsilon(T)$), in conjunction with (A.23), (A.35), (A.6), (A.9), (A.11), we get for $k=0,1,...,m-1$ and $H(k)=\max_{i=0,...,N-1}(|\omega_i((k\delta)|)$:

$$H(k+1) \le \max\left((1+2\delta\|v_x\|)H(k)+\delta\|v_{xx}\|Y, (1-\lambda v_{\min})H(k)+\lambda(\|v_x\|Y+M)\right) \quad (A.48)$$

It follows from (A.10) and Lemma 5.1 for $k=0,1,...,m$:

$$\max_{i=0,...,N-1}(|\omega_i((k\delta)|) \le \exp(2T\|v_x\|)\left(\max_{i=0,...,N-1}(|\omega_i(0)|) + \frac{\|v_x\|Y+M}{v_{\min}} + T\|v_{xx}\|Y\right) \quad (A.49)$$

Definitions (A.3), (A.4), (A.44), (A.18), (A.19) with $a(0)=\varphi(0)$ imply $|\omega_i(0)|\le\|\varphi''\|_\infty$ for $i=0,...,N-1$, where $\|\varphi''\|_\infty := ess\sup\{|\varphi''(x)| : x\in[0,1]\}$. It follows from (A.49) that the following estimate holds for $k=0,1,...,m$:

$$\max_{i=0,...,N-1}(|\omega_i((k\delta)|) \le \Omega := \exp(2T\|v_x\|)\left(\|\varphi''\|_\infty + \frac{\|v_x\|Y+M}{v_{\min}} + T\|v_{xx}\|Y\right). \quad (A.50)$$

Next define for $i=0,1,...,N$, $k=0,1,...,m-1$

$$\eta_i(k\delta) := \delta^{-1}(y_i((k+1)\delta) - y_i(k\delta)). \quad (A.51)$$

Equations (A.20), (A.21),(A.29),(A.44),(A.51), imply that

$$\eta_i(k\delta) = -v_{i+1}(k\delta)\omega_{i-1}(k\delta) - h^{-1}\left(v_{i+1}(k\delta) - v_i(k\delta)\right)y_{i-1}(k\delta),$$
$$\text{for } i = 1,...,N-1,\ k = 0,1,...,m-1 \qquad (A.52)$$

$$\eta_0(k\delta) = -\psi(k\delta), \text{ for } k = 0,1,...,m-1 \qquad (A.53)$$

which combined with (A.9), (A.19), (A.50), (A.35) and the fact that $\left|v_{i+1}(k\delta) - v_i(k\delta)\right| \le h\|v_x\|$ for all $i = 0,...,N-1$, $k = 0,1,...,m$, give for $k = 0,1,...,m-1$:

$$\max_{i=0,...,N}\left(\left|\eta_i((k\delta)\right|\right) \le v_{\max}\Omega + \|v_x\|Y + M. \qquad (A.54)$$

Definitions (A.44), (A.51) with (A.50), (A.54) imply the following estimate for $i, j = 0,1,...,N$, $k,l = 0,1,...,m$:

$$\left|y_i(k\delta) - y_j(l\delta)\right| \le h\left|i-j\right|\Omega + \delta\left|k-l\right|\left(v_{\max}\Omega + M + Y\|v_x\|\right). \qquad (A.55)$$

Estimate (A.55) in conjunction with (A.36), (A.37), (A.38) imply that there exists $L_2 = L_2(T,a,v,\varphi)$ such that the following inequality holds for $N \ge N^*$, $t,\tau \in [0,T]$ and $x,z \in [0,1]$:

$$\left|y(t,x;N) - y(\tau,z;N)\right| \le L_2\left(\left|x-z\right| + \left|t-\tau\right|\right). \qquad (A.56)$$

Following a similar procedure, it is shown that there exists $L_3 = L_3(T,a,v,\varphi)$ such that the following inequality holds for $N \ge N^*$, $t,\tau \in [0,T]$, $x,z \in [0,1]$:

$$\left|p(t,x;N) - p(\tau,z;N)\right| \le L_3\left(\left|x-z\right| + \left|t-\tau\right|\right). \qquad (A.57)$$

It follows from (A.17), (A.28), (A.39), (A.56), (A.43), (A.57) that the sequences $\{w(\cdot\,;N)\}_{N=N^*}^{\infty}$, $\{y(\cdot\,;N)\}_{N=N^*}^{\infty}$, $\{p(\cdot\,;N)\}_{N=N^*}^{\infty}$ are uniformly bounded and equicontinuous. Compactness of $[0,T]\times[0,1]$ and the Arzela-Ascoli theorem implies that there exist Lipschitz functions $w,y,p:[0,T]\times[0,1]\to\Re$ and subsequences $\{w(\cdot\,;N_q)\}_{q=1}^{\infty}$, $\{y(\cdot\,;N_q)\}_{q=1}^{\infty}$, $\{p(\cdot\,;N_q)\}_{q=1}^{\infty}$ for an increasing sequence $\{N_q\}_{q=1}^{\infty}$, which converge uniformly on $[0,T]\times[0,1]$ to w,y,p.

It is shown next that since $\{y(\cdot\,;N_q)\}_{q=1}^{\infty}$, $\{w(\cdot\,;N_q)\}_{q=1}^{\infty}$ converge uniformly to y,w, as $q \to +\infty$, it follows that $y(t,x) = \frac{\partial w}{\partial x}(t,x)$, $p(t,x) = \frac{\partial w}{\partial t}(t,x)$ and $p(t,x) + v(t,x)y(t,x) = 0$, for $(t,x) \in [0,T]\times[0,1]$. Uniqueness follows use of Gronwall's lemma for the functional $V(t) = \int_0^1 e^2(t,x)dx$, where $e = w - \tilde{w}$ and $w,\tilde{w}$ are solutions of (5.3), (5.4), (5.5).

Finally, we assume that there exists a constant $v_{\min} > 0$ such that $v(t,x) \ge v_{\min}$ for all $t \ge 0$, $x \in [0,1]$ and that $a \equiv 0$. Let $T > v_{\min}^{-1}$ be given (arbitrary). Consider the functionals $V_\sigma(t) = \int_0^1 \exp(-\sigma x)w^2(t,x)dx$ on $[0,T]$ with parameter $\sigma > 0$. Using (5.3), (5.5) and the fact that $v(t,x) \ge v_{\min}$, we get:

$$\dot{V}_\sigma(t) = -v(t,1)\exp(-\sigma)w^2(t,1) + \int_0^1 \exp(-\sigma x)w^2(t,x)\frac{\partial v}{\partial x}(t,x)dx$$
$$-\sigma\int_0^1 \exp(-\sigma x)w^2(t,x)v(t,x)dx \le -\left(\sigma v_{\min} - \|v_x\|\right)V_\sigma(t)$$

It follows that $V_\sigma(t) \le \exp\left(-\left(\sigma v_{\min} - \|v_x\|\right)t\right)V_\sigma(0)$, for $t \in [0,T]$. The previous inequality implies the estimate $\|w[t]\|_2^2 \le \exp\left(-\sigma\left(\left(v_{\min} - \sigma^{-1}\|v_x\|\right)t - 1\right)\right)\|\varphi\|_2^2$, for $t \in [0,T]$. Since $\lim_{\sigma\to+\infty}\left(-\sigma\left(\left(v_{\min} - \sigma^{-1}\|v_x\|\right)t - 1\right)\right) = -\infty$ for each $t \in \left(v_{\min}^{-1}, T\right]$, we get $\|w[t]\|_2 = 0$, for $t \in \left(v_{\min}^{-1}, T\right]$. Therefore, by continuity of w and since $T > v_{\min}^{-1}$ is arbitrary, we conclude that $w(t,x) = 0$ for $x \in [0,1]$ and $t \ge v_{\min}^{-1}$. $\triangleleft$

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