STABILITY INVESTIGATION FOR SIMPLE PI-CONTROLLED TRAFFIC SYSTEMS

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ABSTRACT

This paper provides sufficient conditions for the Input-to-State Stability property of simple uncertain vehicular-traffic network models under the effect of a PI-regulator. Local stability properties for vehicular-traffic networks under the effect of PI-regulator control are studied as well: the region of attraction of a locally exponentially stable equilibrium point is estimated by means of Lyapunov functions.

Index Terms— Nonlinear systems, discrete-time systems, PI-regulator.

1. INTRODUCTION

There are a number of relatively simple controlled processes within vehicular-traffic networks or water networks, which share the following characteristics:

- The kernel of the process is some sort of "reservoir" (e.g. an urban road network, a freeway stretch, a water reservoir or basin) which accumulates inflows and outflows; the reservoir features a limited storage capacity.
- There is a controllable but constrained inflow; the inflow may be released at some distance from the reservoir, in which case it reaches the reservoir with a corresponding time-delay.
- There may be additional uncontrollable inflows.
- The outflow depends on the reservoir storage in a nonlinear way; there may be some modeling uncertainty in the related function.
- The control goal is to operate the system near a pre-specified storage level.

Examples of such controlled processes include local freeway ramp metering [12], gating control of urban network parts [8], merging traffic control [13], variable speed limit control on freeways [3], water level and water flow control [10,11]. In some cases, these elementary systems may be interconnected to form bigger composite systems, as, e.g., in the cases of multiple urban network parts [1] or irrigation networks [2].

The mentioned characteristics indicate that these elementary processes may be modeled as discrete-time time-delayed constrained nonlinear first-order systems. A PI-type regulator is usually employed for system control in practice; whereby the regulator parameters are selected after model linearization around the desired set-value, using classical linear sample-data concepts. It should be noted that, in the case of traffic systems, the nonlinear function connecting the outflow with the reservoir storage is typically a concave uni-modal function featuring a maximum, which usually corresponds to the desired operation state. Although these systems are usually operating reasonably well in practice, it is interesting to have a second look at them from a nonlinear analysis point of view. Specifically, we are interested in deriving local and global stability results for the PI-controlled nonlinear models, which is the main scope of this paper. Eventually, we are interested in deriving nonlinear stabilizers and, finally, in considering control of bigger composite systems.

Consider the following 1-dimensional discrete-time control system, which is representative for all examples of elementary practical systems mentioned earlier:

$$\begin{aligned} & x^{+} = x - f(d,x) + \min(u+v, a-x) \\ & x \in [0,a], u \in [b_{\min}, b_{\max}], d \in D, v \in \Re_{+} \end{aligned} \tag{1}$$

where $D \subseteq \Re$ is a compact set, $a > 0$, $0 \le b_{\min} < b_{\max}$ are constants and $f \in C^{0}(D \times [0,a]; \Re)$ satisfies

$$0 \le f(d,x) \le x \quad \text{for all } (d,x) \in D \times [0,a]. \tag{2}$$

System (1) describes the time evolution of a traffic (or water) system, where $x \in [0,a]$ is the current number of vehicles (storage) in the network, $f(d,x)$ is the (uncertain) outflow function, $a > 0$ is the capacity of the network and $v \in \Re_{+}$ is the input that reflects the uncontrollable inflow. System (1) under assumption (2) is a well-defined control system which satisfies $x^{+} \in [0,a]$ for all

$(x,u,d,v)\in[0,a]\times[b_{\min},b_{\max}]\times D\times\Re_+$. In order to state the control problem, we assume that:

(H1) *There exists* $(x^*,u^*,v^*)\in(0,a)\times(b_{\min},b_{\max})\times\Re_+$ *such that* $f(d,x^*)=u^*+v^*<a-x^*$ *for all* $d\in D$.

In other words, $x^*\in(0,a)$ is an equilibrium point for system (1) with $v\equiv v^*$ and $u\equiv u^*$. The PI regulator is the dynamic feedback law given by the equation:

$$u(t)=\max\left(b_{\min},\min\left(b_{\max},u(t-1)-k_1\left(x(t)-x(t-1)\right)-k_2\left(x(t)-x^*\right)\right)\right) \tag{3}$$

where k_1,k_2 are constants. The closed-loop system (1) with (3) is described by the 3-dimensional discrete-time system:

$$\begin{aligned}&x^+=x-f(d,x)+\min(u+v,a-x)\\&y^+=x\quad,\quad w^+=u\\&u=\max\left(b_{\min},\min\left(w-k_1(x-y)-k_2\left(x-x^*\right),b_{\max}\right)\right)\end{aligned} \tag{4}$$

with state space $S=[0,a]^2\times\Re$ (i.e., $(x,y,w)\in[0,a]\times[0,a]\times\Re$) and inputs $(v,d)\in\Re_+\times D$. The point $(x^*,x^*,u^*)\in(0,a)^2\times(b_{\min},b_{\max})$ is an equilibrium point of system (4). In this work, we answer the following questions concerning the PI regulator:

1) What are the conditions that guarantee the (global) Input-to-State Stability (ISS) property with respect to the external input v uniformly in $d\in D$ for system (4)?

2) What are the conditions that guarantee local exponential stability for $(x^*,x^*,u^*)\in(0,a)^2\times(b_{\min},b_{\max})$ in the disturbance-free case, i.e., when $f(d,x)$ is independent of $d\in D$ and $v\equiv v^*$?

3) What is the region of attraction when the equilibrium point $(x^*,x^*,u^*)\in(0,a)^2\times(b_{\min},b_{\max})$ is locally exponentially stable in the disturbance-free case?

As expected, the answers to the above questions are related. The notion of ISS for discrete-time systems was studied in [5] and this notion is adopted here, although the system that we study (namely system (4)) evolves in a restricted state space (in $S=[0,a]^2\times\Re$) and not in $\Re^3$ (all the results in [5] are for discrete-time systems evolving in $\Re^n$). When the ISS property is applied to system (4) with $v\equiv v^*$, then it becomes identical to the notion of the Robust Global Asymptotic Stability (see [6]).

All proofs are omitted due to space limitation. The proofs can be found in [7].

Notation. $\Re_+:=[0,+\infty)$. By $C^0(A;\Omega)$, we denote the class of continuous functions on $A\subseteq\Re^n$, which take values in $\Omega\subseteq\Re^m$. By $C^k(A;\Omega)$, where $k\geq 1$ is an integer, we denote the class of functions on $A\subseteq\Re^n$ with continuous derivatives of order k, which take values in $\Omega\subseteq\Re^m$.

2. LOCAL RESULTS

This section is devoted to the analysis of local exponential stability for the disturbance-free version of the closed-loop system (1) with (3). The local stability analysis of the disturbance-free version of the closed-loop system (1) with (3) is equivalent to the stability analysis of the system:

$$\begin{aligned}&x^+=x-f(x)+\min(w+v^*,a-x)\\&w^+=P\left(w+\sigma f(x)-\sigma\min(w+v^*,a-x)-k_2(x-x^*)\right)\\&(x,w)\in[0,a]\times\Re\end{aligned} \tag{5}$$

where $f\in C^1\left([0,a];\Re\right)$ satisfies (H1), $\sigma=k_1+k_2$ and

$$P(x)=\max\left(b_{\min},\min\left(x,b_{\max}\right)\right),\text{ for all } x\in\Re. \tag{6}$$

Indeed, it should be noticed that the solution $(x(t),y(t),w(t))$ of (4) satisfies equations (5) for $t\geq 1$, since the following equations hold for $t\geq 1$:

$$y(t+1)=y(t)-f(y(t))+\min(w(t)+v^*,a-y(t))$$

$$w(t+1)=P\left(w(t)+\sigma f(y(t))-\sigma\min(w(t)+v^*,a-y(t))-k_2(y(t)-x^*)\right)$$

The equilibrium point (x^*,u^*) of (5) is in the interior of

$$\left\{\begin{aligned}&(x,w)\in[0,a]\times\Re:w+x\leq a-v^*,\\&b_{\min}\leq(1-\sigma)w+\sigma f(x)-\sigma v^*-k_2(x-x^*)\leq b_{\max}\end{aligned}\right\}$$

(recall (H1); notice that the right hand side of (5) is continuously differentiable on the interior of the previously mentioned region), and therefore it follows that (x^*,u^*) is locally exponentially stable iff all roots of the equation

$$s^2-(2-f'(x^*)-\sigma)s+(1-f'(x^*)-\sigma+k_2)=0 \tag{7}$$

are strictly inside the unit ball (see Chapter 5 in [14], Chapter 4 in [9] and the necessary extensions to the case of local exponential stability). In other words, one of the following conditions is equivalent to local exponential stability of the equilibrium point (x^*,u^*) of (5):

$(2-f'(x^*)-\sigma)^2<4(1-f'(x^*)-\sigma+k_2)<4$ (complex roots) or $\left|2-f'(x^*)-\sigma\right|<2$ and $\left|2-f'(x^*)-\sigma\right|-1<1-f'(x^*)-\sigma+k_2\leq\frac{\left(2-f'(x^*)-\sigma\right)^2}{4}$ (real roots).

In order to give an estimation of the region of attraction of the equilibrium point (x^*,u^*) of (5), we need to perform a Lyapunov analysis. The following function

$$V(x,w):=\left|x-x^*\right|+M\left|w+v^*-f(x)+(1-g)(x-x^*)\right|,\quad\text{for all } (x,w)\in[0,a]\times\Re \tag{8}$$

where $M>1$ and $g\in\Re$ with $|g|<1$ are constants to be determined, was selected to be a candidate Lyapunov function for (5). An estimate of the region of attraction $A\subseteq[0,a]\times\Re$ of (x^*,u^*) is the sublevel set

$$\Omega_\rho=\left\{(x,w)\in[0,a]\times\Re:V(x,w)<\rho\right\}\subseteq A \tag{9}$$

where $\rho > 0$ is a constant, for which the inequality $V(x^+, w^+) < V(x, w)$ holds for all $(x, w) \in \Omega_\rho$ (see Chapter 4 in [9] and [4]). Proposition 2.1 gives an estimate of the region of attraction for certain values of k_1, k_2.

Proposition 2.1: *Suppose that the equilibrium point (x^*, u^*) of (5) is locally exponentially stable. Moreover, suppose that $\left|k_2 - q^2\right| < \left(\left|1-q\right|-1\right)^2$, where $q = \frac{\sigma + f'(x^*)}{2} \in (0,2)$ and $\sigma = k_1 + k_2$. Let $\eta > 0$ be a constant for which $\max\left\{\left|f'(x) - f'(x^*)\right| : x \in [0,a], \left|x - x^*\right| \le \eta\right\} = L$ and such that $\left|f'(x) - f'(x^*)\right| < L$, for all $x \in [0,a]$ with $\left|x - x^*\right| < \eta$, where $L := \frac{\left(\left|1-q\right|-1\right)^2 - \left|k_2 - q^2\right|}{\left|q\right| + 1 - \left|1-q\right|}$. Define*

$$\rho = \min\left(\eta, G_1, G_2\right) \tag{10}$$

where $G_1 = \frac{a - v^* - u^* - x^*}{1 + L + \left|q - f'(x^*)\right|}$, $M = \frac{\left|q\right| + 1 - \left|1-q\right|}{\left|q\right|\left(1 - \left|1-q\right|\right) + \left|k_2 - q^2\right|}$ and

$$G_2 = \frac{\min\left(b_{\max} - u^*, u^* - b_{\min}\right)}{\max\left(\frac{\left|1 - 2q + f'(x^*)\right|}{M}, L + \left|k_2 + (1-2q)q + (q-1)f'(x^*)\right|\right)}$$

. Consider the solution $(x(t), w(t)) \in [0,a] \times \Re$ of (5) with initial condition $(x(0), w(0)) \in \Omega_\rho$. Then $\lim_{t \to +\infty} (x(t), w(t)) = (x^*, u^*)$.

Proposition 2.1 provides a conservative estimation of the region of attraction for only a subset of all pairs of values of the parameters k_1, k_2, for which (x^*, u^*) is locally exponentially stable. In order to obtain a less conservative estimation of the region of attraction, we can also use the following Proposition 2.2, which delivers a different region of attraction. Thus the overall estimation of the region of attraction corresponds to the union of the regions of attraction resulting from each proposition.

Proposition 2.2: *Suppose that the equilibrium point (x^*, u^*) of (5) is locally exponentially stable. Moreover, suppose that $0 < k_2 < 2$ and $\left|f'(x^*) - 1 + k_1\right| < \frac{1 - \left|1 - k_2\right|}{k_2 + 1 - \left|1 - k_2\right|}$. Let $\eta > 0$ be a constant for which $\max\left\{\left|f'(x) - 1 + k_1\right| : x \in [0,a], \left|x - x^*\right| \le \eta\right\} = \frac{1 - \left|1 - k_2\right|}{k_2 + 1 - \left|1 - k_2\right|}$ and such that $\left|f'(x) - 1 + k_1\right| < \frac{1 - \left|1 - k_2\right|}{k_2 + 1 - \left|1 - k_2\right|}$, for all $x \in [0,a]$ with $\left|x - x^*\right| < \eta$. Define ρ by means of (10), where $G_1 = \frac{a - v^* - u^* - x^*}{1 + \frac{1 - \left|1 - k_2\right|}{k_2 + 1 - \left|1 - k_2\right|} + \left|\sigma - 1\right|}$ and*

$$G_2 = \frac{\min\left(b_{\max} - u^*, u^* - b_{\min}\right)}{\max\left(\frac{k_2\left|1 - \sigma\right|}{k_2 + 1 - \left|1 - k_2\right|}, \frac{1 - \left|1 - k_2\right|}{k_2 + 1 - \left|1 - k_2\right|} + \left|(1-\sigma)(k_2 - 1)\right|\right)}$$

Consider the solution $(x(t), w(t)) \in [0,a] \times \Re$ of (5) with initial condition $(x(0), w(0)) \in \Omega_\rho$. Then $\lim_{t \to +\infty} (x(t), w(t)) = (x^*, u^*)$.

3. GLOBAL RESULTS

This section is devoted to the study of the (global) Input-to-State Stability (ISS) of system (4) with respect to the input $v \in \Re_+$ (uncontrollable inflow). More specifically, we study system (4) under the assumption

$$x^* + v^* + b_{\max} < a \tag{11}$$

We also assume that the uncertain function $f(d, x)$ satisfies the following assumption:

(H2) *There exist constants $r \le \frac{b_{\min} - u^*}{a - v^* - x^* - b_{\max}}$ with $0 < k_1 + k_2 + r < 2$, $\lambda_i \in [0,1)$, $\gamma_i \in (0, 1 - \lambda_i)$ $(i = 1,2)$, $(L, q, M) \in [0,1) \times (0,1] \times (1, +\infty)$, such that:*

$$\left(f(d,x) + x^* - a\right)\left(\beta - M^{-1}\right) \le \left(LM^{-1} - k_1\right)\left(x - x^*\right) + u^* - \lambda b_{\min} - (1-\lambda) b_{\max} \quad \text{and}$$

$$-\frac{k_1 + LM^{-1}}{\beta + M^{-1}}\left(x - x^*\right) + \frac{\lambda b_{\min} + u^* - (1+\lambda)\max\left(b_{\min}, a - v^* - x\right)}{\beta + M^{-1}} \le f(d,x) + x^* - a \le 0$$

$$\text{for all } (d,x) \in D \times [a - v^* - b_{\max}, a] \tag{12}$$

$$-\lambda_1 q^{-1} \le 1 + \frac{P\left(u^* + r(x - x^*)\right) - f(d,x) + v^*}{x - x^*} \le \lambda_1 \quad \text{and}$$

$$\left|(1-\beta)P\left(u^* + r(x - x^*)\right) + \beta f(d,x) - \beta v^* - (\beta - k_1)(x - x^*) - u^*\right| \le \gamma_1 (x - x^*)$$

$$\text{for all } (d,x) \in D \times (x^*, a - v^* - b_{\min}] \tag{13}$$

$$\lambda_2 \ge 1 + \frac{P\left(u^* + r(x - x^*)\right) - f(d,x) + v^*}{x - x^*} \ge -\lambda_2 q \quad \text{and}$$

$$\left|(1-\beta)P\left(u^* + r(x - x^*)\right) + \beta f(d,x) - \beta v^* - (\beta - k_1)(x - x^*) - u^*\right| \le \gamma_2 q \left|x - x^*\right|$$

$$\forall (d,x) \in D \times [0, x^*) \tag{14}$$

where $\beta := k_1 + k_2 + r$ and $P : \Re \to \Re$ is defined in (6).

Assumption (H2) is a set of sector-like conditions for the uncertain function $f(d,x)$. The following result provides sufficient conditions for the ISS property with respect to the input v uniformly in $d \in D$ for the closed-loop system (4). Notice that the gain of the input $v \in \Re_+$ is linear and is explicitly given. Moreover, for $v \equiv v^*$ we have exponential convergence with rate which is explicitly estimated.

Theorem 3.1: *Let $a > 0$, $0 \le b_{\min} < b_{\max}$ be constants and let $f \in C^0\left(D \times [0,a]; \Re\right)$ be a function satisfying (2) for which assumptions (H1), (H2) and inequality (11) hold. Consider system (4) and suppose that*

$$1+\min_{i=1,2}\left(\frac{1-\lambda_i}{\gamma_i}\right)\left|1-\beta\right| < \min_{i=1,2}\left(\frac{1-\lambda_i}{\gamma_i}\right),\ \frac{1}{1-\left|1-\beta\right|} < M < \min_{i=1,2}\left(\frac{1-\lambda_i}{\gamma_i}\right) \quad (15)$$

where $\beta := k_1 + k_2 + r$. Then the following estimate holds for the solution of (4) corresponding to arbitrary inputs $\{d(i) \in D\}_{i=0}^{\infty}$ and $\{v(i) \in \Re_+\}_{i=0}^{\infty}$ for all $t \ge 1$:

$$\begin{aligned}&\max\left(q(x^* - x(t)), x(t) - x^*\right) \le \\ &\lambda^t M(1+|r|+|k_1|+|k_2|)\left(\left|x(0)-x^*\right| + \left|y(0)-x^*\right| + \left|w(0)-u^*\right|\right), \\ &+\frac{\gamma}{1-\lambda} \max_{i=0,\dots,t-1}\left(\left|v(i)-v^*\right|\right)\end{aligned}$$

where $\lambda := \max\left(M^{-1} + |1-\beta|, L, \max_{i=1,2}\left(\lambda_i + M\gamma_i\right)\right)$ *and* $\gamma := 1 + M|k_1 + k_2| + M|r|$.

Remark 3.2: Theorem 3.1 guarantees the ISS property for system (4) with respect to the input $v \in \Re_+$ uniformly in $d \in D$. However, since we are most interested in the component x of the solution of (4), we have provided only an estimate for x. Similar Sontag-like estimates hold for all components of the solution of (4).

Remark 3.3: Saturation of traffic systems is a very important phenomenon, which must be avoided. It is reasonable to adopt the following definition for the saturation phenomenon of the traffic system (1) under the effect of the PI-regulator (3):

"The traffic network becomes saturated for inputs $\{(d(i), v(i)) \in D \times \Re_+\}_{i=0}^{\infty}$ and initial condition $(x(0), y(0), w(0)) \in [0,a]^2 \times \Re$ if for every $N \ge 1$ there exists $t \ge N$ such that the corresponding solution of (4) satisfies $x(t) = a$."

Theorem 3.1 guarantees that:

"for every pair of inputs $\{d(i) \in D\}_{i=0}^{\infty}$, $\{v(i) \in \Re_+\}_{i=0}^{\infty}$ with $\sup_{i\ge 0}\left(\left|v(i)-v^*\right|\right) < \frac{1-\lambda}{\gamma} a$ and for every initial condition $(x(0), y(0), w(0)) \in [0,a]^2 \times \Re$, the traffic network cannot become saturated, provided that the assumptions of Theorem 3.1 hold."

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5. REFERENCES

[1] Aboudolas, K. and N. Geroliminis, "Perimeter and Boundary Flow Control in Multi-Reservoir Heterogeneous Networks", *Transportation Research Part B*, 55, pp. 265-281, 2013.

[2] Cantoni, M., E. Weyer, Y. Li, K. O. Su, I. Mareels and M. Ryan, "Control of Large-Scale Irrigation Networks", *Proceedings of the IEEE*, 95(1), pp. 75-91, 2007.

[3] Carlson, R.C., I. Papamichail and M. Papageorgiou, "Local Feedback-Based Mainstream Traffic Flow Control On Motorways Using Variable Speed Limits", *IEEE Transactions on Intelligent Transportation Systems,* 12, pp. 1261-1276, 2011.

[4] Diamond, P., "Finite Stability Domains for Difference Equations", *Journal of the Australian Mathematical Society (Series A),* pp. 177-181, 1976.

[5] Jiang, Z.-P., and Y. Wang, "Input-to-State Stability for Discrete-Time Nonlinear Systems", *Automatica*, 37, pp. 857-869, 2001.

[6] Karafyllis, I., and Z.-P. Jiang, *Stability and Stabilization of Nonlinear Systems,* Springer-Verlag London (Series: Communications and Control Engineering), 2011.

[7] Karafyllis, I., and M. Papageorgiou, "Stability Results for Simple Traffic Models Under PI-Regulator Control", arXiv:1308.2505 [math.OC].

[8] Keyvan-Ekbatani, M., A. Kouvelas, I. Papamichail and M. Papageorgiou, "Exploiting the Fundamental Diagram of Urban Networks for Feedback-Based Gating", *Transportation Research Part B*, 46, pp. 1393–1403, 2012.

[9] Lakshmikantham, V. and D. Trigiante, *Theory of Difference Equations Numerical Methods and Applications*, 2nd Edition, Marcel Dekker, New York, 2002.

[10] Papageorgiou, M., and A. Messmer, "Continuous-Time and Discrete-Time Design of Water Flow and Water Level Regulators", *Automatica*, 21, pp. 649-661, 1985.

[11] Papageorgiou, M., and A. Messmer, "Flow Control of a Long River Stretch", *Automatica*, 25, 1989, pp. 177-183.

[12] Papageorgiou, M., H. Hadj-Salem, J.-M Blosseville, "ALINEA: A Local Feedback Control Law for On-Ramp Metering", *Transportation Research Record*, 1320, pp. 58-64, 1991.

[13] Papageorgiou, M., I. Papamichail, A.D. Spiliopoulou, and A.F. Lentzakis, "Real-Time Merging Traffic Control With Applications to Toll Plaza and Work Zone Management", *Transportation Research Part C*, 16, pp. 535-553, 2008.

[14] Sontag, E. D., *Mathematical Control Theory Deterministic Finite Dimensional Systems*, 2nd Edition, Springer-Verlag, New York, 1998.